AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **72(1)** (2018), Pages 128–137

Cycle packings of the complete multigraph

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Abstract

Bryant, Horsley, Maenhaut and Smith recently gave necessary and sufficient conditions for the existence of a decomposition of the complete multigraph into cycles of specified lengths $m_1, m_2, \ldots, m_\tau$. In this paper we find necessary and sufficient conditions for when the complete mulitgraph admits a packing with cycles of specified lengths $m_1, m_2, \ldots, m_\tau$. While some cycle packings can be obtained by removing cycles from a suitable cycle decomposition, in general it has not been previously known when there exists a packing of the complete multigraph with cycles of various specified lengths.

1 Introduction

A *decomposition* of a multigraph G is a collection $\mathcal{D}$ of submultigraphs of G such that each edge of G is in exactly one of the multigraphs in $\mathcal{D}$. A *packing* of a multigraph G is a collection $\mathcal{P}$ of submultigraphs of G such that each edge of G is in at most one of the multigraphs in $\mathcal{P}$. The *leave* of a packing $\mathcal{P}$ is the multigraph obtained by removing the edges in multigraphs in $\mathcal{P}$ from G. A *cycle packing* of a multigraph G is a packing $\mathcal{P}$ of G such that each submultigraph in $\mathcal{P}$ is a cycle. For positive integers λ and v, λK_v denotes the complete multigraph with λ parallel edges between each pair of v distinct vertices. Here we give necessary and sufficient conditions for the existence of a packing of λK_v with cycles of specified lengths $m_1, m_2, \ldots, m_\tau$. Note that for $v \geqslant 2$ and $\lambda \geqslant 2$, the multigraph λK_v contains 2-cycles (pairs of parallel edges).

Theorem 1.1. *Let $m_1, m_2, \ldots, m_\tau$, λ and v be positive integers. There exists a packing of λK_v with τ cycles of lengths $m_1, m_2, \ldots, m_\tau$ if and only if*

(i) $2 \leqslant m_i \leqslant v$ *for* $i \in \{1, \ldots, \tau\}$*;*

(ii) $m_1 + m_2 + \cdots + m_\tau = \lambda\binom{v}{2} - \delta$, *where δ is a nonnegative integer such that $\delta \neq 1$ and $(\delta, \lambda) \neq (2, 1)$ when $\lambda(v-1)$ is even, and $\delta \geqslant \frac{v}{2}$ when $\lambda(v-1)$ is odd;*

(iii) $\sum_{m_i=2} m_i \leqslant \begin{cases} (\lambda-1)\binom{v}{2}-2 & \text{if } \lambda \text{ and } v \text{ are odd and } \delta=2, \\ (\lambda-1)\binom{v}{2} & \text{if } \lambda \text{ is odd; and} \end{cases}$

(iv) $\max(m_1,m_2,\ldots,m_\tau) \leqslant \begin{cases} \frac{\lambda}{2}\binom{v}{2}-\tau+2 & \text{if } \lambda \text{ is even and } \delta=0, \\ \frac{\lambda}{2}\binom{v}{2}-\tau+1 & \text{if } \lambda \text{ is even and} \\ & 2\leqslant\delta<\max(m_1,m_2,\ldots,m_\tau). \end{cases}$

Bryant, Horsley, Maenhaut and Smith [5] recently characterised the complete multigraphs λK_v that admit a decomposition into cycles of specified lengths $m_1, m_2, \ldots, m_\tau$ (see also [4, 12]). Since a decomposition of a multigraph is a packing whose leave contains no edges, many instances of the cycle packing problem can be solved by removing cycles from a suitable cycle decomposition of λK_v. However there are cases which cannot be solved in this manner. These cases occur when $\lambda(v-1)$ is odd and there are $\frac{v}{2}+1$ or $\frac{v}{2}+2$ edges in the leave of the required packing, and they are addressed in Case 2 in the proof of Lemma 3.3.

When $\lambda = 1$, it had previously been found exactly when there exist decompositions of the complete graph K_v into cycles of specified lengths [6]. Furthermore, Horsley [10] found conditions for the existence of packings of the complete graph with uniform length cycles. These results built on earlier results for cycle decompositions and packings of the complete graph [1, 2, 9, 11] (see [7] for a survey). However, even in the $\lambda = 1$ case, necessary and sufficient conditions for the existence of a packing of K_v with cycles of lengths $m_1, m_2, \ldots, m_\tau$ had not previously been obtained.

We will show that the necessity of conditions (i)–(iv) in Theorem 1.1 follows from known results for cycle decompositions of λK_v. The sufficiency of these conditions is proved by first decomposing λK_v into cycles of suitable lengths, and a 1-factor if $\lambda(v-1)$ is even. We then remove some of these cycles and modify the resulting packing to obtain the one that we require. The existence of these cycle decompositions of λK_v was obtained by Bryant et al. [5] and the exact result is stated as Theorem 3.1 in Section 3. Section 2 contains the results required for modifying cycle packings.

1.1 Notation

The following definitions and notation will be used throughout this paper. For a list of integers $(m_1, m_2, \ldots, m_\tau)$, an $(m_1, m_2, \ldots, m_\tau)$-decomposition of λK_v is a decomposition of λK_v into τ cycles of lengths $m_1, m_2, \ldots, m_\tau$. Similarly, an $(m_1, m_2, \ldots, m_\tau)$-packing of λK_v is a packing of λK_v with τ cycles of lengths $m_1, m_2, \ldots, m_\tau$.

For lists $M = (m_1, m_2, \ldots, m_\tau)$ and $N = (n_1, n_2, \ldots, n_s)$, we define the list $MN = (m_1, m_2, \ldots, m_\tau, n_1, n_2, \ldots, n_s)$. If $s \leqslant \tau$ and M and N can be reordered so that $M = (m_1, m_2, \ldots, m_\tau)$ and $N = (m_1, m_2, \ldots, m_s)$, then let $M \setminus N = (m_{s+1}, m_{s+2}, \ldots, m_\tau)$. We shall also write $(m_1^{\ell_1}, m_2^{\ell_2}, \ldots, m_\tau^{\ell_\tau})$ to denote the list of integers $(\underbrace{m_1, \ldots, m_1}_{\ell_1}, \underbrace{m_2, \ldots, m_2}_{\ell_2}, \ldots, \underbrace{m_\tau, \ldots, m_\tau}_{\ell_\tau})$.

For vertices x and y in a multigraph G, the *multiplicity* of xy is the number of edges in G which have x and y as their endpoints, denoted $\mu_G(xy)$. If $\mu_G(xy) \leqslant 1$ for all pairs of vertices in $V(G)$ then we say that G is a *simple* graph. A multigraph is said to be *even* if every vertex has even degree and is said to be *odd* if every vertex has odd degree.

Given a permutation π of a set V, a subset S of V and a multigraph G with $V(G) \subseteq V$, $\pi(S)$ is defined to be the set $\{\pi(x) : x \in S\}$ and $\pi(G)$ is defined to be the multigraph with vertex set $\pi(V(G))$ and edge set $\{\pi(x)\pi(y) : xy \in E(G)\}$. The m-cycle with vertices $x_0, x_1, \ldots, x_{m-1}$ and edges $x_i x_{i+1}$ for $i \in \{0, \ldots, m-1\}$ (with subscripts modulo m) is denoted by $(x_0, x_1, \ldots, x_{m-1})$, and the n-path with vertices $y_0, y_1, \ldots, y_n$ and edges $y_j y_{j+1}$ for $j \in \{0, 1, \ldots, n-1\}$ is denoted by $[y_0, y_1, \ldots, y_n]$.

A *chord* of a cycle is an edge which is incident with two vertices of the cycle but is not in the cycle. Note that in a multigraph, a chord may be an edge parallel to an edge of the cycle. For integers $p \geqslant 2$ and $q \geqslant 1$, a (p, q)-*lasso* is the union of a p-cycle and a q-path such that the cycle and the path share exactly one vertex and that vertex is an end-vertex of the path. A (p, q)-lasso with cycle $(x_1, x_2, \ldots, x_p)$ and path $[x_p, y_1, y_2, \ldots, y_q]$ is denoted by $(x_1, x_2, \ldots, x_p)[x_p, y_1, y_2, \ldots, y_q]$. The *order* of a (p, q)-lasso is $p + q$.

2 Modifying cycle packings of λK_v

The aim of this section is to obtain some useful tools for modifying cycle packings of the complete multigraph, namely Lemmas 2.2 and 2.3. The simple graph versions of Lemmas 2.2 and 2.3 are due to Bryant and Horsley [8] and have been applied to prove the result on maximum packings of the simple complete graph with uniform length cycles [10].

We require the following cycle switching lemma for cycle packings of multigraphs. Lemma 2.1 is similar to [4, Lemma 2.1] and is also closely related to the cycle switching method which has been applied to simple graphs (see for example [3]).

Lemma 2.1 ([4, Lemma 2.1]). *Let v and λ be positive integers, let M be a list of integers, let $\mathcal{P}$ be an M-packing of λK_v, let L be the leave of $\mathcal{P}$, let α and β be distinct vertices of L, and let π be the transposition $(\alpha\beta)$. Let E be a subset of $E(L)$ such that, for each vertex $x \in V(L) \backslash \{\alpha, \beta\}$, E contains precisely $\max(0, \mu_L(x\alpha) - \mu_L(x\beta))$ edges with endpoints x and α, and precisely $\max(0, \mu_L(x\beta) - \mu_L(x\alpha))$ edges with endpoints x and β (so E may contain multiple edges with the same endpoints), and E contains no other edges. Then there exists a partition of E into pairs such that for each pair $\{x_1y_1, x_2y_2\}$ of the partition, there exists an M-packing $\mathcal{P}'$ of λK_v with leave $L' = (L - \{x_1y_1, x_2y_2\}) + \{\pi(x_1)\pi(y_1), \pi(x_2)\pi(y_2)\}$.*

Furthermore, if $\mathcal{P} = \{C_1, \ldots, C_t\}$, then $\mathcal{P}' = \{C'_1, \ldots, C'_t\}$ where for $i \in \{1, \ldots, t\}$, C'_i is a cycle of the same length as C_i such that for $i \in \{1, \ldots, t\}$

- *If neither α nor β is in $V(C_i)$, then $C'_i = C_i$;*
- *If exactly one of α and β is in $V(C_i)$, then $C'_i = C_i$ or $C'_i = \pi(C_i)$; and*
- *If both α and β are in $V(C_i)$, then $C'_i = Q_i \cup Q^*_i$ where $Q_i = P_i$ or $\pi(P_i)$, $Q^*_i = P^*_i$ or $\pi(P^*_i)$, and P_i and P^*_i are the two paths from α to β in C_i.*

When $\lambda(v-1)$ is even, Lemma 2.1 reduces to [4, Lemma 2.1]. Note that $\mathcal{P}$ is a cycle packing of λK_v regardless of the parity of $\lambda(v-1)$, whereas when $\lambda(v-1)$ is odd [4, Lemma 2.1] concerns a cycle packing of $\lambda K_v - I$, where I is a 1-factor of λK_v. Nevertheless, the proof of Lemma 2.1 follows from similar arguments to those used in the corresponding case of the proof in [4].

We will use the following notation when we apply Lemma 2.1. For vertices x and y so that $\{x_1, y_1, x_2, y_2\} \subseteq \{\alpha, \beta, x, y\}$ for some pair $\{x_1y_1, x_2y_2\}$ in the partition of E given by Lemma 2.1, we say that we are performing an (α, β)-switch with origin x and terminus y. Note that when $\lambda \geqslant 2$, x_1y_1 and x_2y_2 may be parallel edges, in which case $x = y$.

Lemma 2.2. *Let v, s and λ be positive integers such that $s \geqslant 3$, and let $(m_1, m_2, \ldots, m_\tau)$ be a list of integers. Suppose there exists an $(m_1, m_2, \ldots, m_\tau)$-packing $\mathcal{P}$ of λK_v whose leave contains a lasso of order at least $s+2$ and suppose that if s is even then the cycle of the lasso has even length. Then there exists an $(m_1, m_2, \ldots, m_\tau, s)$-packing of λK_v.*

Proof. Let L be the leave of $\mathcal{P}$. Suppose that L contains a (p, q)-lasso $(x_1, x_2, \ldots, x_p)[x_p, y_1, y_2, \ldots, y_q]$ such that $p + q \geqslant s + 2$ and p is even if s is even. If L contains an s-cycle then we add it to the packing to complete the proof, so assume L does not contain an s-cycle and hence $p \neq s$. In each of the following applications of an (α, β)-switch with origin x, we observe that $\mu_L(\alpha x) > 0$ and, because L does not contain an s-cycle, we can assume that $\mu_L(\beta x) = 0$ and hence the switch exists.
Case 1. Suppose $2 \leqslant p < s$ and either $p = 2$ or $p \equiv s \pmod 2$. We can assume that $p + q = s + 2$ since L contains a $(p, s + 2 - p)$-lasso.

Note that $\mu_L(x_2y_{q-1}) = 0$ since L does not contain an s-cycle. Then let L' be the leave of the packing $\mathcal{P}'$ obtained from $\mathcal{P}$ by applying an (x_1, y_{q-1})-switch with origin x_2. If the terminus of the switch is not y_{q-2} then L' contains an s-cycle which we add to $\mathcal{P}'$ to obtain an $(m_1, m_2, \ldots, m_\tau, s)$-packing of λK_v. Otherwise, the terminus of the switch is y_{q-2} and L' contains a (q, p)-lasso $(x'_1, x'_2, \ldots, x'_q)[x'_q, y'_1, y'_2, \ldots, y'_p]$. If $p = 2$ then L' contains an s-cycle which we add to the packing to obtain the required $(m_1, m_2, \ldots, m_\tau, s)$-packing of λK_v, so assume $p \geqslant 3$ and that L' contains no s-cycle.

We apply an (x'_2, y'_p)-switch with origin x'_3 to $\mathcal{P}'$. As before, we observe that $\mu_{L'}(x'_3y'_p) = 0$, for otherwise L' contains an s-cycle. Let L'' be the leave of the resulting packing $\mathcal{P}''$. If the terminus of this switch is not y'_{p-1} then L'' contains a $(p+q-2)$-cycle, and since $s = p+q-2$, we add this cycle to $\mathcal{P}''$ to obtain the required packing. If y'_{p-1} is the terminus of the switch, then L'' contains a $(p+2, q-2)$-lasso. Since $p < s$ and $p \equiv s \pmod 2$, after $\frac{p-s}{2}$ iterations of this case we will obtain a packing whose leave contains an s-cycle, and hence the required packing exists.
Case 2. Suppose $3 \leqslant p < s$ and $p \not\equiv s \pmod 2$. As above, assume $p + q = s + 2$. Then s is odd, $p \geqslant 4$ is even and q is odd by our hypotheses.

Let L' be the leave of the packing $\mathcal{P}'$ obtained from $\mathcal{P}$ by applying an (x_2, y_q)-switch with origin x_3. If the terminus of the switch is not y_{q-1} then L' contains an s-cycle which we add to $\mathcal{P}'$ to obtain the required packing. Otherwise, the terminus of the switch is y_{q-1} and L' contains a $(q+2, p-2)$-lasso. Note that $q+2 \equiv s \pmod 2$ and $q + 2 \leqslant s$ because $p + q = s + 2$ and $p \geqslant 4$. If $q + 2 = s$ then we can add the

s-cycle to $\mathcal{P}'$ to obtain the required packing, otherwise $q+2 < s$ and we can proceed as in Case 1.

Case 3. Suppose $3 \leqslant s < p$. We apply an (x_{p-s+1}, y_1)-switch with origin x_{p-s+2} to $\mathcal{P}$. Let $\mathcal{P}'$ be the resulting packing, with leave L'. If the terminus of the switch is not x_p then L' contains an s-cycle which completes the proof. Otherwise, L' contains a $(p-s+2, q+s-2)$-lasso. By repeating this process we obtain an $(m_1, m_2, \ldots, m_\tau)$-packing of λK_v whose leave contains a $(p', p+q-p')$-lasso such that $2 \leqslant p' \leqslant s$ and $p' \equiv p \pmod{(s-2)}$. If $p' = s$ then we include this cycle to obtain an $(m_1, m_2, \ldots, m_\tau, s)$-packing of λK_v, and if $p' < s$ then we apply Case 1 or Case 2. □

Lemma 2.3. *Let v, s and λ be positive integers with $s \geqslant 3$, and let M be a list of integers. Suppose there exists an M-packing of λK_v whose leave L has a component H containing an $(s+1)$-cycle with a chord. Then there exists an M-packing of λK_v with a leave L' such that $E(L') = (E(L) \setminus E(H)) \cup E(H')$, where H' is a graph with $V(H') = V(H)$ and $|E(H')| = |E(H)|$ which contains an $(s,1)$-lasso. Furthermore, $\deg_{H'}(x) \geqslant \deg_H(x)$ for each vertex x in the s-cycle of this lasso.*

Proof. Let $(x_1, \ldots, x_{s+1})$ be an $(s+1)$-cycle in H with chord x_1x_e for some $e \in \{2, 3, \ldots, s\}$ and note that L is not necessarily a simple graph. If H contains an $(s,1)$-lasso then this is the required leave, so suppose otherwise.

If $e = 2$, then perform an (x_3, x_2)-switch with origin x_4. This switch exists because H contains no $(s,1)$-lasso and hence $\mu_L(x_2x_4) = 0$. Regardless of the terminus, the leave of the resulting packing contains the $(s,1)$-lasso $(x_4, \ldots, x_{s+1}, x_1, x_2)[x_2, x_3]$, and $\deg_{H'}(x_i) \geqslant \deg_H(x_i)$ for $i \in \{1, \ldots, s+1\} \setminus \{3\}$. If $e = 3$, then H contains an $(s,1)$-lasso. In either case we obtain an M-packing of λK_v with the required leave.

Suppose $e \geqslant 4$ and, since H does not contain a $(s,1)$-lasso, $\mu_L(x_{e-2}x_e) = 0$. Let $\mathcal{P}^*$ be the packing with leave L^* obtained from $\mathcal{P}$ by applying an (x_{e-1}, x_e)-switch with origin x_{e-2}. If the terminus of the switch is not x_{e+1} then $E(L^*) = (E(L) \setminus E(H)) \cup E(H^*)$, where H^* is a graph with $V(H^*) = V(H)$ and $|E(H^*)| = |E(H)|$ which contains the $(s,1)$-lasso $(x_{e+1}, \ldots, x_{s+1}, x_1, \ldots, x_{e-2}, x_e)[x_e, x_{e-1}]$. Also note that $\deg_{H^*}(x_e) \geqslant \deg_H(x_e)$ and $\deg_{H^*}(x_i) = \deg_H(x_i)$ for $i \in \{1, \ldots, s+1\} \setminus \{e, e-1\}$ as required. Otherwise x_{e+1} is the terminus of the switch and hence $E(L^*) = (E(L) \setminus E(H)) \cup E(H^*)$, where H^* is a graph with $V(H^*) = V(H)$ and $|E(H^*)| = |E(H)|$ which contains an $(s+1)$-cycle $(x_1^*, \ldots, x_{s+1}^*)$ with chord $x_1^* x_{e-1}^*$. Furthermore, the degree of each vertex in this $(s+1)$-cycle remains unchanged in H^*. We can repeat this process until we obtain a packing whose leave has the required $(s,1)$-lasso. □

3 Main result

This section contains the proof of Theorem 1.1. We first use Theorem 3.1 to prove Lemma 3.2 which shows the necessity of the conditions in Theorem 1.1. The sufficiency of these conditions is then established for odd λ and even λ in Lemmas 3.3 and 3.4 respectively. Lemmas 3.3 and 3.4 rely on using Lemmas 2.2 and 2.3 to modify suitable cycle packings of λK_v obtained via Theorem 3.1.

Theorem 3.1 ([5]). *Let $(m_1, m_2, \ldots, m_\tau)$ be a list of integers and let λ and v be positive integers. There is an $(m_1, m_2, \ldots, m_\tau)$-decomposition of λK_v if and only if*

- $\lambda(v-1)$ *is even;*
- $2 \leqslant m_i \leqslant v$ *for* $i \in \{1, \ldots, \tau\}$*;*
- $m_1 + m_2 + \cdots + m_\tau = \lambda\binom{v}{2}$*;*
- $\max(m_1, m_2, \ldots, m_\tau) + \tau - 2 \leqslant \frac{\lambda}{2}\binom{v}{2}$ *when λ is even; and*
- $\sum_{m_i=2} m_i \leqslant (\lambda - 1)\binom{v}{2}$ *when λ is odd.*

There is an $(m_1, m_2, \ldots, m_\tau)$-decomposition of $\lambda K_v - I$, where I is a 1-factor in λK_v, if and only if

- $\lambda(v-1)$ *is odd;*
- $2 \leqslant m_i \leqslant v$ *for* $i \in \{1, \ldots, \tau\}$*;*
- $m_1 + m_2 + \cdots + m_\tau = \lambda\binom{v}{2} - \frac{v}{2}$*; and*
- $\sum_{m_i=2} m_i \leqslant (\lambda - 1)\binom{v}{2}$.

The necessity of conditions (i)–(iv) in Theorem 1.1 follows from Theorem 3.1 as we now show.

Lemma 3.2. *Let $(m_1, m_2, \ldots, m_\tau)$ be a list of integers and let λ and v be positive integers. If there exists an $(m_1, m_2, \ldots, m_\tau)$-packing of λK_v then*

(i) $2 \leqslant m_i \leqslant v$ *for* $i \in \{1, \ldots, \tau\}$*;*

(ii) $m_1 + m_2 + \cdots + m_\tau = \lambda\binom{v}{2} - \delta$*, where δ is a nonnegative integer such that $\delta \neq 1$ and $(\delta, \lambda) \neq (2, 1)$ when $\lambda(v-1)$ is even, and $\delta \geqslant \frac{v}{2}$ when $\lambda(v-1)$ is odd;*

(iii) $$\sum_{m_i=2} m_i \leqslant \begin{cases} (\lambda-1)\binom{v}{2} - 2 & \text{if } \lambda \text{ and } v \text{ are odd and } \delta = 2, \\ (\lambda-1)\binom{v}{2} & \text{if } \lambda \text{ is odd; and} \end{cases}$$

(iv) $$\max(m_1, m_2, \ldots, m_\tau) \leqslant \begin{cases} \frac{\lambda}{2}\binom{v}{2} - \tau + 2 & \text{if } \lambda \text{ is even and } \delta = 0, \\ \frac{\lambda}{2}\binom{v}{2} - \tau + 1 & \text{if } \lambda \text{ is even and} \\ & \quad 2 \leqslant \delta < \max(m_1, m_2, \ldots, m_\tau). \end{cases}$$

Proof. Let $M = (m_1, m_2, \ldots, m_\tau)$. Suppose there exists an M-packing $\mathcal{P}$ of λK_v with leave L. Condition (i) is obvious. The degree of each vertex in λK_v is $\lambda(v-1)$, so if $\lambda(v-1)$ is even then L is an even multigraph and if $\lambda(v-1)$ is odd then L is an odd multigraph. Hence (ii) follows because a loopless even graph cannot have a single edge, an even simple graph cannot have two edges, and an odd graph on v vertices has at least $\frac{v}{2}$ edges. To see that condition (iii) holds, note that there are at most $\lfloor\frac{\lambda}{2}\rfloor\binom{v}{2}$ edge-disjoint 2-cycles in λK_v. Furthermore, note that if λ and v are both odd and $\delta = 2$ then L is an even multigraph with two edges and hence

L is a 2-cycle. If λ is even and $\delta = 0$ then (iv) follows directly from Theorem 3.1, so suppose λ is even and $2 \leqslant \delta < \max(m_1, m_2, \ldots, m_\tau)$. Then L contains at least one cycle so there exists an MN-decomposition of λK_v for some list N containing at least one entry. So (iv) follows from Theorem 3.1. □

It remains to prove the sufficiency of conditions (i)–(iv) in Theorem 1.1 for the existence of cycle packings of λK_v.

Lemma 3.3. *Let $(m_1, m_2, \ldots, m_\tau)$ be a list of integers and let λ and v be positive integers with λ odd. Then there exists an $(m_1, m_2, \ldots, m_\tau)$-packing of λK_v if and only if*

(i) $2 \leqslant m_i \leqslant v$ *for* $i \in \{1, \ldots, \tau\}$;

(ii) $m_1 + m_2 + \cdots + m_\tau = \lambda\binom{v}{2} - \delta$, *where δ is a nonnegative integer such that $\delta \neq 1$ and $(\delta, \lambda) \neq (2, 1)$ if v is odd, and $\delta \geqslant \frac{v}{2}$ if v is even; and*

(iii) $\sum_{m_i=2} m_i \leqslant \begin{cases} (\lambda-1)\binom{v}{2} - 2 & \text{if } v \text{ is odd and } \delta = 2, \\ (\lambda-1)\binom{v}{2} & \text{otherwise.} \end{cases}$

Proof. Let $M = (m_1, m_2, \ldots, m_\tau)$. If there exists an M-packing of λK_v, then conditions (i)–(iii) hold by Lemma 3.2. So it remains to show that if λ, v and M satisfy (i)–(iii), then there is an M-packing of λK_v.

Let $\varepsilon = \delta$ if v is odd, and $\varepsilon = \delta - \frac{v}{2}$ if v is even. If $\varepsilon = 0$ then there exists an M-packing of λK_v by Theorem 3.1. If $v = 2$, then ε is even by (i) and (ii) and there exists a 2-cycle decomposition of $\lambda K_2 - I$, where I is a 1-factor of λK_2, so we obtain an M-packing of λK_v by removing all but τ of the 2-cycles from this decomposition. So suppose $\varepsilon \geqslant 1$ and $v \geqslant 3$, and note that if v is odd then $\varepsilon \neq 1$ and $(\lambda, \varepsilon) \neq (1, 2)$.

Case 1. Suppose v is odd or $\varepsilon \geqslant 3$. Note that if v is odd and $\varepsilon = 2$ then $2 + \sum_{m_i=2} m_i \leqslant (\lambda - 1)\binom{v}{2}$ by (iii).

We will show that there exists a list N such that $2 \leqslant n \leqslant v$ for all $n \in N$, $\sum_{n\in N} n = \varepsilon$ and $\sum_{n\in N, n=2} n + \sum_{m_j=2} m_j \leqslant (\lambda - 1)\binom{v}{2}$. We then show that by Theorem 3.1 there exists an MN-decomposition $\mathcal{D}$ of λK_v (if v is odd) or $\lambda K_v - I$ (if v is even), where I is a 1-factor of λK_v. We obtain an M-packing of λK_v by removing cycles of the lengths in N from $\mathcal{D}$.

We first consider $v = 3$. If $v = 3$ and ε is even, then $m_i = 3$ for some $i \in \{1, \ldots, \tau\}$ by (i) and (ii). Then $\varepsilon + \sum_{m_i=2} m_i \leqslant (\lambda - 1)\binom{v}{2}$ by (ii) and we take $N = (2^{\varepsilon/2})$. If $v = 3$ and ε is odd then $\varepsilon - 3 + \sum_{m_i=2} m_i \leqslant (\lambda - 1)\binom{v}{2}$ by (ii) and we take $N = (2^{(\varepsilon-3)/2}, 3)$. In each of these cases we can see that there exists an MN-decomposition of λK_v since the assumptions of Theorem 3.1 are satisfied by (i)–(iii).

Now assume $v \geqslant 4$ and let q and r be nonnegative integers such that $\varepsilon = vq+r$ and $0 \leqslant r < v$. If $q = 0$ or $r \notin \{1, 2\}$ then we take $N = (r, v^q)$. If $q \geqslant 1$ and $r \in \{1, 2\}$, note that either $v - 3 + r \geqslant 3$, or $v = 4$ and $r = 1$ and let $N = (3, v - 3 + r, v^{q-1})$. If $\varepsilon = 2$ or $(v, r) = (4, 1)$, then N contains exactly one entry equal to 2 and otherwise $n \geqslant 3$ for all $n \in N$. By (iii) and the assumption that v is odd or $\varepsilon \geqslant 3$, if $\varepsilon = 2$ then $2 + \sum_{m_i=2} m_i \leqslant (\lambda - 1)\binom{v}{2}$. Further, if $v = 4$ and $\varepsilon = 4q + 1$ for some $q \geqslant 1$ then (i) and (ii) imply that $m_i = 3$ for some $i \in \{1, \dots, \tau\}$ so again $2+\sum_{m_i=2} m_i \leqslant (\lambda-1)\binom{v}{2}$. We can therefore see that there exists an MN-decomposition of λK_v (or $\lambda K_v - I$) since the assumptions of Theorem 3.1 are satisfied by (i)–(iii) and the fact that $\sum_{n\in N} n = \varepsilon$.

Case 2. Suppose $v \geqslant 4$ is even and $\varepsilon \in \{1, 2\}$. Let m be the least odd entry in M if M contains an odd entry, otherwise let m be the least entry in M such that $m \geqslant 4$. It follows from (iii) that such an entry exists. Note that if $\varepsilon = 1$ then it follows from (ii) that M contains an odd entry and hence m is odd.

Case 2a. Suppose $m + \varepsilon \leqslant v$. By Theorem 3.1 there exists an M'-decomposition $\mathcal{D}$ of $\lambda K_v - I$, where I is a 1-factor of λK_v and $M' = (m_1, m_2, \dots, m_\tau, m + \varepsilon) \setminus (m)$. Let $\mathcal{P}$ be the $M \setminus (m)$-packing of λK_v that is obtained by removing an $(m+\varepsilon)$-cycle from $\mathcal{D}$. Let L be the leave of $\mathcal{P}$ and note that L is an edge-disjoint union of an $(m + \varepsilon)$-cycle and the 1-factor I.

If L contains an $(m + \varepsilon, 1)$-lasso then we apply Lemma 2.2 to $\mathcal{P}$ with $s = m$ to obtain an M-packing of λK_v. The assumptions of Lemma 2.2 are satisfied because $\varepsilon + 1 \geqslant 2$, and if m is even then M contains no odd entries so $\varepsilon = 2$ by (ii).

So suppose L does not contain an $(m + \varepsilon, 1)$-lasso. Then $m + \varepsilon$ is even and L contains a component H such that H is the edge-disjoint union of an $(m+\varepsilon)$-cycle and a 1-factor on the vertex set of this cycle. We apply Lemma 2.3 to $\mathcal{P}$ with $s = m+\varepsilon-1$ to obtain an $M \setminus (m)$-packing $\mathcal{P}'$ of λK_v whose leave L' contains a component H' on $m + \varepsilon$ vertices that has $\frac{3}{2}(m + \varepsilon)$ edges and contains an $(m + \varepsilon - 1, 1)$-lasso. If $\varepsilon = 1$ then adding the m-cycle of this lasso to $\mathcal{P}'$ results in the required M-packing of λK_v. Otherwise $\varepsilon = 2$ and H' contains an $(m+1)$-cycle with a chord because $m \geqslant 3$ and any vertex in this cycle has degree at least 3. Then we can apply Lemma 2.3 with $s = m$ to $\mathcal{P}'$ to obtain an $M \setminus (m)$-packing $\mathcal{P}''$ of λK_v whose leave contains an $(m, 1)$-lasso. We add the m-cycle of this lasso to $\mathcal{P}''$ to obtain the required packing.

Case 2b. Suppose $m+\varepsilon > v$. Then $m \geqslant v-1$ and $\varepsilon = 2$ because ε is even if $m = v$.

If $m = v$ then $m_i \in \{2, v\}$ for all $i \in \{1, \dots, \tau\}$, so

$$\lambda\binom{v}{2} - \frac{v}{2} \equiv 2 + \sum_{m_i=2} m_i \pmod{v}$$

by (ii) and hence $2 + \sum_{m_i=2} m_i \leqslant (\lambda - 1)\binom{v}{2}$ by (iii). Then by Theorem 3.1 there exists an $(m_1, m_2, \dots, m_\tau, 2)$-decomposition $\mathcal{D}$ of $\lambda K_v - I$. We remove a 2-cycle from $\mathcal{D}$ to complete the proof.

Now suppose that $m = v - 1$. It follows from (ii) and the definition of ε that $\varepsilon+\sum_{i=1}^{\tau} m_i = \frac{v}{2}(\lambda(v-1)-1)$, so since ε and v are even, $\varepsilon+\sum_{i=1}^{\tau} m_i$ is also even and at least two entries of M are equal to $v-1$. Let $M' = (m_1, m_2, \dots, m_\tau, v, v)\setminus(v-1, v-1)$, then by Theorem 3.1 there exists an M'-decomposition $\mathcal{D}_0$ of $\lambda K_v - I$. Let $\mathcal{P}_0$ be the $M' \setminus (v)$-packing of λK_v formed by removing a v-cycle from $\mathcal{D}_0$. The leave L_0 of $\mathcal{P}_0$ is the edge-disjoint union of a v-cycle and the 1-factor I. Let $\mathcal{P}_1$ be the $M'\setminus(v)$-packing of λK_v obtained by applying Lemma 2.3 to $\mathcal{P}_0$ with $s = v - 1$. Then the leave of $\mathcal{P}_1$ contains a $(v - 1, 1)$-lasso. We add the $(v - 1)$-cycle of this lasso to $\mathcal{P}_1$ and remove a v-cycle to obtain an $M \setminus (v - 1)$-packing $\mathcal{P}_2$ of λK_v. The leave of $\mathcal{P}_2$ has size $3\frac{v}{2} + 1$.

By applying Lemma 2.3 to $\mathcal{P}_2$ with $s = v-1$ we obtain an $M \setminus (v-1)$-packing $\mathcal{P}_3$ of λK_v whose leave contains a $(v-1,1)$-lasso. We add the $(v-1)$-cycle of this lasso to $\mathcal{P}_3$ to obtain the required M-packing of λK_v. □

Lemma 3.4. *Let $(m_1, m_2, \ldots, m_\tau)$ be a list of integers and let λ and v be positive integers with λ even. Then there exists an $(m_1, m_2, \ldots, m_\tau)$-packing of λK_v if and only if*

(i) $2 \leqslant m_i \leqslant v$ *for* $i \in \{1, \ldots, \tau\}$*;*

(ii) $m_1 + m_2 + \cdots + m_\tau = \lambda\binom{v}{2} - \delta$*, where δ is a nonnegative integer such that* $\delta \neq 1$*; and*

(iii) $\max(m_1, m_2, \ldots, m_\tau) \leqslant \begin{cases} \frac{\lambda}{2}\binom{v}{2} - \tau + 2 & \text{if } \delta = 0, \\ \frac{\lambda}{2}\binom{v}{2} - \tau + 1 & \text{if } 2 \leqslant \delta < \max(m_1, m_2, \ldots, m_\tau). \end{cases}$

Proof. Let $M = (m_1, m_2, \ldots, m_\tau)$ and, without loss of generality, reorder M so that m_τ is a maximal entry in M. If there exists an M-packing $\mathcal{P}$ of λK_v with leave L, then conditions (i)–(iii) hold by Lemma 3.2. So it remains to show that if λ, v and M satisfy (i)–(iii), then there exists an M-packing of λK_v. If $\delta = 0$ then the result follows immediately from Theorem 3.1, so suppose $\delta \geqslant 2$.

Let

$$N = \begin{cases} (\delta) & \text{if } 2 \leqslant \delta < m_\tau, \\ (2^{(\delta - m_\tau)/2}, m_\tau) & \text{if } \delta \geqslant m_\tau \text{ and } \delta \equiv m_\tau \pmod 2, \\ (2^{(\delta - m_\tau + 1)/2}, m_\tau - 1) & \text{if } \delta \geqslant m_\tau \text{ and } \delta \not\equiv m_\tau \pmod 2. \end{cases}$$

Note that in each case $\sum_{n \in N} n = \delta$. Also note that $n \geqslant 2$ for each $n \in N$ because $\delta \geqslant 2$ and, it follows from (ii) that δ is even when $m_\tau = 2$. We now show that MN, λ and v satisfy the conditions of Theorem 3.1, giving an MN-decomposition of λK_v from which we obtain the required M-packing of λK_v.

Let s be the number of entries in N. First observe that $\sum_{m \in M} m + \sum_{n \in N} n = \lambda\binom{v}{2}$ by (ii) and since $\sum_{n \in N} n = \delta$. By (i) and the definition of N it also holds that $2 \leqslant n \leqslant m_\tau \leqslant v$ for all $n \in N$. If $2 \leqslant \delta < m_\tau$, then $m_\tau \leqslant \frac{\lambda}{2}\binom{v}{2} - \tau - s + 2$ by (iii) and because $s = 1$. If $\delta \geqslant m_\tau$, then because $\sum_{m \in M} m \geqslant m_\tau + 2(\tau - 1)$ and $\sum_{n \in N} n \geqslant m_\tau - 1 + 2(s-1)$, it follows that

$$\begin{aligned} \tfrac{\lambda}{2}\tbinom{v}{2} - \tau - s + 2 &= \tfrac{1}{2}\left(\textstyle\sum_{m \in M} m + \sum_{n \in N} n\right) - \tau - s + 2 \\ &\geqslant \tfrac{1}{2}(m_\tau + 2(\tau - 1) + m_\tau - 1 + 2(s-1)) - \tau - s + 2 \\ &= m_\tau - \tfrac{1}{2}. \end{aligned}$$

Therefore $m_\tau \leqslant \frac{\lambda}{2}\binom{v}{2} - \tau - s + 2$ is a maximal entry in MN because $\frac{\lambda}{2}\binom{v}{2} - \tau - s + 2$ is an integer. So by Theorem 3.1 we can see that there exists an MN-decomposition of λK_v from which we remove cycles of the lengths in N. □

Acknowledgements

The author was supported by a Monash University Faculty of Science Postgraduate Publication Award.

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(Received 20 Dec 2017; revised 9 Apr 2018)