AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **73(1)** (2019), Pages 42–70

Refinements of Hall's condition

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Abstract

Let $\mathcal{C}$ be an infinite set of symbols. A function L is a list assignment to a graph G if L assigns to each vertex of G a non-empty finite subset of $\mathcal{C}$, called a list. A proper L-coloring of G is an assignment of "colors"to the vertices of G, from their lists, so that adjacent vertices are colored with different colors. Interpreted as a theorem about proper list colorings of complete graphs, P. Hall's theorem on systems of distinct representatives inspires a generalization, a necessary condition for proper colorings, known as Hall's Condition (HC). In this paper we present several refinements of HC, and for each of them, we look into the question: for which graphs is satisfaction of this condition by the graph and a list assignment L sufficient for the existence of a proper L-coloring of the graph?

1 Introduction

Graph list coloring was introduced in the late 1970s independently by Vizing [17] with the intention to study total colorings, and Erdős, Rubin and Taylor [5], with motivation from Dinitz's conjecture on $n \times n$ arrays. In [10] and [11] Hilton and

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ISSN: 2202-3518

Johnson brought forth a necessary condition on a graph with a list assignment L for the existence of a proper L-coloring, and proceeded to raise many questions and give very few answers. In this paper we will raise questions and give a few answers concerning certain strengthenings, or refinements, of that necessary condition. But first, here are some definitions in graph theory. The reader can refer to [18].

Throughout this paper, the graph $G = (V, E)$ will be a finite simple graph with vertex set $V = V(G)$ and edge set $E = E(G)$.

A *list assignment* to the graph G is a function L which assigns a non-empty finite set (list) $L(v)$ to each vertex $v \in V(G)$.

A *proper L-coloring* of G is a function $\psi : V(G) \to \bigcup_{v\in V(G)} L(v)$ satisfying, for every $u, v \in V(G)$,

(i) $\psi(v) \in L(v)$,

(ii) $uv \in E(G) \Rightarrow \psi(v) \neq \psi(u)$.

The *choice number* or *list-chromatic number of* G, denoted by $ch(G)$, is the smallest integer k such that there is always a proper L-coloring of G if L satisfies $|L(v)| \geq k$ for every $v \in V(G)$. With χ denoting the chromatic number, it is easy to see, and well known, that $\chi(G) \leq ch(G)$. The extremal equation $\chi(G) = ch(G)$ is a major research interest; see, for instance, [4] and [5].

Theorem 1.1 (P. Hall [9])**.** *Suppose $A_1, A_2, \ldots, A_n$ are (not necessarily distinct) finite sets. There exist distinct elements $a_1, a_2, \ldots, a_n$ such that $a_i \in A_i$, $i = 1, 2, \ldots, n$, if and only if for each $J \subseteq \{1, 2, \ldots, n\}$, $|\bigcup_{j\in J} A_j| \geq |J|$.*

A list of distinct elements $a_1, \ldots, a_n$ such that $a_i \in A_i$, $i = 1, \ldots, n$, is called a *system of distinct representatives* of the sets $A_1, \ldots, A_n$. A proper L-coloring of a complete graph K_n is simply a system of distinct representatives of the finite lists $L(v)$, $v \in V$. Therefore, as noted in [10], Hall's theorem can be restated as:

Theorem 1.2. (Hall's theorem restated). *Suppose that L is a list assignment to K_n. There is a proper L-coloring of K_n if and only if, for all $U \subseteq V(K_n)$, $|L(U)| = |\bigcup_{u\in U} L(u)| \geq |U|$.*

Let L be a list assignment to a simple graph G, H a subgraph of G and $\mathcal{C}$ the set of possible colors. If $\psi : V(G) \to \mathcal{C}$ is a proper L-coloring of G, then for any subgraph $H \subset G$, ψ restricted to $V(H)$ is a proper L-coloring of H.

For any $\sigma \in \mathcal{C}$, let $H(\sigma, L) = <\{v \in V(H) \mid \sigma \in L(v)\}>$ denote the subgraph of H induced by the support set $\{v \in V(H) \mid \sigma \in L(v)\}$. For convenience, we sometimes simply write H_σ.

For each $\sigma \in \mathcal{C}$, $\psi^{-1}(\sigma) = \{v \in V(G) \mid \psi(v) = \sigma\} \subseteq V(G_\sigma)$; $\psi^{-1}(\sigma)$ is an independent set because ψ is a proper L-coloring. Further, $\psi^{-1}(\sigma) \cap V(H) \subseteq V(H_\sigma)$. So, $|\psi^{-1}(\sigma) \cap V(H)| \leq \alpha(H_\sigma)$, where α denotes the vertex independence number. This implies that

$$\sum_{\sigma\in\mathcal{C}} \alpha(H_\sigma) \geq \sum_{\sigma\in\mathcal{C}} |\psi^{-1}(\sigma) \cap V(H)| = |V(H)| \text{ for all } H \subseteq G.$$

When G and L satisfy the inequality

$$\sum_{\sigma\in\mathcal{C}} \alpha(H_\sigma) \geq |V(H)| \tag{1}$$

for each subgraph H of G, they are said to satisfy *Hall's Condition*. By the discussion preceding, Hall's Condition is a necessary condition for a proper L-coloring of G. Because removing edges does not diminish the vertex independence number, for G and L to satisfy Hall's Condition it suffices that (1) holds for all induced subgraphs H of G.

Hall's Condition is sufficient for a proper coloring when $G = K_n$, because if H is an induced subgraph of K_n then for each $\sigma \in \mathcal{C}$,

$$\alpha(H_\sigma) = \begin{cases} 1 & if\ \sigma \in \bigcup_{v\in V(H)} L(v) \\ 0, & otherwise. \end{cases}$$

So

$$\sum_{\sigma\in\mathcal{C}} \alpha(H_\sigma) = |\bigcup_{v\in V(H)} L(v)| \ ;$$

therefore Hall's Condition, that

$$\sum_{\sigma\in\mathcal{C}} \alpha(H_\sigma) \ \geq \ |V(H)|$$

for every such H, is just a restatement of the condition in Theorem 1.2. (It is necessary to point out here that if $\sigma \notin L(v)$ for all $v \in V(H)$ then H_σ is the null graph, and $\alpha(H_\sigma) = 0$.) Consequently, Hall's theorem may be restated: For complete graphs, Hall's Condition on the graph and a list assignment suffices for a proper coloring.

The class of graphs for which Hall's condition suffices for the existence of proper list colorings is small; see Theorems 1.3 and 2.3 . Figure 1 shows the smallest graph with a list assignment L_0 for which Hall's Condition holds, and yet G has no proper L_0-coloring.

Remark 1.1. It is clear that if H is an induced subgraph of G and $H \neq G$, then $H \subseteq G - v$ for some $v \in V(G)$. So, if $G - v$ has a proper L-coloring, then $H \subseteq G - v$ and L must satisfy (by necessity) (1). Thus, in practice, in order to show that G and L satisfy Hall's Condition, it suffices to verify that $G - v$ is properly L-colorable for each $v \in V(G)$ and that G itself satisfies the inequality (1).

Denoted by $h(G)$, the *Hall number* of a graph G is the smallest positive integer k such that there is a proper L-coloring of G whenever G and L satisfy Hall's Condition and $|L(v)| \geq k$ for each $v \in V(G)$. Clearly, $h(G) = 1$ if and only if the satisfaction of Hall's Condition by G and a list assignment L is sufficient for the existence of a proper L-coloring of G. Therefore, $h(K_n) = 1$ for each $n = 1, 2, \ldots$. It is well-known (see, for instance, [5] or [12]) that for every n-cycle C_n, $n \geq 4$, $h(C_n) = 2$.

Example 1.1. The following example originally appeared in [10]. Consider the four-cycle, C_4, in Figure 1, a bipartite graph with parts $V_i = \{u_i, v_i\}$, $i = 1, 2$ and L_0 the list assignment indicated.

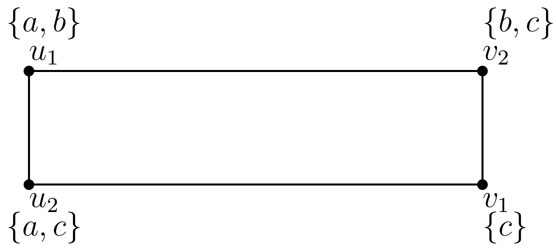

Figure 1: A list assignment to C_4.

It is straightforward to see that there is no proper L_0-coloring of C_4, even though C_4 and L_0 satisfy Hall's Condition. (Use the method suggested in Remark 1.1 to verify the latter claim.)

Next, we present a couple of results that can be found in [11].

Theorem 1.3. *A graph G has the property that for all L, if G and L satisfy Hall's Condition then there is a proper L-coloring of G, if and only if every block of G is a clique.*

Lemma 1.1. *Hall's Condition holds for G and L if and only if the inequality (1) holds for each connected induced subgraph of G.*

Suppose G and L satisfy Hall's Condition and let H be a subgraph of G. H is said to be an *L-tight* subgraph of G if

$$\sum_{\sigma \in \mathcal{C}} \alpha(H(\sigma, L)) = |V(H)|.$$

If H is non-induced but is L-tight, then the subgraph induced by the vertices of H will also be L-tight. To see this, let $H' = G[V(H)]$, the subgraph of G induced by the vertices of H. Obviously, $\alpha(H'(\sigma, L)) \leq \alpha(H(\sigma, L))$ for all $\sigma \in \mathcal{C}$ and all $L : V(G) \to \mathcal{F}(C) = \{\mathit{finite\ subsets\ of\ } \mathcal{C}\}$. If G and L satisfy Hall's condition and H is L-tight, then

$$|V(H)| = |V(H')| \leq \sum_{\sigma \in \mathcal{C}} \alpha(H'(\sigma, L)) \leq \sum_{\sigma \in \mathcal{C}} \alpha(H(\sigma, L)) = |V(H)|.$$

Although we will be considering L-tight subgraphs of G only under the assumption that G and L satisfy Hall's condition, we can allow a subgraph H of G to be L-tight even in the absence of this assumption. When H is L-tight, then in every proper L-coloring of H, should any exist, every color σ has to appear $\alpha(H(\sigma, L))$ times; i.e., for each $\sigma \in \mathcal{C}$, a maximum independent subset of $V(H_\sigma)$ is colored σ. This observation will be much used in the proofs in sections 2, 3, and 4.

The careful reader may have noticed that our definitions so far do not make sense unless we allow the existence of a *null graph* Γ, a graph with no vertices and, therefore, no edges. If $\sigma \in \mathcal{C}$ appears on no list $L(v)$, $v \in V(G)$, then $H(\sigma, L) = \Gamma$ for every subgraph H of G. We consider Γ to be a subgraph of every simple graph. Since, clearly, $\alpha(\Gamma) = 0 = |V(\Gamma)|$, Γ is therefore an L-tight subgraph of every subgraph of a graph G with list assignment L.

In the next section we introduce 4 refinements of Hall's condition, of two distinct types, and verify that each condition on a graph G and a list assignment L is necessary for the existence of a proper L-coloring of G. For each condition HCX on G and L we define a graph G to be HallX if the satisfying of the condition HCX on G and L is sufficient for the existence of a proper L-coloring of G. We show that the families HallX are closed under taking induced subgraphs.

In sections 3 and 4 we show that each of the families HallX are closed under the operation of attaching a clique at a single vertex; this is also a property possessed by the family Hall associated with HC, our abbreviation of Hall's condition. We also show that cycles of length ≥ 4, and K_4-minus-an-edge, which are not Hall, are HallX for each of the 4 new families.

In section 5 we introduce the Sudoku-Hall conditions, of which we know little. In section 6 we pose some questions.

2 Refinements of Hall's Condition

From here on, we will abbreviate Hall's Condition to *HC*. The following definitions are of conditions on a pair (G, L), where L is a list assignment to the graph G. In each case, the condition is necessary for the existence of a proper L-coloring of G, and in each case, the condition is at least as strong as HC, meaning that if G and L satisfy the condition, then they must satisfy HC. This will be obvious in all but Definitions 2.3 and 2.4; in the other definitions, HC will be an explicit requirement. In these definitions, $\mathcal{C}$ will stand for an infinite set of colors such that all list assignments assign finite subsets of $\mathcal{C}$ to vertices of graphs. When L is a list assignment to G and $S \subseteq V(G)$, let $L(S) = \bigcup_{v \in S} L(v)$. When $\sigma \in \mathcal{C} \backslash L(V(G))$ then $G(\sigma, L) = \Gamma$, the null graph.

Definition 2.1. G and L are said to satisfy **Hall's Condition plus**, denoted **HC+**, if they satisfy HC and there is an indexed family $\{S_\sigma : \sigma \in \mathcal{C}\}$ of independent subsets of $V(G)$ satisfying:

(i) $S_\sigma \subseteq V(G(\sigma, L))$ for all $\sigma \in \mathcal{C}$ (i.e., for all vertices $v \in S_\sigma$, $\sigma \in L(v)$).

(ii) For each L-tight subgraph H of G, $|S_\sigma \cap V(H)| = \alpha(H(\sigma, L))$.

In the previous definition, any collection $\{S_\sigma : \sigma \in \mathcal{C}\}$ of independent subsets of $V(G)$ satisfying (i) and (ii) is called an *HC+ -satisfying family* with respect to the pair (G, L).

Definition 2.2. G and L are said to satisfy **Hall's Condition plus plus**, denoted **HC++**, if they satisfy HC+, and in some HC+ -satisfying family $\{S_\sigma : \sigma \in \mathcal{C}\}$ of independent subsets of $V(G)$, the S_σ's are pairwise disjoint. (That is, $S_\sigma \cap S_\tau = \emptyset$ for $\sigma \neq \tau$).

In both definitions, when $V(G(\sigma, L)) = \emptyset$—i.e., when σ does not appear in the L-lists on G—then $S_\sigma = \emptyset$ is forced. But it can happen that $S_\sigma = \emptyset$ even when $G(\sigma, L) \neq \Gamma$.

$HC++$-satisfying families are defined analogously to $HC+$ - satisfying families.

If ψ is a proper L-coloring of G, then the ψ supports $S_\sigma = \psi^{-1}(\sigma) = \{v \in V | \psi(v) = \sigma\}$, $\sigma \in \mathcal{C}$ form an *HC++* -satisfying family for G and L. So, HC++ (and hence HC+) is a necessary condition for a proper L-coloring of G.

For an induced subgraph H of G, the *conditional independence number* of G with respect to H, denoted $\alpha(G|H)$, is the maximum cardinality of an independent set I of $V(G)$ such that $|I \cap V(H)| = \alpha(H)$.

Definition 2.3. A graph G with list assignment L is said to satisfy **Hall's Condition star**, denoted **HC***, if

$$\sum_{\sigma \in \mathcal{C}} \alpha(H_\sigma | T_\sigma) \geq |V(H)|, \tag{2}$$

for every induced subgraph H of G and every induced L-tight subgraph T of H.

Definition 2.4. A graph G with a list assignment L is said to satisfy **Hall's Condition star star**, denoted **HC****, if

$$\sum_{\sigma \in \mathcal{C}} \min_{T \lhd H} \alpha(H_\sigma | T_\sigma) \geq |V(H)|, \tag{3}$$

for every induced subgraph H of G, where the minimum is taken over all L-tight induced subgraphs T of H. ($T \lhd H$ means: T is an induced subgraph of H.)

As mentioned in section 1, we consider the null graph Γ to be an L-tight subgraph of any induced subgraph H of G; $\Gamma_\sigma = \Gamma$ for each $\sigma \in \mathcal{C}$. Since $\alpha(H|\Gamma) = \alpha(H)$ for all H, it follows that HC**$\Longrightarrow$ HC* $\Longrightarrow$ HC.

Theorem 2.1. *HC** is a necessary condition for the existence of a proper L-coloring of a graph G.*

Proof. Suppose ψ is a proper L-coloring of G. We proceed to show that HC** holds. Let H be an induced subgraph of G. Suppose T is an L-tight induced subgraph of H. Then

$$\sum_{\sigma \in \mathcal{C}} \alpha(T_\sigma) = |V(T)|, \tag{4}$$

and the number of times that $\sigma \in \mathcal{C}$ appears as a color assigned by the function ψ on T, i.e., $|\psi^{-1}(\sigma) \cap V(T)|$, is actually $\alpha(T_\sigma)$. Therefore the number of times an arbitrary color $\sigma \in \mathcal{C}$ appears as a color in H must satisfy the inequality

$$|\psi^{-1}(\sigma) \cap V(H)| \leq \alpha(H_\sigma|T_\sigma) \tag{5}$$

because $\psi^{-1}(\sigma) \cap V(H) \subseteq V(H_\sigma)$ is an independent set of vertices of H_σ which extends (by T being L-tight) a maximum independent set of vertices of T_σ. Since inequality (5) holds for every L-tight induced subgraph T of H, it must be that

$$|V(H)| = \sum_{\sigma \in \mathcal{C}} |\psi^{-1}(\sigma) \cap V(H)| \quad \leq \quad \sum_{\sigma \in \mathcal{C}} \min_{T \triangleleft H} \alpha(H_\sigma|T_\sigma)$$

which establishes the inequality (3). □

Definition 2.5. For $\diamondsuit \in \{$ empty string, +, ++, *, ** $\}$, G is a **Hall**$\diamondsuit$ graph if, whenever G, L satisfy HC$\diamondsuit$, there is a proper L-coloring of G.

We convene that "G is a Hall$\diamondsuit$ graph", "G is Hall$\diamondsuit$", and "$G \in Hall\diamondsuit$" all mean the same thing. Since HC++ $\Longrightarrow$ HC+ $\Longrightarrow$ HC and HC** $\Longrightarrow$ HC* $\Longrightarrow$ HC, it follows that *Hall++* $\supseteq$ *Hall+* $\supseteq$ *Hall* and that *Hall*** $\supseteq$ *Hall** $\supseteq$ *Hall*.

The rest of this paper is mainly devoted to proving some fundamental results about the graph families Hall$\diamondsuit$. There are two main results, in each case:

1. If $G \in Hall\diamondsuit$ and H is an induced subgraph of G, then $H \in Hall\diamondsuit$.
2. If $G \in Hall\diamondsuit$ and X is obtained from G by attaching a clique to G at a single vertex of G, then $X \in Hall\diamondsuit$.

We will dispose the first of these right here.

Theorem 2.2. *Suppose that $\diamondsuit \in \{$ empty string, +, ++, *, ** $\}$, $G \in Hall\diamondsuit$ and H is an induced subgraph of G. Then $H \in Hall\diamondsuit$.*

Proof. Suppose that L is a list assignment to H such that H and L satisfy $HC\diamondsuit$. Extend L to a list assignment $\tilde{L}$ to G such that for all $v \in V(G)\backslash V(H)$, $|\tilde{L}(v)| > |V(G)|$, and the sets $\tilde{L}(v)$, $v \in V(G)\backslash V(H)$, are pairwise disjoint, and each disjoint from $L(V(H)) = \bigcup_{u \in V(H)} L(u)$. Then the only $\tilde{L}$-tight induced subgraphs of G are L-tight induced subgraphs of H. It is then straightforward to verify, from the assumption that H and L satisfy $HC\diamondsuit$, that G and $\tilde{L}$ satisfy $HC\diamondsuit$. Therefore, because $G \in Hall\diamondsuit$, there is a proper $\tilde{L}$-coloring of G. The restriction of such a coloring of H is a proper L-coloring of H. Since L was an arbitrary list assignment to H such that H and L satisfy $HC\diamondsuit$, it follows that $H \in Hall\diamondsuit$. □

A family $\mathcal{G}$ of (finite simple) graphs has a *forbidden-induced-subgraph characterization* if and only if there is a collection $\mathscr{F}$ of graphs such that $G \in \mathcal{G}$ if and only $\mathcal{G}$ contains no element of $\mathscr{F}$ as an induced subgraph. The following lemma is folklorically well known.

Lemma 2.1. *A family $\mathcal{G}$ of graphs has a forbidden-induced-subgraph characterization if and only if $\mathcal{G}$ is closed under the operation of taking induced subgraphs.*

Proof. Suppose that $G \in \mathcal{G}$ if and only if G contains no subgraph in $\mathscr{F}$ as an induced subgraph. Since an induced subgraph of an induced subgraph is an induced subgraph, it follows that every induced subgraph of any $G \in \mathcal{G}$ is in the family $\mathcal{G}$.

Now suppose that $\mathcal{G}$ is closed under the operation of taking induced subgraphs. Take $\mathscr{F} = \{F \mid F\ is\ a\ finite\ simple\ graph\ and\ F \notin \mathcal{G}\}$. Then $G \in \mathcal{G}$ if and only if G has no induced subgraph from $\mathscr{F}$. □

Given $\mathcal{G}$, closed under taking induced subgraphs, one would like as small a collection of forbidden induced subgraphs characterizing $\mathcal{G}$ as can be found. The family given in the proof of Lemma 2.1 is the largest such characterizing family, the opposite of what is desired. In every case the smallest characterizing family is the family of "smallest" graphs not in $\mathcal{G}$, otherwise known as the family of graphs *vertex-critical* with respect to the property of not being in $\mathcal{G}$: $\mathscr{F} = \{H \mid H\ is\ a\ finite,\ simple\ graph,\ H \notin \mathcal{G},\ and\ for\ every\ v \in V(H),\ H - v \in \mathcal{G}\}$. The following improvement of Theorem 1.3 gives the optimal forbidden-induced-subgraph characterization of the Hall graphs. A proof can be found in [12].

Theorem 2.3. *The following are equivalent.*

(a) $G \in Hall$.

(b) Every block of G is a clique.

(c) G has no induced cycle subgraph C_t, $t > 3$, nor any induced K_4-minus-an-edge.

In the next to last section of this paper we will give two more refinements of Hall's Condition, both referring to tight subgraphs of a graph with a list assignment satisfying Hall's Condition. We call these two the Sudoku-Hall condition and the generalized Sudoku-Hall condition. Any mathematician who often attempts Sudoku puzzles will, upon reading the definitions, quickly understand why we have chosen the terminology. But why have we relegated these refinements to the next to last section? Because we have nothing to say about them! For now we leave the exploration of this new territory to others.

However, we can answer a burning question about these and all possible refinements of HC which are distinguished from HC only by requirements concerning tight subgraphs. The burning question: are any of these refinements *sufficient* as well as necessary conditions on G and L for a proper L-coloring of G? In other words, could there be such a condition, HCX, such that every finite simple graph is $HallX$?

We will show that the answer is no by giving one of many examples of a graph G and a list assignment L satisfying HC such that G is not properly L-colorable and there are no non-null L-tight subgraphs of G. [Perhaps we should stipulate that "distinguished from HC only by requirements concerning tight subgraphs" includes that whenever HC is satisfied and Γ is the only L-tight subgraph of G, then HCX is satisfied.]

Example 2.1.

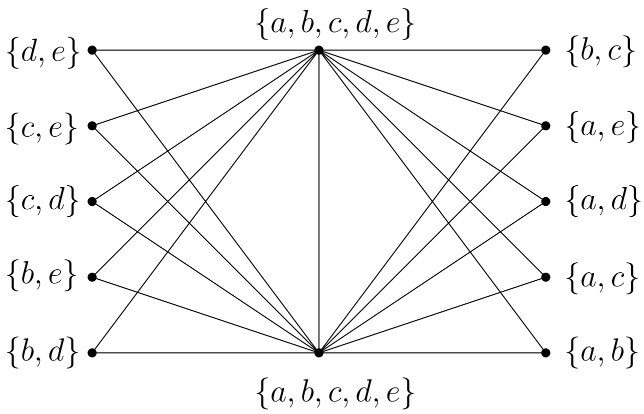

Figure 2: $G = K_2 \vee \overline{K_{10}}$ with a list assignment.

The graph G in Figure 2 is $K_2 \vee \overline{K_{10}}$, the *join* of K_2 with the empty graph on 10 vertices. The list assignment L assigns $\{a,b,c,d,e\}$ to each vertex of K_2 and distributes the 10 two-subsets of $\{a,b,c,d,e\}$ to the 10 vertices of $\overline{K_{10}}$. Therefore G has no proper L-coloring, for in any such coloring, the colors on the vertices of K_2 would form a two-subset of $\{a,b,c,d,e\}$, and thus one of the vertices of $\overline{K_{10}}$ would not be colorable.

It is clear from that explanation that for every $v \in V(G)$, $G - v$ is properly L-colorable. Further, $\displaystyle\sum_{\sigma\in\{a,b,c,d,e\}} \alpha(G_\sigma) = 4 \cdot 5 = 20 > |V(G)| = 12$. Therefore G and L satisfy Hall's Condition. To see that G has no L-tight induced subgraph other than Γ, the null graph, involves some checking that we leave to the reader. (The smallest $\displaystyle\sum_{\sigma\in\mathcal{C}} \alpha(H_\sigma) - |V(H)|$ can be, as H ranges over non-null induced subgraphs of G, is $1 = 5 - 4 = 6 - 5$, achieved in two different ways, one with $|V(H)| = 4$, and the other with $|V(H)| = 5$.)

Is $K_2 \vee \overline{K_{10}}$ the smallest graph with a list assignment satisfying the list of pathological requirements given above? We do not know, but it is the smallest such graph of the form $K_2 \vee \overline{K_m}$, if the list assignments are confined to those of the type exhibited in Figure 2 : the vertices of the K_2 are assigned the same list L, and the lists on $\overline{K_m}$ are 2-subsets of L.

We end this section with a rather technical lemma which allows us, in the proofs to come, to consider only list assignments L to a graph G such that $G(\sigma, L)$ is connected, for all $\sigma \in \mathcal{C}$.

Lemma 2.2. *Suppose that L is a list assignment to G, $\sigma \in \mathcal{C}$ and $\tilde{L}$ is obtained from L by replacing σ in the lists on some components of $G(\sigma, L)$ by a symbol $\tau \in \mathcal{C}\backslash L(V(G))$. Then:*

(a) there is a proper L-coloring of G if and only if there is a proper $\tilde{L}$-coloring of G; and,

*(b) for $\diamondsuit \in \{$ empty string, +, ++, * $\}$, G and L satisfy HC$\diamondsuit$ if and only if G and $\tilde{L}$ satisfy HC$\diamondsuit$. If G and $\tilde{L}$ satisfy HC**, then so do G and L.*

Proof. Claim (a) is easy to see, so we prove claim (b).

Suppose that H is an induced subgraph of G. Because σ is replaced by τ on entire components of $G(\sigma, L)$, and thus on entire components of $H(\sigma, L)$, which are subgraphs of components of $G(\sigma, L)$, it is clear that $\alpha(H(\sigma, L)) = \alpha(H(\sigma, \tilde{L})) + \alpha(H(\tau, \tilde{L}))$, and therefore, since $\tau \notin L(V(G))$, $\sum\limits_{\mu \in \mathcal{C}} \alpha(H(\mu, L)) = \sum\limits_{\mu \in \mathcal{C}} \alpha(H(\mu, \tilde{L}))$.

Since this holds for arbitrary H, it follows that G and L satisfy HC if and only if G and $\tilde{L}$ satisfy HC. Suppose that G and L satisfy HC and $\{S_\mu : \mu \in \mathcal{C}\}$ is an HC+-satisfying (HC++-satisfying) family with respect to the pair (G, L). Define $\{\tilde{S}_\mu : \mu \in \mathcal{C}\}$ by $\tilde{S}_\mu = S_\mu$ if $\mu \in \mathcal{C}\backslash\{\sigma, \tau\}$, $\tilde{S}_\tau = \{v \in S_\sigma | \tilde{L}(v)$ is obtained from $L(v)$ by replacing σ by $\tau\}$, and $\tilde{S}_\sigma = S_\sigma \backslash \tilde{S}_\tau$. If the S_μ are pairwise disjoint, then so are the $\tilde{S}_\mu$. If H is an $\tilde{L}$-tight subgraph of G, then H is also L-tight, since $\sum\limits_{\mu \in \mathcal{C}} \alpha(H(\mu, L)) = \sum\limits_{\mu \in \mathcal{C}} \alpha(H(\mu, \tilde{L}))$. Therefore, since $\{\tilde{S}_\mu : \mu \in \mathcal{C}\}$ is HC+-satisfying with respect to (G, L), for $\mu \in \mathcal{C}\backslash\{\sigma, \tau\}$, $|S_\mu \cap V(H)| = |\tilde{S}_\mu \cap V(H)| = \alpha(H(\mu, L)) = \alpha(H(\mu, \tilde{L}))$. Because τ replaces σ on entire components of $H(\sigma, L)$, $\alpha(H(\sigma, L)) = \alpha(H(\sigma, \tilde{L})) + \alpha(H(\tau, \tilde{L})) = |S_\sigma \cap V(H)| = |\tilde{S}_\sigma \cap V(H)| + |\tilde{S}_\tau \cap V(H)|$. Since $\tilde{S}_\sigma \cap V(H)$ is an independent set of vertices of $H(\sigma, \tilde{L})$, $|\tilde{S}_\sigma \cap V(H)| \leq \alpha(H(\sigma, \tilde{L}))$; similarly, $|\tilde{S}_\tau \cap V(H)| \leq \alpha(H(\tau, \tilde{L}))$. Therefore, $|\tilde{S}_\sigma \cap V(H)| = \alpha(H(\sigma, \tilde{L}))$ and $|\tilde{S}_\tau \cap V(H)| = \alpha(H(\tau, \tilde{L}))$. This completes the proof that if G and L satisfy HC+ (HC++), then so do G and $\tilde{L}$. In the other direction: If $\{\tilde{S}_\mu : \mu \in \mathcal{C}\}$ is an HC+-satisfying (HC++-satisfying) family with respect to G and $\tilde{L}$, define $\{S_\mu : \mu \in \mathcal{C}\}$, by $S_\mu = \tilde{S}_\mu$, $\mu \in \mathcal{C}\backslash\{\sigma, \tau\}$, $S_\tau = \emptyset$, and $S_\sigma = \tilde{S}_\sigma \cup \tilde{S}_\tau$. It is straightforward to see that $\{S_\mu : \mu \in \mathcal{C}\}$ is an HC+-satisfying (HC++-satisfying) family with respect to G and L.

Suppose that H is an induced subgraph of G and T is an L-tight subgraph of H. As above, it follows that T is $\tilde{L}$-tight. Conversely, if T is $\tilde{L}$-tight, then it is L-tight, as we now show. Clearly $\alpha(H(\mu, L)|T(\mu, L)) = \alpha(H(\mu, \tilde{L})|T(\mu, \tilde{L}))$ for all $\mu \in \mathcal{C}\backslash\{\sigma, \tau\}$. Because τ replaces σ on components of $G(\sigma, L)$, in the formation of $\tilde{L}$, if $A \subseteq V(G(\sigma, \tilde{L})), B \subseteq V(G(\tau, \tilde{L}))$ are independent sets in G, then $A \cup B \subseteq$

$V(G(\sigma, L))$ is independent. Further, every independent set of vertices of $G(\sigma, L)$ is uniquely representable as such a union. From these remarks, it can be seen that $\alpha(H(\sigma, L)|T(\sigma, L)) = \alpha(H(\sigma, \tilde{L})|T(\sigma, \tilde{L})) + \alpha(H(\tau, \tilde{L})|T(\tau, \tilde{L}))$. From this equation we have that $\sum_{\mu \in \mathcal{C}} \alpha(H(\mu, L)|T(\mu, L)) = \sum_{\mu \in \mathcal{C}} \alpha(H(\mu, \tilde{L})|T(\mu, \tilde{L}))$, and thus, since H and $T \subseteq H$ were arbitrary, that G and L satisfy HC* if and only if G and $\tilde{L}$ satisfy HC*. From the equation above, we also conclude that

$$\min_{T \subseteq H} \alpha(H(\sigma, L)|T(\sigma, L)) \geq \min_{T \subseteq H} \alpha(H(\sigma, \tilde{L})|T(\sigma, \tilde{L})) + \min_{T \subseteq H} \alpha(H(\tau, \tilde{L})|T(\tau, \tilde{L})).$$

Therefore if G and $\tilde{L}$ satisfy HC**, then so do G and L. □

Corollary 2.1. *Suppose $\diamondsuit \in \{emptystring, +, ++, *\}$. $G \in Hall\diamondsuit \iff G$ is properly L-colorable for every L such that G and L satisfy $HC\diamondsuit$ and, for all colors $\sigma \in \mathcal{C}$, $G(\sigma, L)$ is connected.*

Proof. The "$\Longrightarrow$" claim is trivial. Conversely, suppose that G is properly $\tilde{L}$-colorable for every $\tilde{L}$ such that G and $\tilde{L}$ satisfy $HC\diamondsuit$ and, for all $\sigma \in \mathcal{C}$, $G(\sigma, \tilde{L})$ is connected. Suppose that L is a list assignment to G such that G and L satisfy $HC\diamondsuit$. To show that $G \in Hall\diamondsuit$, it suffices to show that for every such L, G is properly L-colorable. Using the convenience of the infinitude of $\mathcal{C}$, we replace L by a list assignment $\tilde{L}$ obtained by repeated replacements of symbols $\sigma \in L(V(G))$ by new symbols, with each replacement satisfying the hypothesis of Lemma 2.2 with respect to G and the current list assignment. By repeated application of Lemma 2.2, part (b), G and $\tilde{L}$ satisfy $HC\diamondsuit$. We can arrange for $\tilde{L}$ to satisfy: $G(\tau, \tilde{L})$ is connected, for every $\tau \in \mathcal{C}$. By assumption, G is properly $\tilde{L}$-colorable. By repeated application of part (a) of Lemma 2.2, G is properly L-colorable. □

We leave open the questions: In Lemma 2.2(b), does the "only if" conclusion hold when $\diamondsuit$ = ** ? In Corollary 2.1, does the backward implication hold when $\diamondsuit$ = ** ? Clearly a "yes" to the first question implies a "yes" to the second.

3 Hall, Hall+ and Hall++ Graphs

A particular case of the next lemma has already been used, in the proof of Theorem 2.2, in an extreme case where the conclusion was obvious. The finer version will be useful in this section.

Lemma 3.1. *Suppose that H is an induced subgraph of G, L is a list assignment to G, L_0 is the restriction of L to $V(H)$, and $\diamondsuit \in \{+, ++\}$. Suppose that G and L satisfy HC, H and L_0 satisfy $HC\diamondsuit$, and the only L-tight subgraphs of G are subgraphs of H. Then every $HC\diamondsuit$-satisfying family for H and L_0 is an $HC\diamondsuit$-satisfying family for G and L; so G and L satisfy $HC\diamondsuit$.*

Proof. Let $\{S_\sigma | \sigma \in \mathcal{C}\}$ be an $HC\diamondsuit$-satisfying family for H and L_0. For each $\sigma \in \mathcal{C}$, $S_\sigma \subseteq V(H(\sigma, L_0)) \subseteq V(G(\sigma, L))$, and for each L-tight subgraph X of G, since X is an L_0-tight subgraph of H, $|S_\sigma \cap V(X)| = \alpha(X(\sigma, L_0)) = \alpha(X(\sigma, L))$. Therefore, $\{S_\sigma | \sigma \in \mathcal{C}\}$ is an $HC\diamondsuit$-satisfying family for H and L. $\square$

Theorem 1.3 implies that the class of Hall graphs is closed under the operation of attaching a clique at a vertex.

Theorem 3.1. *Suppose that $\diamondsuit \in \{+, ++\}$. If a graph $H \in Hall\diamondsuit$, and G is obtained from H by attaching a clique to a vertex v of H, then $G \in Hall\diamondsuit$. Thus the class of $Hall\diamondsuit$ graphs is closed under attachment of cliques at single vertices.*

The proof of this theorem will require a few more lemmas. In what follows, if $S \subseteq V(G)$, then the subgraph of G induced by S will be denoted by $G[S]$. If X and Y are subgraphs of G, then $G[X \cup Y]$ stands for $G[V(X) \cup V(Y)]$.

Lemma 3.2. *Suppose K is a clique, L a list assignment to the graph K, and K and L satisfy Hall's Condition. Suppose further that H_1, H_2 are L-tight sub-cliques of K. Then*

i.) $H_0 = H_1 \cap H_2$ *is also L-tight.*

ii.) $H_3 = K[H_1 \cup H_2]$ *is also L-tight.*

Proof. As before, we will use the notation $L(U) = \bigcup_{u \in U} L(u)$ for $U \subseteq V(G)$. Since H_3 is a clique, $\alpha(H_3(\sigma, L))$ as a function of the symbol σ is the characteristic function of $L(V(H_3))$. That is,

$$\alpha(H_3(\sigma, L)) = \begin{cases} 1, & \text{if } \sigma \in L(V(H_3)); \\ 0, & \text{otherwise.} \end{cases}$$

Therefore (and this holds for any clique, not just H_3), $\sum_{\sigma \in \mathcal{C}} \alpha(H_3(\sigma, L)) = |L(V(H_3))|$.

By Hall's Condition on K and L, and the fact that $V(H_3) = V(H_1) \cup V(H_2)$, we have:

$$\sum_{\sigma \in \mathcal{C}} \alpha(H_3(\sigma, L)) \geq |V(H_3)| = |V(H_1) \cup V(H_2)|. \tag{6}$$

Also,

$$\begin{aligned} \sum_{\sigma \in \mathcal{C}} \alpha(H_3(\sigma, L)) &= |L(V(H_3))| = |L(V(H_1) \cup V(H_2))| = |L(V(H_1)) \cup L(V(H_2))| \\ &= |L(V(H_1))| + |L(V(H_2))| - |L(V(H_1)) \cap L(V(H_2))| \\ &\leq |L(V(H_1))| + |L(V(H_2))| - |L(V(H_1) \cap V(H_2))| \\ &\leq |V(H_1)| + |V(H_2)| - |V(H_1) \cap V(H_2)| \qquad (7) \\ &= |V(H_1) \cup V(H_2)| = |V(H_3)| \end{aligned}$$

Inequalities (7) hold because, for $i = 1, 2$,

$$\begin{aligned} \sum_{\sigma \in \mathcal{C}} \alpha(H_i(\sigma, L)) &= |L(V(H_i))| \quad \text{(because } H_i \text{ is a clique)} \\ &= |V(H_i)|, \end{aligned}$$

because H_i is L-tight; and, because K and L satisfy Hall's Condition,

$$\begin{aligned}\sum_{\sigma\in\mathcal{C}} \alpha(H_0(\sigma, L)) &= |L(V(H_0))| = |L(V(H_1) \cap V(H_2))| \\ &\geq |V(H_0)| = |V(H_1) \cap V(H_2)|. \end{aligned} \tag{8}$$

From inequalities (6) and (7), we see that both inequalities must be equalities, and that both H_0 and H_3 must be L-tight. $\square$

Lemma 3.3. *If $\{S_\sigma : \sigma \in \mathcal{C}\}$ is an HC+ (or HC++)-satisfying family for G and L, then for any induced subgraph H of G, $\{S_\sigma \cap V(H) : \sigma \in \mathcal{C}\}$ is also an HC+ (respectively HC++)-satisfying family for H and the restriction of L to $V(H)$.*

The proof is straightforward.

Lemma 3.4. *Suppose K is a clique with a list assignment L, and K and L satisfy HC. Suppose that for some color τ, removing τ from $L(V(K))$ wherever it appears results in a list assignment which does not satisfy HC with K. Then some subclique K_τ of K is L-tight. Further, $\tau \in L(V(K_\tau))$.*

Proof. Let L' be the list assignment obtained from L by removing all occurrences of τ from the L-lists on K, and let $K_\tau = H$ be an induced subgraph, and thus a subclique, of K such that $\sum_{\sigma\in\mathcal{C}} \alpha(H(\sigma, L')) < |V(H)|$. It is straightforward to see that K_τ satisfies the claims of the lemma. $\square$

Proof of Theorem 3.1

Suppose that $\diamondsuit \in \{+, ++\}$, and that L is a list assignment to G such that G and L satisfy $HC\diamondsuit$. We will show that G has a proper L-coloring, if $H \in Hall\diamondsuit$. Let $\widehat{K} \cong K_n$, $n \geq 2$, denote the clique attached to H at v. Let $\{S_\sigma : \sigma \in \mathcal{C}\}$ be an $HC\diamondsuit$ - satisfying family for G and L. By Lemma 3.3, $\{S_\sigma \cap V(H) : \sigma \in \mathcal{C}\}$ is an $HC\diamondsuit$ - satisfying family for H and L. (L here is short for L-restricted-to-$V(H)$.) Because $H \in Hall\diamondsuit$, there is a proper L-coloring of H. Let $K = \widehat{K} - v$, and let $\mathcal{P} = \{\tau \in \mathcal{C} \mid \text{for some proper } L\text{-coloring } \psi \text{ of } H,\ \psi(v) = \tau\ \}$. For $\tau \in \mathcal{P}$, if we remove τ from all the lists on K, then we may as well assume that the resulting list assignment on K does not satisfy Hall's Condition with K; for, if it did, then there would be a proper L-coloring of K with τ not appearing, which, when put together with a proper L-coloring ψ of H such that $\psi(v) = \tau$, would give a proper L-coloring of G, and we would be done. By Lemma 3.4, therefore, we may as well assume that for each $\tau \in \mathcal{P}$, there is a subclique K_τ of K such that $\tau \in L(V(K_\tau))$ and K_τ is L-tight. Applying Lemma 3.2(ii) $|\mathcal{P}|$ - 1 times, we see that $K_\mathcal{P} = K[\cup_{\tau\in\mathcal{P}} K_\tau]$ is an L-tight subclique of K, and $\mathcal{P} \subseteq L(V(K_\mathcal{P}))$. Therefore, because $\{S_\sigma : \sigma \in \mathcal{C}\}$ is an $HC\diamondsuit$ - satisfying family for G and L, we have, for each $\tau \in \mathcal{P}$, $|S_\tau \cap V(K_\mathcal{P})| = \alpha(K_\mathcal{P}(\tau, L)) = 1$; recollect that $K_\mathcal{P}$ is a clique. Therefore, each such S_τ has a vertex in K. Because S_τ is an independent set of vertices in G, $v \notin S_\tau$, and this holds for all $\tau \in \mathcal{P}$. Define a new list assignment L' on the graph H as follows:

$L'(u) = L(u)$ for all $u \in V(H)\backslash\{v\}$, and $L'(v) = L(v)\backslash\mathcal{P}$. If H and L' satisfy $HC\diamondsuit$, then, because $H \in Hall\diamondsuit$, there exists a proper L'-coloring of H. But because $L'(u) = L(u)$ for all $u \in V(H)\backslash\{v\}$, a proper L'-coloring of H would also be a proper L-coloring for H, so the color on v would have to be an element of $\mathcal{P}$. This is impossible, since $\mathcal{P} \cap L'(v) = \emptyset$. Therefore, H and L' do not satisfy $HC\diamondsuit$. Therefore, either H and L' do not satisfy HC, or H and L' do satisfy HC, but there does not exist an $HC\diamondsuit$-satisfying family for H and L'.

Case 1: Suppose that H and L' do not satisfy HC. Then for some induced subgraph H_1 of H, it must be that

$$\sum_{\sigma\in\mathcal{C}} \alpha(H_1(\sigma, L')) < |V(H_1)|\,. \tag{9}$$

Now, L and L' are the same except at v, and H and L satisfy HC. Therefore, $v \in V(H_1)$.
Let $H_2 = G[H_1 \cup K_{\mathcal{P}}]$. Note that:

i.) $V(H_1) \cap V(K_{\mathcal{P}}) = \emptyset$;

ii.) if $\sigma \in \mathcal{C}\backslash\mathcal{P}$, then $\alpha(H_1(\sigma, L')) = \alpha(H_1(\sigma, L))$ and $\alpha(H_2(\sigma, L)) \leq \alpha(H_1(\sigma, L)) + \alpha(K_{\mathcal{P}}(\sigma, L)) = \alpha(H_1(\sigma, L')) + \alpha(K_{\mathcal{P}}(\sigma, L))$; and

iii.) if $\sigma \in \mathcal{P}$ then $\alpha(H_2(\sigma, L)) \leq \alpha(H_1(\sigma, L')) + 1 = \alpha(H_1(\sigma, L')) + \alpha(K_{\mathcal{P}}(\sigma, L))$.

Therefore, because G and L satisfy HC, we have

$$\begin{aligned}
|V(H_1)| + |V(K_{\mathcal{P}})| &= |V(H_2)| \leq \sum_{\sigma\in\mathcal{C}} \alpha(H_2(\sigma, L)) \qquad (10)\\
&\leq \sum_{\sigma\in\mathcal{C}} \alpha(H_1(\sigma, L')) + \sum_{\sigma\in\mathcal{C}} \alpha(K_{\mathcal{P}}(\sigma, L)) \qquad (11)\\
&= \sum_{\sigma\in\mathcal{C}} \alpha(H_1(\sigma, L')) + |V(K_{\mathcal{P}})| \qquad (12)\\
&< |V(H_1)| + |V(K_{\mathcal{P}})|\,. \qquad (13)
\end{aligned}$$

Thus, $|V(H_1)| + |V(K_{\mathcal{P}})| < |V(H_1)| + |V(K_{\mathcal{P}})|$, a contradiction. This absurd conclusion allows us to conclude that H and L' satisfy HC. [The equality in (12) follows because $K_{\mathcal{P}}$ is L-tight. Finally the strict equality in (13) follows from (9)].

Case 2: H and L' do satisfy HC but there is no $HC\diamondsuit$-satisfying family for H and L'. Therefore, $\{S_\sigma \cap V(H) : \sigma \in \mathcal{C}\}$, which is an $HC\diamondsuit$ - satisfying family for H and L, is not an $HC\diamondsuit$-satisfying family for H and L'. Consequently, there is an L'-tight subgraph T of H such that for some $\tau \in \mathcal{C}$, $|S_\tau \cap V(T)| < \alpha(T(\tau, L'))$. Because $L(u) = L'(u)$ for all $u \in V(H)\backslash\{v\}$, and $\{S_\sigma \cap V(H) : \sigma \in \mathcal{C}\}$ is an $HC\diamondsuit$ - satisfying family for H and L, it must be that $v \in V(T)$. For a similar reason, it must be that $\tau \in \mathcal{P}$. As concluded earlier, for every $\sigma \in \mathcal{P}$, $|S_\sigma \cap V(K_{\mathcal{P}})| = 1 = \alpha(K_{\mathcal{P}}(\sigma, L))$. Let $G_1 = G[T \cup K_{\mathcal{P}}]$.

Claim: G_1 is L-tight.

Proof.

$$\begin{aligned}
|V(T)| + |V(K_{\mathcal{P}})| &= |V(G_1)| \\
&\leq \sum_{\sigma\in\mathcal{C}} \alpha(G_1(\sigma, L)) \qquad (14)\\
&= \sum_{\sigma\in\mathcal{C}\setminus\mathcal{P}} \alpha(G_1(\sigma, L)) + \sum_{\sigma\in\mathcal{P}} \alpha(G_1(\sigma, L)) \\
&\leq \sum_{\sigma\in\mathcal{C}\setminus\mathcal{P}} [\alpha(T(\sigma, L')) + \alpha(K_{\mathcal{P}}(\sigma, L))] \\
&\qquad + \sum_{\sigma\in\mathcal{P}} [\alpha(T(\sigma, L')) + \alpha(K_{\mathcal{P}}(\sigma, L))] \qquad (15)\\
&= \sum_{\sigma\in\mathcal{C}} \alpha(T(\sigma, L')) + \sum_{\sigma\in\mathcal{C}} \alpha(K_{\mathcal{P}}(\sigma, L)) \\
&= |V(T)| + |V(K_{\mathcal{P}})| \qquad (16)\\
&= |V(G_1)|
\end{aligned}$$

The equality in (16) follows from the L-tightness of $K_{\mathcal{P}}$ and the L'-tightness of T. The inequality (14) follows from the fact that G and L satisfy HC. The inequality (15) follows from :

$$\alpha(G_1(\sigma, L)) \;\leq\; \alpha(T(\sigma, L')) + \alpha(K_{\mathcal{P}}(\sigma, L)), \qquad (17)$$

which holds for all $\sigma \in \mathcal{C}$, but for different reasons depending on whether $\sigma \in \mathcal{C}\setminus\mathcal{P}$ or $\sigma \in \mathcal{P}$. If $\sigma \in \mathcal{C} \setminus \mathcal{P}$ then $G_1(\sigma, L) = T(\sigma, L') \cup K_{\mathcal{P}}(\sigma, L)$, so (17) follows from general principles. If $\sigma \in \mathcal{P}$ then $\sigma \notin L'(v)$ and $K_{\mathcal{P}}(\sigma, L)$ is a non-null clique not containing v, but in $N_G(v)$. If a maximum independent set I of vertices in $G_1(\sigma, L)$ contains v, then $I \setminus \{v\} \cup \{u\}$, for some (any) $u \in V(K_{\mathcal{P}}(\sigma, L))$ is an independent set in $G_1(\sigma, L)$ with the same number of vertices as I and is the disjoint union of an independent set in $T(\sigma, L')$ with one in $K_{\mathcal{P}}(\sigma, L)$. From this (17) follows. If no maximum independent set of vertices in $G_1(\sigma, L)$ contains v, then obviously (17) holds, with equality. This concludes the proof that G_1 is L-tight.

Because G_1 is L-tight and $\{S_\sigma : \sigma \in \mathcal{C}\}$ is an $HC\diamondsuit$-satisfying family for G and L,

$$\alpha(G_1(\tau, L)) = |S_\tau \cap V(G_1)|\,. \qquad (18)$$

On the other hand,

$$\begin{aligned}
|S_\tau \cap V(G_1)| &= |S_\tau \cap V(T)| + |S_\tau \cap V(K_{\mathcal{P}})| < \alpha(T(\tau, L')) + 1 \qquad (19)\\
&= \alpha(G_1(\tau, L)) \qquad (20)
\end{aligned}$$

The inequality (19) follows from the assumption about S_τ and T, the L-tightness of the clique $K_{\mathcal{P}}$, and the fact that $\tau \in \mathcal{P}$. To see equality (20), suppose that A is a maximum independent set of vertices in $G_1(\tau, L)$. If $v \notin A$, then $A \cap V(T)$ is a maximum independent set of vertices in $T(\tau, L')$ and $|A \cap V(K_{\mathcal{P}})| = 1$, so

$\alpha(G_1(\tau, L)) = |A \cap V(T)| + |A \cap V(K_{\mathcal{P}})| = \alpha(T(\tau, L')) + 1$. If $v \in A$, then $|A \cap V(K_{\mathcal{P}})| = \emptyset$, and $|A \backslash \{v\}| \leq \alpha(T(\tau, L')$. Replacing A with $B \cup \{w\}$, where B is a maximum independent set of vertices i in $T(\tau, L')$ and w is a vertex of $K_{\mathcal{P}}$ with $\tau \in L(w)$, gives us a maximum independent set of vertices in $G_1(\tau, L)$ not containing v, so, again, (20) holds.

The contradiction arrived at in (18), (19), and (20) establishes that $\{S_\sigma \cap V(H) : \sigma \in \mathcal{C}\}$ is an $HC\diamondsuit$ - satisfying family for H and L after all. Therefore, H is properly L' colorable, so H is properly L-colorable with v receiving a color not in $\mathcal{P}$, which contradicts the definition of $\mathcal{P}$. Thus G is $HC\diamondsuit$, after all. □

We end this section with two results that bear on the minimal forbidden-induced-subgraph characterizations of the classes Hall+ and Hall++.

Proposition 3.1. *All cycles on* 4 *or more vertices are Hall+.*

Proof. Suppose $n \geq 4$ and $G \cong C_n$ is a cycle on n vertices. Suppose that L is a list assignment to G such that G and L satisfy $HC+$. We aim to show that there is a proper L-coloring of G. Let $\{S_\sigma; \sigma \in \mathcal{C}\}$ be an HC+ - satisfying family for G and L. Since $h(G) = 2$ [12], and G and L satisfy HC, we can assume that some list is a singleton. Let $L(v) = \{\sigma\}$ and let u and w be the vertices on either side of v, on the cycle G.

$P = G - uv$ is a path on n vertices. Because P is a subgraph of G, and G satisfies HC with L, P and L satisfy HC. Because the blocks of P are its edges, $P \in Hall$, by Theorem 1.3. Therefore there is a proper L-coloring of P.

If, for such a coloring, u is colored with something other than σ, then the coloring is also a proper L-coloring of G. Therefore, we may as well assume that u is colored with σ in every proper L-coloring of P.

Let $L' : V(G) \to C$ be defined by $L'(x) = L(x)$ for all $x \in V(G) \backslash \{u\}$, and $L'(u) = L(u) \backslash \{\sigma\}$. Since $P \in Hall$ and there is no proper L'-coloring of P, it must be that P and L' do not satisfy Hall's Condition. So, there is an induced subgraph H of P such that $\sum_{\tau \in \mathcal{C}} \alpha(H(\tau, L')) < |V(H)|$. Since

$$\begin{aligned} |V(H)| &\leq \sum_{\tau \in \mathcal{C}} \alpha(H(\tau, L)) && (21) \\ &\leq 1 + \sum_{\tau \in \mathcal{C}} \alpha(H(\tau, L')) && (22) \\ &\leq |V(H)|, && (23) \end{aligned}$$

and $L = L'$ except at u, it follows that $u \in V(H)$, H is L-tight, and u is in every maximum independent set of vertices in $H(\sigma, L)$. Because H is L-tight and $\{S_\mu : \mu \in \mathcal{C}\}$ is an HC+-satisfying family for G and L, $|S_\sigma \cap V(H)| = \alpha(H(\sigma, L))$. That is, $S_\sigma \cap V(H)$ is a maximum independent set of vertices in $H(\sigma, L)$. Therefore, $u \in S_\sigma$.

But v alone is also an L-tight subgraph of G, and $L(v) = \{\sigma\}$, so $v \in S_\sigma$. This cannot be, because u and v are adjacent in G and S_σ is an independent set of vertices in G. This contradiction establishes that G is properly L-colorable, after all. L was arbitrary, so $G \in Hall+$.

□

Corollary 3.1. *$Hall \subsetneq Hall+$.*

Proof. By either Theorem 1.3 or Theorem 2.3, and the proposition preceding, for $n \geq 4$ $C_n \in Hall + \backslash Hall$. □

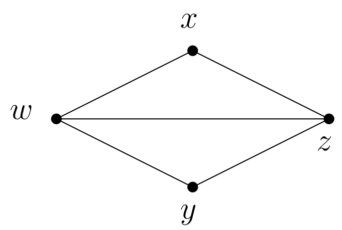

Figure 3: $\theta(1, 2, 2)$

The graph K_4-minus-an-edge is also known as $\theta(1, 2, 2)$. It is known that $\theta(1, 2, 2)$ has Hall number $h = 2$ [12].

Proposition 3.2. *$\theta(1, 2, 2) \in Hall+$.*

Proof. Let $G = \theta(1, 2, 2)$, with vertices labeled as in Figure 3. Suppose L is a list assignment to G such that G and L satisfy $HC+$. We aim to see that there is a proper L-coloring of G. Let the collection $\{\mathcal{S}_\sigma : \sigma \in \mathcal{C}\}$ be an HC+ - satisfying family for G and L. Since G and L satisfy HC and $h(G) = 2$ [12], we may as well suppose that $L(v) = \{\sigma\}$ for some vertex $v \in V(G)$ and $\sigma \in \mathcal{C}$. Since $\{v\}$ is a trivial L-tight subgraph, we have $v \in S_\sigma$. There are essentially two cases to consider: $v = w$ and $v = x$.

Case 1: $L(x) = \{\sigma\}$

Let $H = G[\{w, y, z\}]$. Since H and L satisfy HC (because G and L do) and $H \cong K_3$, by Theorem 1.3, H is properly L-colorable. We may assume that for every proper L-coloring of H, either w or z is colored σ. Therefore, if L' is defined on H by $L'(y) = L(y)$ and $L'(v) = L(v) \backslash \{\sigma\}$ if $v \in \{w, z\}$, then H is not properly L'-colorable, and so H and L' do not satisfy HC: there must be an induced subgraph T of H such that $|\bigcup_{v \in V(T)} L'(v)| = \sum_{\mu \in \mathcal{C}} \alpha(T(\mu, L')) < |V(T)|$. As in similar circumstances in the proof of Proposition 3.1, it follows that T is L-tight. Also, $\emptyset \neq V(T(\sigma, L)) \subseteq \{w, z\}$. But then S_σ must contain one of w, z, which cannot be, because the independent set S_σ contains $v = x$.

Case 2: $L(w) = \{\sigma\}$

By case 1, we may suppose that $|L(v)| \geq 2$ for $v \in \{x, y\}$. Further, if, say, $|L(x)| \geq 3$, then a proper L-coloring of G may be obtained by taking a proper L-coloring of $H = G[\{w, y, z\}]$ and then coloring x with an element of $L(x)$ other than the colors on w and z. So, we may assume that $2 = |L(x)| = |L(y)|$. If $\sigma \notin L(x)$ then, again, G can be properly colored as suggested above, because in every proper L-coloring of H, w will be colored σ, and $L(x)$ contains two elements different from σ. Therefore, we may assume that $L(x) = \{\sigma, a\}$ and $L(y) = \{\sigma, b\}$ for some $a, b \in \mathcal{C}\backslash\{\sigma\}$, not necessarily distinct. If $L(z)$ contains anything other than σ, a, b, then, clearly, there is a proper L-coloring of G. Therefore, we may assume that $L(z) \subseteq \{\sigma, a, b\}$. Then G itself is L-tight. Since the only maximum independent set of vertices in $G(\sigma, L)$ is $\{x, y\}$, it follows that $x, y \in S_\sigma$, which cannot be, because the independent set S_σ contains $v = w$. $\square$

The next example shows that attaching two Hall+ graphs at one vertex does not necessarily result in a Hall+ graph.

Example 3.1. The graph G and list assignment L in Figure 4 satisfy HC+ (there are no non-null L-tight subgraphs of G) and yet G cannot be properly L-colored.

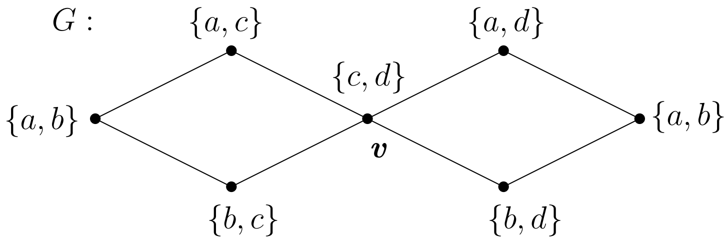

Figure 4: Hall+ graphs attached at a vertex, resulting in a graph which is not Hall+

4 Hall* and Hall** Graphs

The main result in this section will be similar to that of Theorem 3.1.

Theorem 4.1. *Suppose that $\diamondsuit \in \{*, **\}$. If $H \in Hall\diamondsuit$, and G is obtained from H by attaching a clique $\widehat{K}$ to H at a vertex v of H, then $G \in Hall\diamondsuit$.*

Proof: Let $K = \widehat{K} - v \cong K_{n-1}$, for some $n \geq 2$ and suppose L is a list assignment on G such that G and L satisfy HC*. We show that there is a proper L-coloring of G. Suppose not.

Since H is an induced subgraph of G, and G and L satisfy HC*, it must be that H and L restricted to $V(H)$ will satisfy HC*. Since $H \in$ Hall*, it follows that there is a proper L-coloring of H. However, because there is no proper L-coloring of G, for every proper L-coloring of H, if the color on the vertex v is removed from the lists on $V(K)$, then K and the new list assignment do not satisfy HC. Let $\mathcal{P} = \{\tau \in \mathcal{C} \mid$ for some proper L-coloring ψ of H, $\psi(v) = \tau\}$. We see from Lemma 3.4 that for every symbol $\sigma \in \mathcal{P}$, there is an L-tight subclique of the clique K with σ appearing on its lists. Hence there is an L-tight complete subgraph $K_{\mathcal{P}}$ of K with all colors of $\mathcal{P}$ on its lists, by Lemma 3.2. Now we define a new list assignment L' on H: $L'(v) = L(v) \backslash \mathcal{P}$, and $L'(u) = L(u)$ for all $u \in V(H - v)$. Then it must be that H and L' do not satisfy HC* because there is no proper L'-coloring of H (but they do satisfy Hall's Condition by the argument given in the proof of Theorem 3.1). Therefore, there is some induced subgraph H' of H and some L'-tight subgraph T' of H' such that

$$\sum_{\sigma \in \mathcal{C}} \alpha(H'(\sigma, L') \mid T'(\sigma, L')) < |V(H')| \,. \tag{24}$$

The vertex v must belong to H' since $L' = L$ on the graph $H - v$ and H and L satisfy HC*.

With $v \in V(H')$ and $T' \subseteq H'$, two cases arise: $v \in V(T')$ and $v \notin V(T')$.

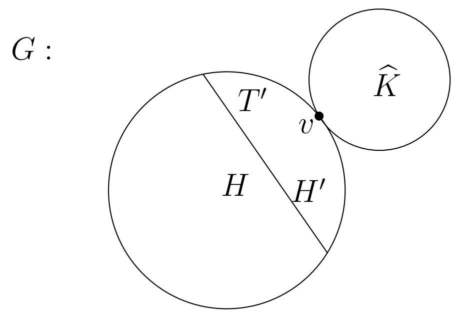

Figure 5: Hall* graph H $\supset H' \supset T'$, and clique $\widehat{K}$

Let $H'' = G[V(H') \cup V(K_{\mathcal{P}})]$ be the subgraph induced by the disjoint union of $V(H')$ and $V(K_{\mathcal{P}})$ and let $T'' = G[V(T') \cup V(K_{\mathcal{P}})]$ be the subgraph induced by the disjoint union of $V(T')$ and $V(K_{\mathcal{P}})$. We have the following two claims:

Claim 1: T'' is L-tight;

Claim 2: $\sum_{\sigma \in \mathcal{C}} \alpha(H''(\sigma, L) \mid T''(\sigma, L)) < |V(H'')| \,.$

If both claims hold, the hypothesis that G and L satisfy HC* is contradicted; hence, the claim of Theorem 4.1 concerning Hall* is true, if both claims hold.

Remark 4.1. It is easy to see that if each component of a graph G and a list assignment L satisfy HC, then G and L will satisfy HC. Moreover, if each component is L-tight, then so too is the whole graph.

Proof of Claim 1:

If $v \notin V(T')$, then $L' = L$ on T' and so T'' must also be L-tight as the disjoint union of two L-tight subgraphs. Now suppose $v \in V(T')$. If $\sigma \notin \mathcal{P}$, then

$$\alpha(T''(\sigma, L)) \leq \alpha(T'(\sigma, L)) + \alpha(K_{\mathcal{P}}(\sigma, L)) = \alpha(T'(\sigma, L')) + \alpha(K_{\mathcal{P}}(\sigma, L)). \tag{25}$$

Suppose $\sigma \in \mathcal{P}$ and let U be a maximum independent set of vertices of $T''(\sigma, L)$. If $v \notin U$, then

$$\begin{aligned} |U| = \alpha(T''(\sigma, L)) &= \alpha(T'(\sigma, L')) + 1 \\ &= \alpha(T'(\sigma, L')) + \alpha(K_{\mathcal{P}}(\sigma, L)). \end{aligned} \tag{26}$$

If $v \in U$, then $V(K_{\mathcal{P}}) \cap U = \emptyset$; since $U \backslash \{v\}$ is an independent set of vertices in $T'(\sigma, L')$, we have

$$\begin{aligned} \alpha(T''(\sigma, L)) = |U| &\leq \alpha(T'(\sigma, L')) + 1 \\ &= \alpha(T'(\sigma, L')) + \alpha(K_{\mathcal{P}}(\sigma, L)). \end{aligned} \tag{27}$$

Putting together equations (25), (26), and (27), we have that if $v \in V(T')$, then for all $\sigma \in \mathcal{C}$,

$$\alpha(T''(\sigma, L)) \leq \alpha(T'(\sigma, L')) + \alpha(K_{\mathcal{P}}(\sigma, L)). \tag{28}$$

Note that (28) holds, with equality, when $v \notin V(T')$, for all $\sigma \in \mathcal{C}$.
Finally, because Hall's Condition is satisfied by H and L', and by G and L, we have

$$\begin{aligned} |V(T'')| &\leq \sum_{\sigma \in \mathcal{C}} \alpha(T''(\sigma, L)) && (29) \\ &\leq \sum_{\sigma \in \mathcal{C}} \alpha(T'(\sigma, L')) + \sum_{\sigma \in \mathcal{C}} \alpha(K_{\mathcal{P}}(\sigma, L)) && (30) \\ &= |V(T')| + |V(K_{\mathcal{P}})| && (31) \\ &= |V(T'')|. && (32) \end{aligned}$$

where (31) holds because T' is L'-tight and $K_{\mathcal{P}}$ is L-tight, and (32) holds because $V(T'')$ is the disjoint union of $V(T')$ and $V(K_{\mathcal{P}})$. Thus T'' is L-tight. Note that by (29) - (32) and previous remarks, (28) holds with equality whether $v \in V(T')$ or not, for all $\sigma \in \mathcal{C}$.

Proof of Claim 2:

It is easy to see that if $\sigma \in \mathcal{C} \backslash L(K_{\mathcal{P}})$, then

$$\alpha(H''(\sigma, L) \mid T''(\sigma, L)) = \alpha(H'(\sigma, L') \mid T'(\sigma, L')). \tag{33}$$

We claim that if $\sigma \in L(K_{\mathcal{P}})$, then

$$\begin{aligned} \alpha(H''(\sigma, L) \mid T''(\sigma, L)) &\leq \alpha(H'(\sigma, L') \mid T'(\sigma, L')) + 1 \\ &= \alpha(H'(\sigma, L') \mid T'(\sigma, L')) + \alpha(K_{\mathcal{P}}(\sigma, L)). \end{aligned} \tag{34}$$

Let U be a maximum independent set of vertices in $T''(\sigma, L)$, let W be an independent set of vertices in $H''(\sigma, L)$ containing U, such that $|W| = \alpha(H''(\sigma, L) \mid T''(\sigma, L))$. By a previous remark about (28) holding with equality,

$$|U| = \alpha(T''(\sigma, L)) = \alpha(T'(\sigma, L')) + \alpha(K_{\mathcal{P}}(\sigma, L)). \tag{35}$$

[In fact, (35) holds for all $\sigma \in \mathcal{C}$, not just $\sigma \in L(K_{\mathcal{P}})$.] If $v \in U$, then neither U nor W contains vertices of $K_{\mathcal{P}}$, so $U \subseteq W \subseteq V(H'(\sigma, L))$. We have $\alpha(K_{\mathcal{P}}(\sigma, L)) = 1$ because $\sigma \in L(K_{\mathcal{P}})$. Therefore, letting $U_1 = U\backslash\{v\} \subseteq V(T'(\sigma, L'))$, by (35) we see that $|U| = |U_1| + 1 = \alpha(T'(\sigma, L')) + 1$, which implies that U_1 is a maximum independent set of vertices in $T'(\sigma, L')$. Since $W_1 = W\backslash\{v\}$ is an independent set of vertices in $H'(\sigma, L')$, it follows that $|W| = \alpha(H''(\sigma, L) \mid T''(\sigma, L)) = |W_1| + 1 \leq \alpha(H'(\sigma, L'))|(T'(\sigma, L')) + 1$, which establishes (34) when $v \in U$.

Now suppose that $v \notin U$. Since $\sigma \in L(K_{\mathcal{P}})$ and U is a maximum independent set in $T''(\sigma, L) = G[V(T'(\sigma, L)) \cup V(K_{\mathcal{P}}(\sigma, L))]$, it must be that $U = U_1 \cup \{x\}$, where U_1 is an independent set of vertices in $T'(\sigma, L) - v = T'(\sigma, L') - v$ and $x \in V(K_{\mathcal{P}}(\sigma, L))$. Because W is independent and $x \in U \subseteq W$, $v \notin W = W_1 \cup \{x\}$, where W_1 is an independent set of vertices of $H'(\sigma, L) - v = H'(\sigma, L') - v$, and $U_1 \subseteq W_1$. If $|U_1| = \alpha(T'(\sigma, L'))$, then we would have, as above,

$$|W| = \alpha(H''(\sigma, L) \mid T''(\sigma, L)) = |W_1| + 1 \leq \alpha(H'(\sigma, L') \mid T'(\sigma, L')) + 1,$$

once again establishing (34), and the proof of (34) would be finished. But by (35), which holds whether or not $v \in V(T')$, $|U| = |U_1| + 1 = \alpha(T'(\sigma, L')) + \alpha(K_{\mathcal{P}}(\sigma, L)) = \alpha(T'(\sigma, L')) + 1$, so (34) is established.

The proof of Claim 2, and thereby the proof of the assertion of Theorem 4.1 for Hall*, is concluded as follows.

$$\begin{aligned}
&\textstyle\sum_{\sigma \in \mathcal{C}} \alpha(H''(\sigma, L) \mid T''(\sigma, L)) \\
&\quad = \sum_{\sigma \in \mathcal{C} \backslash L(K_{\mathcal{P}})} \alpha(H''(\sigma, L) \mid T''(\sigma, L)) + \sum_{\sigma \in L(K_{\mathcal{P}})} \alpha(H''(\sigma, L) \mid T''(\sigma, L)) \qquad (36) \\
&\quad \leq \sum_{\sigma \in \mathcal{C} \backslash L(K_{\mathcal{P}})} \alpha(H'(\sigma, L') \mid T'(\sigma, L')) + \sum_{\sigma \in L(K_{\mathcal{P}})} \alpha(H'(\sigma, L') \mid T'(\sigma, L')) \\
&\qquad\quad + \sum_{\sigma \in L(K_{\mathcal{P}})} 1 \\
&\quad = \sum_{\sigma \in \mathcal{C}} \alpha(H'(\sigma, L') \mid T'(\sigma, L')) + |L(K_{\mathcal{P}})| \\
&\quad < |V(H')| + |V(K_{\mathcal{P}})| \qquad (37) \\
&\quad = |V(H'')|.
\end{aligned}$$

Inequality (37) follows from (33) and (34); (37) follows from the assumption about H' and the L-tightness of the clique $K_{\mathcal{P}}$.

The proof of the theorem's claim for $Hall{**}$ follows the preceding proof for $Hall{*}$ until the very end. Here is a recapitulation of the main points.

1. Suppose L is a list assignment to G such that G and L satisfy $HC{**}$. We proceed towards a contradiction from the assumption that there is no proper L-coloring of G.

2. Since H and L (restricted to $V(H)$) satisfy $HC{**}$ and H is $Hall{**}$, by assumption, there is a proper L-coloring of H. Let $\mathcal{P} = \{\tau \in \mathcal{C}|$ for some proper

L-coloring ψ of H, $\psi(v) = \tau\}$.

3. For each $\tau \in \mathcal{P}$, K is not properly $(L\backslash\{\tau\})$-colorable, so, by Lemma 3.4, there is an L-tight subclique K_τ of K with $\tau \in L(K_\tau)$. Applying Lemma 3.2 repeatedly we see that $K_{\mathcal{P}} = K[\bigcup_{\tau\in\mathcal{P}} V(K_\tau)]$ is L-tight and $\mathcal{P} \subseteq L(V(K_{\mathcal{P}}))$.

4. Let L' be defined on H (or on G!) by $L' = L$ except at v : $L'(v) = L(v)\backslash\mathcal{P}$. As in the proof of Theorem 4.1 so far, H and L' satisfy HC. Since they cannot possibly satisfy $HC{*}{*}$, there must exist an induced subgraph H' of H, containing v, such that
$$\sum_{\sigma\in\mathcal{C}} \min_{\widetilde{T}\lhd H'} \alpha(H'(\sigma, L')|\widetilde{T}(\sigma, L')) < |V(H')|, \tag{38}$$
in which the minimum is taken over L'-tight subgraphs $\widetilde{T}$ of H'.

5. Let $H'' = G[V(H') \cup V(K_{\mathcal{P}})]$ and for each L'-tight subgraph $\widetilde{T}$ of H', let $\widetilde{T}' = G[V(\widetilde{T}) \cup V(K_{\mathcal{P}})]$. As in an earlier part of this proof, each $\widetilde{T}'$ is L-tight, and
$$\alpha(\widetilde{T}'(\sigma, L)) = \alpha(\widetilde{T}'(\sigma, L')) + \alpha(K_{\mathcal{P}}(\sigma, L)) \tag{39}$$
for each $\sigma \in \mathcal{C}$.

6. From (39) and by arguments advanced in the earlier proof of Theorem 4.1 for Hall$*$, we obtain that for every L'-tight subgraph $\widetilde{T}$ of H', if $\sigma \in \mathcal{C}\backslash L(K_{\mathcal{P}})$ then
$$\alpha(H''(\sigma, L)|\widetilde{T}'(\sigma, L)) = \alpha(H'(\sigma, L')|\widetilde{T}(\sigma, L')), \tag{40}$$
and if $\sigma \in L(K_{\mathcal{P}})$ then
$$\alpha(H''(\sigma, L)|\widetilde{T}'(\sigma, L)) \le \alpha(H'(\sigma, L')|\widetilde{T}(\sigma, L')) + 1. \tag{41}$$

Then
$$\begin{aligned}
\sum_{\sigma\in\mathcal{C}} \min_{T\lhd H''} \alpha(H''(\sigma, L)|T(\sigma, L)) &\le \sum_{\sigma\in\mathcal{C}} \min_{\widetilde{T}\lhd H'} \alpha(H''(\sigma, L)|\widetilde{T}'(\sigma, L)) \\
&\le \sum_{\sigma\in\mathcal{C}\backslash L(K_{\mathcal{P}})} \min_{\widetilde{T}\lhd H'} \alpha(H'(\sigma, L')|\widetilde{T}(\sigma, L')) \\
&\quad + \sum_{\sigma\in L(K_{\mathcal{P}})} \min_{\widetilde{T}\lhd H'} [\alpha(H'(\sigma, L')|\widetilde{T}(\sigma, L')) + 1] \\
&= \sum_{\sigma\in\mathcal{C}} \min_{\widetilde{T}\lhd H'} \alpha(H'(\sigma, L')|\widetilde{T}(\sigma, L')) + |L(K_{\mathcal{P}})| \\
&< |V(H')| + |V(K_{\mathcal{P}})| \\
&= |V(H'')|,
\end{aligned} \tag{42}$$
contradicting the assumption that G and L satisfy $HC{*}{*}$. ☐

Proposition 4.1. *Every cycle of order at least* 4 *is* $Hall*$.

Proof. Suppose that $n \geq 4$ and L is a list assignment to $G \cong C_n$ such that G and L satisfy $HC*$, and, for every $\sigma \in \mathcal{C}$, $G(\sigma, L)$ is connected. By Corollary 2.1, to prove the proposition it suffices to prove that G has a proper L-coloring. Since $HC* \Longrightarrow HC$, by Theorem 1.3 every subpath P of G is properly L-colorable.

Since the Hall number of $G \cong C_n$ is $h(G) = 2$ [12], we may assume that for some $v \in V(G)$ and $\sigma \in \mathcal{C}$, $L(v) = \{\sigma\}$. Let u be one of v's neighbors in G, and let $P = G - uv \cong P_n$, a path on n vertices. Since P is a subgraph of G, P and L satisfy HC. By Theorem 1.3, P is properly L-colorable. Since in every proper L-coloring of any subgraph of G containing v, v must be colored σ, it must be that in every proper L-coloring of P, u is colored σ; otherwise, we could get a proper L-coloring of G from some proper L-coloring of P, and we would be done.

Let L' be defined on $V(P) = V(G)$ by $L'(u) = L(u)\backslash\{\sigma\}$ and $L'(w) = L(w)$ for all $w \in V(G)\backslash\{u\}$. Since $\sigma \notin L'(u)$, P is not properly L'-colorable; if it were, then P would be properly L-colorable with u not colored σ. By Theorem 1.3, again, it must be that P and L' do not satisfy HC. Therefore, for some induced subgraph H of P,

$$\sum_{\mu \in \mathcal{C}} \alpha(H(\mu, L')) < |V(H)| \leq \sum_{\mu \in \mathcal{C}} \alpha(H(\mu, L)). \tag{43}$$

Since $H(\mu, L) = H(\mu, L')$ for all $\mu \in \mathcal{C}\backslash\{\sigma\}$, and $H(\sigma, L)$ and $H(\sigma, L')$ differ by at most one vertex, u, from (43) we can conclude that

(i) H is L-tight, (ii) $u \in V(H)$, and (iii) u is a member of every maximum independent set of vertices of $H(\sigma, L)$.

Since (because G and L satisfy HC) every component of an L-tight induced subgraph of P is L-tight, we can replace H by its component containing u; (i), (ii), (iii) still hold, and we have the luxury of picturing H as a subpath of P containing u.

Case 1: $v \in V(H)$. Then $G = H \cup uv$. Since $G(\sigma, L)$ is connected and u is in every maximum independent set of vertices in $H(\sigma, L) = P(\sigma, L)$, it must be either that

(a) $G(\sigma, L) = G = H \cup uv$ and $H(\sigma, L) = P \cong P_n$, n odd, or

(b) $G(\sigma, L) = P_k \cup uv \cup P_r$, where P_k and P_r are disjoint paths on k and r vertices, respectively, $k + r < n$, k odd, $u \in V(P_k)$, $v \in V(P_r)$.

In subcase (a), $\alpha(G(\mu, L)) \leq \alpha(H(\mu, L))$ for all $\mu \in \mathcal{C}\backslash\{\sigma\}$ and $\alpha(G(\sigma, L)) = \alpha(C_n) = \frac{n-1}{2} = \frac{n+1}{2} - 1 = \alpha(P_n) - 1 = \alpha(H(\sigma, L)) - 1$, so

$$\begin{aligned} |V(G)| = |V(H)| = \sum_{\mu \in \mathcal{C}} \alpha(H(\mu, L)) &\geq \sum_{\mu \in \mathcal{C}} \alpha(G(\mu, L)) + 1 \\ &\geq |V(G)| + 1. \end{aligned}$$

The last inequality is implied by the fact that G and L satisfy HC. The equality $|V(H)| = \sum_{\mu \in \mathcal{C}} \alpha(H(\mu, L))$ holds because H is L-tight.

The contradiction previously derived takes care of subcase (a) of Case 1.

In subcase (b), let $T = v \cong K_1$, an L-tight subgraph of G. For all $\mu \in \mathcal{C}\backslash\{\sigma\}$,

$\alpha(G(\mu, L)|T(\mu, L)) = \alpha(G(\mu, L)) = \alpha(H(\mu, L))$, because $T(\mu, L) = \Gamma$, the null graph and because $v \notin V(G(\mu, L))$; for $\mu = \sigma$, we have

$\alpha(G(\sigma, L)|T(\sigma, L)) = \alpha(H(\sigma, L)) - 1$. Then,

$$\begin{aligned} |V(G)| &= |V(H)| = \sum_{\mu\in\mathcal{C}} \alpha(H(\mu, L)) = \sum_{\mu\in\mathcal{C}} \alpha(G(\mu, L)|T(\mu, L)) + 1 \\ &\geq |V(G)| + 1. \end{aligned}$$

This time, the contradiction is derived using the L-tightness of H and the assumption that G and L satisfy HC*.

Case 2: $v \notin V(H)$. In this case, H is an L-tight induced subgraph of G, and thus of $H' = G[V(H) \cup \{v\}]$. Again, taking $T = v$, we have

$$\alpha(H'(\mu, L)|T(\mu, L)) = \alpha(H(\mu, L)),\ \mu \in \mathcal{C}\backslash\{\sigma\},$$

and

$$\alpha(H'(\sigma, L)|T(\sigma, L)) \leq \alpha(H(\sigma, L)).$$

Therefore,

$$\begin{aligned} |V(H')| &\leq \sum_{\mu\in\mathcal{C}} \alpha(H'(\sigma, L)|T(\sigma, L)) \\ &\leq \sum_{\mu\in\mathcal{C}} \alpha(H(\sigma, L)) = |V(H)| = |V(H')| - 1. \end{aligned}$$

This last contradiction establishes that if G and L satisfy HC* then there must be a proper L-coloring of G.

□

Corollary 4.1. *Hall* $\subsetneq$ *Hall*$*$.

Proof. By either Theorem 1.3 or Theorem 2.3, and the preceding proposition, for $n \geq 4$ $C_n \in Hall + \backslash Hall$. □

Proposition 4.2. $\theta(1, 2, 2) \in Hall*$.

Proof. Let the vertices of $G = \theta(1, 2, 2)$ be labeled as in Figure 3, and let L be a list assignment to G such that G and L satisfy $HC*$. To prove the proposition, it suffices to show that G must have a proper L-coloring. Since $h(G) = 2$ [12], and G and L satisfy HC, we may as well suppose that $|L(v)| = 1$ for some $v \in V(G)$.

Case 1: $L(x) = \{\sigma\}$.

Let $H = G[\{w, y, z\}]$. As in the proof of Proposition 3.2, the assumption that there is no proper L-coloring of G implies that H has an L-tight subclique T such that $\emptyset \neq V(T(\sigma, L)) \subseteq \{w, z\}$. Let $H' = G[V(T) \cup \{x\}]$, and let x denote the graph consisting of the single vertex x. Then x is an L-tight subgraph of H'. But

$$\begin{aligned} \sum_{\mu \in \mathcal{C}} \alpha(H'(\mu, L) \mid x(\mu, L)) &= 1 + \sum_{\mu \in \mathcal{C} \setminus \{\sigma\}} \alpha(T(\mu, L)) \\ &= 1 + |V(T)| - 1 \\ &= |V(T)| \\ &< |V(H')|, \end{aligned}$$

using the fact that T is an L-tight clique with $\emptyset \neq V(T(\sigma, L)) \subseteq \{w, z\}$.

Case 2: $L(w) = \{\sigma\}$.

As in the proof of Proposition 3.2, from the assumption that G and L satisfy $HC*$, and thus HC, and that there is no proper L-coloring of G, and that $|L(x)|$, $|L(y)| \geq 2$, from case 1, we can assume that for some $a, b \in \mathcal{C} \setminus \{\sigma\}$, not necessarily distinct, we have $L(x) = \{a, \sigma\}$, $L(y) = \{b, \sigma\}$, and $L(z) \subseteq \{a, b, \sigma\}$. But then, $\sum_{\mu \in \mathcal{C}} \alpha(G(\mu, L) \mid w(\mu, L)) = 3 < 4 = |V(G)|$, contradicting the assumption that G and L satisfy $HC*$. □

5 Sudoku-Hall Condition (SHC)

Suppose the pair (G, L) satisfies Hall's Condition. A *list-reducing pair* in G, with respect to L, is a pair (H, U) such that

(i) H is an L-tight clique in G, and

(ii) $U \subseteq V(G) \setminus V(H)$ for all $u \in U$ and $\sigma \in L(u) \cap L(V(H))$, $V(H(\sigma, L)) \subseteq N_G(u)$.

If (H, U) is a list-reducing pair, the *reduced list assignment* resulting from (H, U) is the function $L' : V(G) \to \mathcal{C}$ defined by

$$L'(u) = \begin{cases} L(u), & if\ u \notin U \\ L(u) \setminus L(V(H)) & if\ u \in U. \end{cases}$$

We say that (G, L) satisfies the *(simplified) Sudoku-Hall condition* if and only if G, L satisfy Hall's condition and for every sequence $(H_1, U_1, L_1), \ldots, (H_t, U_t, L_t)$, $t \geq 1$, in which (H_i, U_i) is a list-reducing pair in G, with respect to L_{i-1} ($L_0 = L$), and L_i is the reduced list assignment on G resulting from (H_i, U_i) (with L replaced by L_{i-1} in the definition of reduced list assignment), (G, L_t) satisfies Hall's Condition.

A generalized list-reducing pair (H, U) in G, with respect to L, consists of an L-tight subgraph H of G, not necessarily a clique, and a set $U \subseteq V(G) \setminus V(H)$, such that for each $u \in U$ and $\sigma \in L(u) \cap L(V(H))$, if S is a maximum independent set of vertices in $H(\sigma, L)$, then $S \cap N_G(u) \neq \emptyset$. The reduced list assignment L' resulting from such a pair (H, U) is defined as before: $L'(v) = L(v)$ for all $v \in V(G) \backslash U$, and $L'(u) = L(u) \backslash L(V(H))$ for all $u \in U$. The definition of the *generalized Sudoku-Hall Condition*, a condition to be satisfied, or not, by the pair (G, L), is as given above.

We leave it to the reader to see that the simplified Sudoku-Hall Condition is implied by the generalized Sudoku-Hall Condition, and that the latter is, and thus both are, necessary for the existence of a proper L-coloring of G. We define the family of (generalized) Sudoku-Hall graphs, denoted (G)SH, to be the collection of those graphs G such that, for all list assignments L to G, if G and L satisfy the (generalized) Sudoku-Hall condition then there is a proper L-coloring of G.

As admitted in Section 2, we know very little about the families SH and GSH. It is clear from the proof of Theorem 2.2 that each family is closed under taking induced subgraphs; in the proof of that theorem, let the colors used to extend the list assignment to H to a list assignment to G be disjoint from $L(H)$, so that no sequence of list reductions will create L_t-tight subgraphs that are not subgraphs of H.

We strongly suspect that both families are closed under clique-attachment-at-one-vertex. We wonder if the graph of order 81 that underlies the usual Sudoku puzzles is itself Sudoku-Hall.

6 Open Questions

1. The main results of this paper show that for $\diamondsuit \in \{+, ++, *, **\}$, the family of $Hall\diamondsuit$ graphs shares with the family of Hall graphs the properties of being closed under the operations of

   (a) taking an induced subgraph and
   (b) attaching a clique at a vertex.

   Does the same hold for (c) contracting an edge?

2. For $\diamondsuit \in \{+, ++, *, **\}$, we know that the family $Hall\diamondsuit$ has a forbidden-induced-subgraphs characterization, but we do not know if that characterization is worth pursuing. The graphs which are vertex-critical with respect to the property of not being Hall are C_n, $n \geq 4$, and $\theta(1, 2, 2)$; perhaps the set of graphs which are vertex-critical with respect to the property of not being $Hall\diamondsuit$ is not so neatly described.

   Sometimes a problem is illuminated by attempts on a harder problem. We can define the $Hall\diamondsuit$ **number** $\mathbf{h}_\diamondsuit$ as the Hall number h was defined: for a finite simple graph G, $h_\diamondsuit(G)$ is the smallest positive integer m such that there is a proper L-coloring of G whenever $|L(v)| \geq m$ for all $v \in V(G)$ and G and L satisfy $HC\diamondsuit$. With this definition, $Hall\diamondsuit = \{G \mid h_\diamondsuit(G) = 1\}$. Hard as this collection may be to describe, we can, with a bow to [6], ask for more: give a forbidden-induced-subgraph characterization of $\{G \mid h_\diamondsuit(G) \leq 2\}$.

3. For $\diamondsuit \in \{+, *\}$, is $Hall\diamondsuit$ properly contained in $Hall\diamondsuit\diamondsuit$?

4. Is $Hall + \setminus Hall*$ non-empty? What about $Hall * \setminus Hall+$?

5. The questions in 4 can be attacked by seeking an intelligible description of all the graphs in $Hall * \cap Hall+$, a family of graphs which has a forbidden-induced-subgraph characterization.

6. Here is a class of list-coloring problems of some interest:

   For some $m \geq \chi(G)$, some of the vertices of G are "precolored" with single colors from $\{1, \ldots, m\}$ so that no two vertices bearing the same color are adjacent; and then the problem is to extend this precoloring to a proper coloring of G with the colors $\{1, \ldots, m\}$. The precoloring defines a list assignment to G: The precolored vertices are assigned lists of length 1 and each uncolored vertex v is assigned $L(v) = \{1, \ldots, m\} \setminus \{$colors appearing on precolored vertices adjacent to v in $G\}$.

   Clearly the question of the existence of a proper extension of the precoloring to an m-coloring of G is the same as the question of the existence of a proper L-coloring of G. Hall's condition and its refinements are necessary conditions for the existence of such an L-coloring. Notice that each precolored vertex is, by itself, an L-tight subgraph of G.

   The completion of partial latin squares is an important subclass of these completion-of-proper-precoloring problems. The underlying graph is, for some n, the line graph of $K_{n,n}$, which is also the Cartesian Product of K_n with itself, $K_n \Box K_n$, and the color set is $\{1, \ldots, n\}$. One of the co-authors of this paper, Matt Cropper, asked some time ago (around 2000) if it might be the case that the satisfying of Hall's Condition by $K_n \Box K_n$ and the list assignment induced upon it by a partial latin square would be sufficient for the partial latin square to be completable to a latin square. This question led to [2], [7], and [14]. In [2] it was shown that the answer to Cropper's question is no (this result was due entirely to John Goldwasser), but that several well-known theorems on completing latin squares, including those in [1] and [15], were equivalent to statements of the following form: If the filled-in (precolored) cells of a partial latin square (or a partial commutative latin square—the underlying graph is a bit different for these) lie in such-and-such a formation, then Hall's Condition suffices for the existence of a completion of the partial latin square to a latin square. In [7] and [14] more results of this form are proved, but these are new, not restatements of earlier theorems.

   It is quite surprising that Cropper's question was not asked much earlier, in view of the fact that the first formulation of Hall's Condition, in [10], arose from a reformulation (due to Hilton) of Ryser's famous theorem on partial latin squares. This reformulation was of the form mentioned above.

   We can now revive Cropper's question: Are any of the refinements of Hall's Condition defined in this paper sufficient conditions for the completability of a partial latin square?

References

[1] L. D. Andersen and A. J. W. Hilton, Thank Evans!, *Proc. London Math. Soc.* 47(3) (1983), 507–522.

[2] B. B. Bobga, J. L. Goldwasser, A. J. W. Hilton and P. D. Johnson Jr., Completing partial latin squares: Cropper's question, *Australas. J. Combin.* 49 (2011), 127–151.

[3] M. M. Cropper, J. L. Goldwasser, A. J. W. Hilton, D. G. Hoffman and P. D. Johnson Jr., Extending the disjoint-representatives theorems of Hall, Halmos, and Vaughan to list-multicolorings of graphs, *J. Graph Theory* 33(4) (2000), 199–219.

[4] H. Enomoto, K. Ohba, K. Ota and J. Sakamoto, Choice number of some complete multi-partite graphs, *Discrete Math.* 244 (2002), 55–66.

[5] P. Erdős, A. Rubin and H. Taylor, Choosability in graphs, *Proc. West Coast Conf. Combinatorics, Graph Theory and Computing*, *Congr. Numer.* 26 (1979), 125–157.

[6] C. Eslahchi and M. Johnson, Characterization of graphs with Hall number 2, *J. Graph Theory* 45(2) (2004), 81–100.

[7] J. L. Goldwasser, A. J. W. Hilton, D. G. Hoffman and S. Ozkan, Hall's theorem and extending partial latinized rectangles, *J. Combin. Theory Ser. A* 130 (2015), 26–41.

[8] M. Hall, An existence theorem for latin squares, *Bull. Amer. Math. Soc.* 51 (1945), 387–388.

[9] P. Hall, On representatives of subsets, *J. London Math. Soc.* 10 (1935), 26–30.

[10] A. J. W. Hilton and P. D. Johnson Jr., A variation of Ryser's theorem and a necessary condition for the list-colouring problem, Chap. 10, pp. 135–143, *Graph Colourings*, (Eds.: R. Nelson and R. Wilson), Pitman Research Notes in Mathematics Series 218, Longman Scientific and Technical with John Wiley and Sons, Harlow, Essex, and New York, 1990.

[11] A. J. W. Hilton and P. D. Johnson Jr., Extending Hall's theorem, *Topics in Combinatorics and Graph Theory: Essays in Honour of Gerhard Ringel*, Physica-Verlag, Heidelberg, 1990, 359–371.

[12] A. J. W. Hilton, P. D. Johnson Jr. and E. B. Wantland, The Hall number of a simple graph, *Congr. Numer.* 121 (1996), 161–182.

[13] A. J. W. Hilton and P. D. Johnson Jr., List multicolorings of graphs with measurable sets, *J. Graph Theory* 54(3) (2007), 179i–193.

[14] A. J. W. Hilton and E. R. Vaughan, Hall's condition for partial latin squares, *Congr. Numer.* 223 (2014), 151–174.

[15] D. G. Hoffman, Completing commutative latin squares with prescribed diagonals, *European. J. Combin.* 4 (1983), 33–35.

[16] H. J. Ryser, A combinatorial theorem with an application to Latin squares, *Proc. Amer. Math. Soc.* 2 (1951), 550–552.

[17] V. G. Vizing, Coloring the vertices of a graph in prescribed colors (Russian), *Diskret. Analiz. No. 29 Metody Diskret. Anal. V. Teorii Kodov i Shem* 29 (1976), 3–10, 101 (MR5816371).

[18] D. B. West, *Introduction to Graph Theory*, 2nd Ed., Pearson Edu., 2005.

(Received 10 Nov 2017; revised 6 Oct 2018)