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Block designs and strongly regular graphs admitting a transitive action of the Mathieu group M_{11}

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Abstract

In this paper we construct transitive t-designs, for $t \geq 2$, and strongly regular graphs from the Mathieu group M_{11}. We classify transitive t-designs with 11, 12 and 22 points admitting a transitive action of Mathieu group M_{11}. The most important result of this classification is proving the existence of a 3-design with parameters 3-$(22, 7, 18)$. Additionally, we prove the existence of 2-designs with certain parameters having 55 and 66 points. Furthermore, we classify strongly regular graphs on at most 450 vertices admitting a transitive action of the Mathieu group M_{11}.

1 Introduction

We assume that the reader is familiar with the basic facts of group theory, design theory and theory of strongly regular graphs. We refer the reader to [1, 23] for relevant background reading in design theory, to [5, 21] for relevant background reading in group theory, and to background reading in theory of strongly regular graphs we refer the reader to [1, 2, 23].

An incidence structure is an ordered triple $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ where $\mathcal{P}$ and $\mathcal{B}$ are non-empty disjoint sets and $\mathcal{I} \subseteq \mathcal{P} \times \mathcal{B}$. The elements of the set $\mathcal{P}$ are called points, the elements of the set $\mathcal{B}$ are called blocks and $\mathcal{I}$ is called an incidence relation. If $|\mathcal{P}| = |\mathcal{B}|$, then the incidence structure is called symmetric. The incidence matrix

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of an incidence structure is a $v \times b$ matrix $[m_{ij}]$ where v and b are the numbers of points and blocks respectively, such that $m_{ij} = 1$ if the point P_i and the block x_j are incident, and $m_{ij} = 0$ otherwise. An isomorphism from one incidence structure to another is a bijective mapping of points to points and blocks to blocks which preserves incidence. An isomorphism from an incidence structure $\mathcal{D}$ onto itself is called an automorphism of $\mathcal{D}$. The set of all automorphisms forms a group called the full automorphism group of $\mathcal{D}$ and is denoted by $Aut(\mathcal{D})$.

A t-(v, k, λ) design is a finite incidence structure $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ satisfying the following requirements:

1. $|\mathcal{P}| = v$;
2. every element of $\mathcal{B}$ is incident with exactly k elements of $\mathcal{P}$;
3. every t elements of $\mathcal{P}$ are incident with exactly λ elements of $\mathcal{B}$.

The elements of the set $\mathcal{P}$ are called points, and the elements of the set $\mathcal{B}$ are called blocks. Blocks can be regarded as subsets of the set of points. A t-design is called simple if it does not have repeated blocks. 2-designs are called block designs. If $\mathcal{D}$ is a t-design, then it is also an s-design, for $1 \leq s \leq t - 1$. Hence every t-design, for $t \geq 2$, is a block design. A simple 2-design having $\binom{v}{k}$ blocks is called complete. 2-designs which are not complete are often called balanced incomplete block designs (BIBDs). BIBDs have a wide range of applications, e.g. in experimental design, software testing, coding theory and cryptography. We say that a t-(v, k, λ) design $\mathcal{D}$ is a quasi-symmetric design with intersection numbers x and y ($x < y$) if any two blocks of $\mathcal{D}$ intersect in either x or y points.

A graph is regular if all the vertices have the same degree; a regular graph is strongly regular of type (v, k, λ, μ) if it has v vertices, degree k, and if any two adjacent vertices are together adjacent to λ vertices, while any two non-adjacent vertices are together adjacent to μ vertices. A strongly regular graph of type (v, k, λ, μ) is usually denoted by $\mathrm{SRG}(v, k, \lambda, \mu)$.

We say that an incidence structure $\mathcal{I}$ is transitive if an automorphism group of $\mathcal{I}$ acts transitively on points and blocks. A transitive incidence structure $\mathcal{I}$ is called primitive if an automorphism group acts primitively on points and blocks.

Further, we say that a graph Γ is transitive (primitive) if an automorphism group acts transitively (primitively) on the set of vertices of the graph Γ and that a graph is edge-transitive if an automorphism group acts transitively on the set of edges of the graph.

A flag of a design is an incident pair (point, block). We say that a t-design is flag-transitive if an automorphism group acts transitively on the set of flags of the design.

One of the main problems in design theory is classifying structures with given parameters or/and with a given automorphism group. In this paper we consider t-designs, $t \geq 2$, admitting a transitive action of the Mathieu group M_{11} on points and blocks. Construction of transitive designs from finite simple groups gives additional information on the group acting on a design, which is interesting from the group theoretical point of view. To our best knowledge, this is the first time that the simple group M_{11} was taken into consideration for the construction of combinatorial structures in a way described in this paper.

In this paper we consider t-designs and strongly regular graphs constructed from the Mathieu group M_{11}. The group M_{11} is the simple group of order 7920, the smallest of five Mathieu simple groups, and up to conjugation it has 39 subgroups given in Table 1. Using the method introduced in [8], we classify all t-designs on 11, 12 or 22 points on which the group M_{11} acts transitively on points and blocks. Additionally, we obtain numerous transitive designs, under the action of M_{11}, for $v =$ 55, 66. In many cases we prove the existence of 2-designs with certain parameters. We also prove the existence of a 3-$(22, 7, 18)$ design. In addition, we classify all t-designs, for $t \geq 2$, on which the group M_{11} acts flag-transitively. This can be seen as a contribution to the sections in the Handbook of Combinatorial Designs on t-designs with $t \geq 3$ by Khosrovshahi and Laue and to the section on t-designs with $t = 2$ by Kreher (references [14, 16]). All the designs obtained in this paper are simple.

Further, we construct strongly regular graphs on 55, 66, 144 or 330 vertices admitting a transitive action of the simple group M_{11}. The strongly regular graphs constructed have been known before, but constructed in a different way.

Generators of the group M_{11} are available on the Internet:

`http://brauer.maths.qmul.ac.uk/Atlas/`.

All the structures are obtained by using programs written for Magma [4]. The designs having 11, 12 or 22 points can be found at the link:

`http://www.math.uniri.hr/~asvob/M11designs.txt`.

The strongly regular graphs constructed in this paper can be found at the link:

`http://www.math.uniri.hr/~asvob/SRGs_M11.txt`.

The paper is organized as follows. In Section 2 we describe the method of construction of transitive designs used in this paper, and in Section 3 we describe combinatorial structures constructed under the action of the Mathieu group M_{11}.

2 The method of construction

The method for constructing transitive incidence structures was presented in [10]. Further research of the construction of primitive symmetric 1-designs and regular graphs for which the stabilizer of a point and the stabilizer of a block are conjugate is given in [11], [12] and [13]. In [8], a construction of not necessarily primitive, but still transitive, block designs is presented.

Theorem 1 ([8]) *Let G be a finite permutation group acting transitively on the sets Ω_1 and Ω_2 of size m and n, respectively. Let $\alpha \in \Omega_1$ and $\Delta_2 = \bigcup_{i=1}^{s} \delta_i G_\alpha$, where $G_\alpha = \{g \in G \mid \alpha g = \alpha\}$ is the stabilizer of α and $\delta_1, \ldots, \delta_s \in \Omega_2$ are representatives* *of distinct G_α-orbits on Ω_2. If $\Delta_2 \neq \Omega_2$ and*

$$\mathcal{B} = \{\Delta_2 g : g \in G\},$$

then $\mathcal{D}(G, \alpha, \delta_1, \ldots, \delta_s) = (\Omega_2, \mathcal{B})$ is a 1-$(n, |\Delta_2|, \frac{|G_\alpha|}{|G_{\Delta_2}|} \sum_{i=1}^{s} |\alpha G_{\delta_i}|)$ design with $\frac{m \cdot |G_\alpha|}{|G_{\Delta_2}|}$ blocks. The group $H \cong G / \bigcap_{x \in \Omega_2} G_x$ acts as an automorphism group on $(\Omega_2, \mathcal{B})$, transitively on points and blocks of the design.

If $\Delta_2 = \Omega_2$ then the set $\mathcal{B}$ consists of one block, and $\mathcal{D}(G, \alpha, \delta_1, \ldots, \delta_s)$ is a design with parameters 1-$(n, n, 1)$.

If a group G acts t-homogeneously on the set Ω_2, then the design obtained, $(\Omega_2, \mathcal{B})$, is a t-design (see [8]).

The construction described in Theorem 1 gives us all simple designs on which the group G acts transitively on the points and blocks; i.e. if G acts transitively on the points and blocks of a simple 1-design $\mathcal{D}$, then $\mathcal{D}$ can be obtained as described in Theorem 1. It follows from [6, Proposition 1.3.] that the group H acts flag-transitively on the design $(\Omega_2, \mathcal{B})$ if and only if the base block Δ_2 is a single G_α-orbit.

If a group G acts transitively on Ω, $\alpha \in \Omega$, and Δ is an orbit of G_α, then $\Delta' = \{\alpha g \mid g \in G,\ \alpha g^{-1} \in \Delta\}$ is also an orbit of G_α. Here Δ' is called the orbit of G_α paired with Δ. It is obvious that $\Delta'' = \Delta$ and $|\Delta'| = |\Delta|$. If $\Delta' = \Delta$, then Δ is said to be self-paired.

Corollary 1 *If $\Omega_1 = \Omega_2$ and Δ_2 is a union of self-paired and mutually paired orbits of G_α, then the design $\mathcal{D}(G, \alpha, \delta_1, \ldots, \delta_s)$ is a symmetric self-dual design and the incidence matrix of that design is the adjacency matrix of a $|\Delta_2|-$regular graph.*

Using Theorem 1 and Corollary 1 from [8], we construct t-designs and strongly regular graphs from the Mathieu group M_{11}. Additionally, combining the method given in Theorem 1 with the results presented in [6, Proposition 1.3.] we obtain all flag-transitive designs from the Mathieu group M_{11}.

The method of constructing designs and regular graphs described in Theorem 1 is a generalization of results presented in [7, 11, 12]. Using Corollary 1, one can construct all regular graphs admitting a transitive action of the group G, but we will be interested only in those regular graphs that are strongly regular.

For further details about implementing the construction and obtaining the results, including the isomorph rejection, we refer the reader to [9].

3 Combinatorial structures from M_{11}

The Mathieu group M_{11} is a simple group of order 7920, and up to conjugation it has 39 subgroups. It is the smallest sporadic group and acts 4-transitively on 11 points. There are five simple Mathieu groups, introduced by Emile Mathieu in [17, 18, 19] and M_{11} is the smallest among all Mathieu groups. The t-designs arising from the

Mathieu groups M_{22}, M_{23} and M_{24} have been studied in work by Kramer, Magliveras and Mesner (see [15]), but those arising from M_{11} in the way described in this paper have not been explored so far.

In Table 1 we give the list of all the subgroups, up to conjugation, and some of their properties. Since each transitive action of a group G is permutation isomorphic to an action of G on cosets of its subgroup, the indices of the subgroups in Table 1 give us degrees of all transitive actions of the group M_{11}.

Subgroup	Structure	Order	Index	Subgroup	Structure	Order	Index
H_1	I	1	7920	H_{21}	$Z_5 : Z_4$	20	396
H_2	Z_2	2	3960	H_{22}	$SL(2,3)$	24	330
H_3	Z_3	3	2640	H_{23}	S_4	24	330
H_4	Z_4	4	1980	H_{24}	$E_9 : Z_4$	36	220
H_5	E_4	4	1980	H_{25}	$E_9 : Z_4$	36	220
H_6	Z_5	5	1584	H_{26}	$S_3 \times S_3$	36	220
H_7	S_3	6	1320	H_{27}	$GL(2,3)$	48	165
H_8	S_3	6	1320	H_{28}	$Z_{11} : Z_5$	55	144
H_9	Z_6	6	1320	H_{29}	A_5	60	132
H_{10}	Q_8	8	990	H_{30}	A_5	60	132
H_{11}	D_8	8	990	H_{31}	$E_9 : Q_8$	72	110
H_{12}	Z_8	8	990	H_{32}	$(S_3 \times S_3) : Z_2$	72	110
H_{13}	E_9	9	880	H_{33}	$E_9 : Z_8$	72	110
H_{14}	D_{10}	10	792	H_{34}	S_5	120	66
H_{15}	Z_{11}	11	720	H_{35}	$(E_9 : Z_8) : Z_2$	144	55
H_{16}	A_4	12	660	H_{36}	A_6	360	22
H_{17}	D_{12}	12	660	H_{37}	$PSL(2,11)$	660	12
H_{18}	QD_{16}	16	495	H_{38}	$A_6.Z_2$	720	11
H_{19}	$E_9 : Z_2$	18	440	H_{39}	M_{11}	7920	1
H_{20}	$Z_3 \times S_3$	18	440				

Table 1: Subgroups of the group M_{11}

Generators of all subgroups presented in Table 1 are given in the Appendix.

3.1 t-designs with $v \leq 22$

In this section we give all t-designs with at most 22 points on which the group M_{11} acts transitively. The designs are obtained from the group M_{11} by using Theorem 1. In that case, the stabilizers of points are subgroups of M_{11} having the indices 11, 12 and 22. The list of all designs obtained is given in Table 2. In each table we give the parameters of the constructed structures, the number of non-isomorphic structures and their full automorphism group. The group M_{11} acts 4-transitively on 11 points, hence all designs obtained by Theorem 1 on 11 points are 4-designs. The designs marked with $*$ are flag-transitive. Additionally, we have obtained two more flag-transitive designs that are not mentioned in Table 2, the complements of the designs with parameters 4-$(11,5,1)$ and 3-$(12,4,3)$.

Parameters of designs	# of blocks	# non-isomorphic	Full automorphism group
3-$(11, 3, 1)^*$	165	1	S_{11}
4-$(11, 4, 1)^*$	330	1	S_{11}
4-$(11, 5, 1)^*$	66	1	M_{11}
4-$(11, 5, 6)^*$	396	1	M_{11}
3-$(12, 3, 1)^*$	220	1	S_{12}
3-$(12, 4, 6)^*$	330	1	M_{11}
3-$(12, 4, 3)^*$	165	1	M_{11}
3-$(12, 5, 6)^*$	132	1	M_{11}
3-$(12, 5, 30)$	660	1	M_{11}
3-$(12, 6, 2)^*$	22	1	M_{11}
3-$(12, 6, 10)^*$	110	1	M_{11}
5-$(12, 6, 6)$	792	1	M_{12}
2-$(22, 7, 36)$	396	1	M_{11}
2-$(22, 7, 180)$	1980	1	M_{11}
2-$(22, 7, 360)$	3960	3	M_{11}
2-$(22, 7, 720)$	7920	2	M_{11}
3-$(22, 7, 18)$	792	1	$M_{11} \times Z_2$
3-$(22, 7, 90)$	3960	3	$M_{11} \times Z_2$
3-$(22, 7, 180)$	7920	1	M_{11}

Table 2: t-designs constructed from the group M_{11}, $v \leq 22$

Remark 1 We proved the existence of a 3-$(22, 7, 18)$ design, since it is the first known example of the design with these parameters. The 2-designs with 22 points having parameters $(22, 7, 36)$ and $(22, 7, 180)$ from Table 2 are not mentioned in [20] since $r > 41$. To the best of our knowledge, the designs with these parameters have not been known before. The Steiner system 4-$(11, 5, 1)$ is known as the Witt design W_{11}. For further information on W_{11} we refer the reader to [25, 26]. The t-designs having the parameters of other transitive t-designs described in Table 2 were previously known. For further information on quasi-symmetric 3-$(12, 6, 2)$ we refer the reader to [22], and for others known designs mentioned in Table 2 see [14, 20].

3.2 BIBDs with $v = 55$

In this section we give all 2-designs with 55 points on which the group M_{11} acts transitively. Note that there are no t-designs with $v = 55$ and $t \geq 3$ on which M_{11} acts transitively. The designs are obtained from the group M_{11} by using Theorem 1. In that case, the stabilizer of a point is a subgroup of M_{11} having index 55. The list of all designs obtained is given in Table 3. In Table 3 we give the parameters of the constructed structures, the number of non-isomorphic structures and their full automorphism group.

Parameters of block designs	# non-isomorphic	Full automorphism group
2-(55, 3, 4)	1	M_{11}
2-(55, 3, 8)	1	M_{11}
2-(55, 4, 8)	1	M_{11}
2-(55, 4, 16)	7	M_{11}
2-(55, 6, 10)	1	M_{11}
2-(55, 6, 20)	2	M_{11}
2-(55, 6, 40)	67	M_{11}
2-(55, 7, 14)	1	M_{11}
2-(55, 7, 28)	5	M_{11}
2-(55, 7, 56)	115	M_{11}
2-(55, 9, 8)	1	M_{11}
2-(55, 9, 32)	10	M_{11}
2-(55, 9, 48)	23	M_{11}
2-(55, 9, 64)	16	M_{11}
2-(55, 9, 96)	632	M_{11}
2-(55, 10, 12)	2	M_{11}
2-(55, 10, 20)	1	M_{11}
2-(55, 10, 24)	4	M_{11}
2-(55, 10, 40)	8	M_{11}
2-(55, 10, 48)	6	M_{11}
2-(55, 10, 60)	27	M_{11}
2-(55, 10, 80)	9	M_{11}
2-(55, 10, 120)	1647	M_{11}
2-(55, 12, 44)	5	M_{11}
2-(55, 12, 88)	43	M_{11}
2-(55, 12, 176)	6025	M_{11}
2-(55, 13, 104)	81	M_{11}
2-(55, 13, 208)	9086	M_{11}
2-(55, 15, 56)	6	M_{11}
2-(55, 15, 112)	8	M_{11}
2-(55, 15, 140)	53	M_{11}
2-(55, 15, 280)	23748	M_{11}
2-(55, 16, 80)	3	M_{11}
2-(55, 16, 160)	93	M_{11}
2-(55, 16, 320)	$\geq$ 8500	M_{11}
2-(55, 18, 68)	2	M_{11}
2-(55, 18, 102)	8	M_{11}
2-(55, 18, 136)	39	M_{11}
2-(55, 18, 204)	215	M_{11}
2-(55, 18, 272)	161	M_{11}
2-(55, 18, 408)	$\geq$ 20162	M_{11}
2-(55, 19, 152)	18	M_{11}
2-(55, 19, 228)	173	M_{11}
2-(55, 19, 304)	376	M_{11}
2-(55, 19, 456)	$\geq$ 15310	M_{11}
2-(55, 21, 140)	9	M_{11}
2-(55, 21, 280)	249	M_{11}
2-(55, 21, 560)	$\geq$ 14250	M_{11}
2-(55, 22, 308)	220	M_{11}
2-(55, 22, 616)	$\geq$ 18400	M_{11}
2-(55, 24, 184)	8	M_{11}
2-(55, 24, 368)	256	M_{11}
2-(55, 24, 736)	$\geq$ 12100	M_{11}
2-(55, 25, 200)	8	M_{11}
2-(55, 25, 320)	25	M_{11}
2-(55, 25, 400)	445	M_{11}
2-(55, 25, 800)	$\geq$ 12800	M_{11}
2-(55, 27, 78)	1	S_{11}
2-(55, 27, 234)	8	M_{11}
2-(55, 27, 312)	56	M_{11}
2-(55, 27, 468)	308	M_{11}
2-(55, 27, 624)	626	M_{11}
2-(55, 27, 936)	$\geq$ 16905	M_{11}

Table 3: BIBDs constructed from M_{11}, $v = 55$

Remark 2 Designs from Table 3 are not mentioned in [20] since $r > 41$. To our best knowledge they have not been known before, so we proved the existence of 2-designs with the parameters listed in Table 3. The design with parameters 2-$(55, 4, 8)$ is flag-transitive.

3.3 BIBDs with $v = 66$

In this section we give all 2-designs with 66 points on which the group M_{11} acts transitively. Note that there is no t-designs with $v = 66$ and $t > 2$ on which M_{11} acts transitively. The designs are obtained from the group M_{11} by using Theorem 1. In that case, the stabilizer of a point is subgroup of M_{11} having the index 66.

In Table 4 we give the parameters of the constructed structures, the number of non-isomorphic structures and their full automorphism group.

Parameters of block designs	# non-isomorphic	Full automorphism group
2-$(66, 13, 36)$	1	M_{11}
2-$(66, 13, 48)$	13	M_{11}
2-$(66, 13, 72)$	43	M_{11}
2-$(66, 13, 96)$	79	M_{11}
2-$(66, 13, 144)$	≥ 1210	M_{11}
2-$(66, 14, 56)$	6	M_{11}
2-$(66, 14, 84)$	33	M_{11}
2-$(66, 14, 112)$	105	M_{11}
2-$(66, 14, 168)$	≥ 223	M_{11}
2-$(66, 26, 100)$	2	M_{11}
2-$(66, 26, 120)$	2	M_{11}
2-$(66, 26, 200)$	33	M_{11}
2-$(66, 26, 240)$	4	M_{11}
2-$(66, 26, 300)$	159	M_{11}
2-$(66, 26, 400)$	1799	M_{11}
2-$(66, 26, 600)$	≥ 8910	M_{11}
2-$(66, 27, 36)$	1	M_{11}
2-$(66, 27, 72)$	1	M_{11}
2-$(66, 27, 216)$	60	M_{11}
2-$(66, 27, 324)$	49	M_{11}
2-$(66, 27, 432)$	412	M_{11}
2-$(66, 27, 648)$	≥ 7913	M_{11}

Table 4: BIBDs constructed from M_{11}, $v = 66$

Remark 3 Designs from Table 4 are not mentioned in [20] since $r > 41$. As far as we know, they were not known before, so we proved the existence of 2-designs with the parameters listed in Table 4.

3.4 Strongly regular graphs

Using the method described in Theorem 1 and Corollary 1, we have obtained all regular graphs with at most 450 vertices admitting a transitive action of the group M_{11}. Using a computer search we have obtained strongly regular graphs on 55, 66, 144 or 330 vertices. Finally, we give the full automorphism groups of the constructed SRGs.

Theorem 2 *Up to isomorphism there are exactly* 5 *strongly regular graphs with at most* 450 *vertices, admitting a transitive action of the group* M_{11}. *These strongly regular graphs have parameters* $(55, 18, 9, 4)$, $(66, 20, 10, 4)$, $(144, 55, 22, 20)$, $(144, 66, 30, 30)$ *and* $(330, 63, 24, 9)$. *Details about the strongly regular graphs obtained are given in Table 5.*

Graph Γ	Parameters	$Aut(\Gamma)$
Γ_1	(55,18,9,4)	S_{11}
Γ_2	(66,20,10,4)	S_{12}
Γ_3	(144,55,22,20)	M_{11}
Γ_4	(144,66,30,30)	$M_{12} : Z_2$
Γ_5	(330,63,24,9)	S_{11}

Table 5: SRGs constructed from the Mathieu group M_{11}

Remark 4 The graphs Γ_1 and Γ_2 are the triangular graphs $T(11)$ and $T(12)$, respectively. Strongly regular graphs with parameters $(144, 55, 22, 20)$, $(144, 66, 30, 30)$ and $(330, 63, 24, 9)$ have been known before (see [2, 3]). The adjacency matrix of a $SRG(144, 66, 30, 30)$ is the incidence matrix of a symmetric design with parameters $(144, 66, 30)$, a design with Menon parameters (related to a regular Hadamard matrix of order 144). This symmetric design is described in [24]. The group M_{11} does not act flag transitively on that symmetric $(144, 66, 30)$ design, since the base block is the union of G_α-orbits (see Theorem 1). However, the Mathieu group M_{12} acts flag transitively on that symmetric $(144, 66, 30)$ design. Consequently, the group M_{12} acts edge-transitively on the constructed $SRG(144, 66, 30, 30)$, while the Mathieu group M_{11} does not. The $SRG(330, 63, 24, 9)$ is the distance 1 or 4 graph in the Johnson graph $J(11, 4)$. The Johnson graph $J(11, 4)$ can also be constructed from the Mathieu group M_{11} using Theorem 1. The strongly regular graphs $\Gamma_1, \Gamma_2, \Gamma_3$ and Γ_4 are edge-transitive with respect to their full automorphism group. Additionally, the same holds for the complements of the graphs Γ_1 and Γ_2.

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Appendix

In this section we give generators of the subgroups of M_{11} described in Section 3, Table 1, as permutations on 11 points.

$$
\begin{aligned}
H_1 =& I\\
H_2 =& \langle (1,3)(2,11)(4,9)(6,7)\rangle\\
H_3 =& \langle (1,3,10)(2,7,4)(6,11,9)\rangle\\
H_4 =& \langle (1,7,2,4,6)(3,11,8,10,9)\rangle\\
H_5 =& \langle (1,7,3,11,4,6,5,8,9,10,2)\rangle\\
H_6 =& \langle (1,3)(2,9)(4,11)(8,10),(1,3)(2,11)(4,9)(6,7)\rangle\\
H_7 =& \langle (2,11)(3,6)(4,10)(8,9),(2,6,11,3)(4,8,10,9)\rangle\\
H_8 =& \langle (2,7,11)(3,8,10)(4,9,6),(1,5)(2,11)(3,10)(4,6)\rangle\\
H_9 =& \langle (2,3)(6,9)(7,10)(8,11),(2,7,11)(3,8,10)(4,9,6)\rangle\\
H_{10} =& \langle (1,9)(2,3)(5,6)(8,10),(1,10,6)(4,11,7)(5,9,8)\rangle\\
H_{11} =& \langle (1,8,9)(2,10,3)(4,11,6),(1,4,10)(2,9,6)(3,8,11)\rangle\\
H_{12} =& \langle (1,7,2,4,6)(3,11,8,10,9),(1,7)(2,6)(3,11)(8,9)\rangle\\
H_{13} =& \langle (1,11,9,10,8)(2,6,4,5,3),(1,7,3,11,4,6,5,8,9,10,2)\rangle\\
H_{14} =& \langle (2,11)(3,6)(4,10)(8,9),(2,6,11,3)(4,8,10,9),(2,8,11,9)(3,10,6,4)\rangle\\
H_{15} =& \langle (1,11)(2,7)(3,6)(9,10),(1,7)(2,11)(4,8)(9,10),(2,7)(3,9)(4,8)(6,10)\rangle\\
H_{16} =& \langle (1,11,7,8,4,6,10,9)(2,3),(1,4)(6,11)(7,10)(8,9),(1,7,4,10)(6,9,11,8)\rangle\\
H_{17} =& \langle (1,7,8)(2,4,11)(3,6,10),(1,3)(2,9)(4,11)(8,10),(1,3)(2,11)(4,9)(6,7)\rangle\\
H_{18} =& \langle (1,6)(2,3)(5,9)(7,11),(1,9)(2,3)(5,6)(8,10),(1,10,6)(4,11,7)(5,9,8)\rangle\\
H_{19} =& \langle (2,11)(3,6)(4,10)(8,9),(1,8,9)(2,10,3)(4,11,6),(1,4,10)(2,9,6)(3,8,11)\rangle\\
H_{20} =& \langle (1,5)(2,6)(3,9)(7,10),(1,3,10)(4,11,8)(5,9,7),(1,11,9)(3,8,7)(4,5,10)\rangle\\
H_{21} =& \langle (2,11)(3,6)(4,10)(8,9),(2,4,11,10)(3,8,6,9),(1,8,6,3,9)(2,4,10,11,7)\rangle\\
H_{22} =& \langle (2,3)(6,9)(7,10)(8,11),(1,11,4,6)(7,9,10,8),(1,4)(6,11)(7,10)(8,9),\\
& (1,7,4,10)(6,9,11,8)\rangle\\
H_{23} =& \langle (3,6,10)(4,9,8)(5,11,7),(1,6,2,7)(3,10,11,5),(1,11,2,3)(5,6,10,7),\\
& (1,2)(3,11)(5,10)(6,7)\rangle\\
H_{24} =& \langle (1,11)(2,7)(3,6)(9,10),(1,7)(2,11)(4,8)(9,10),(1,2,8)(3,10,9)(4,11,7),\\
& (2,7)(3,9)(4,8)(6,10)\rangle\\
H_{25} =& \langle (2,11)(3,6)(4,10)(8,9),(1,4,10)(2,9,6)(3,8,11),(2,4,11,10)(3,8,6,9),\\
& (1,6,3)(2,8,4)(9,11,10)\rangle\\
H_{26} =& \langle (2,11)(3,6)(4,10)(8,9),(1,4,10)(2,9,6)(3,8,11),(1,6,3)(2,8,4)(9,11,10),\\
& (2,8,11,9)(3,10,6,4)\rangle
\end{aligned}
$$

$H_{27} = \langle (1,5)(2,6)(3,9)(7,10), (1,3,10)(4,11,8)(5,9,7), (1,5)(3,7)(4,11)(9,10), (1,11,9)(3,8,7)(4,5,10) \rangle$
$H_{28} = \langle (1,5)(2,7)(3,8)(4,9), (1,2,11)(3,4,9)(6,8,10) \rangle$
$H_{29} = \langle (1,7,2)(3,4,10)(6,8,9), (2,10)(3,11)(4,6)(7,8) \rangle$
$H_{30} = \langle (3,5)(6,7)(8,9)(10,11), (3,6,10)(4,9,8)(5,11,7), (1,6,2,7)(3,10,11,5), (1,11,2,3)(5,6,10,7), (1,2)(3,11)(5,10)(6,7) \rangle$
$H_{31} = \langle (2,11)(3,6)(4,10)(8,9), (1,4,10)(2,9,6)(3,8,11), (2,6,11,3)(4,8,10,9), (1,6,3)(2,8,4)(9,11,10), (2,8,11,9)(3,10,6,4) \rangle$
$H_{32} = \langle (1,5)(2,6)(3,9)(7,10), (1,3,10)(4,11,8)(5,9,7), (1,5)(3,7)(4,11)(9,10), (1,4,5,11)(3,9,7,10), (1,11,9)(3,8,7)(4,5,10) \rangle$
$H_{33} = \langle (1,7,11,9,5,3,4,10)(2,6), (1,3,10)(4,11,8)(5,9,7), (1,5)(3,7)(4,11)(9,10), (1,4,5,11)(3,9,7,10), (1,11,9)(3,8,7)(4,5,10) \rangle$
$H_{34} = \langle (1,4)(3,9)(5,8)(6,7), (1,7,6,5,4,8)(2,10,9)(3,11) \rangle$
$H_{35} = \langle (1,6)(2,3)(5,9)(7,11), (1,11,5)(3,6,8)(4,9,10) \rangle$
$H_{36} = \langle (1,5)(2,6)(3,9)(7,10), (1,3,5,7)(4,10,11,9), (1,3,10)(4,11,8)(5,9,7), (1,5)(3,7)(4,11)(9,10), (1,4,5,11)(3,9,7,10), (1,11,9)(3,8,7)(4,5,10) \rangle$
$H_{37} = \langle (1,7)(2,11)(4,8)(9,10), (1,10,8,3)(2,7,9,6) \rangle$
$H_{38} = \langle (1,8)(2,6)(3,4)(10,11), (1,6,8,7)(2,10,9,3) \rangle$
$H_{39} = \langle (1,2)(4,5)(6,11)(8,10), (1,6,9,4)(3,8,7,11) \rangle$

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