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Adjunctions in broadcast domination with a cost function

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Abstract

We generalize the notion of a dominating broadcast by introducing a cost function k^* which assigns to each distance x the cost, $k^*(x)$, for a vertex to broadcast that distance. A cost function k^* has a right adjoint k_*. The right adjoint k_* provides a different approach for generalising broadcast domination. We study the relationship between these two approaches and make use of both perspectives to generalise the bounds found in [B. Brešar and S. Špacapan, *Ars Combin.* 92, (2009), 303–320] on broadcast domination of product graphs.

1 Introduction

For background on graph domination and broadcast domination see [5] and [2] respectively. A broadcast on a graph G was first defined by Erwin in [3] as a function $f\colon V(G) \to \{1, 2, \dots, \mathrm{diam}(G)\}$ which assigns to each vertex the distance it broadcasts. In this paper we use a slightly modified definition, where we instead take the codomain to be $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\} = \{0, 1, 2, \dots\} \cup \{\infty\}$. This decision will be justified below. Given a broadcast $f\colon V(G) \to \overline{\mathbb{N}}$ on a graph G, the f-neighbourhood of v is the set $N_f[v] = \{w \in V(G) \mid d_G(v, w) \leq f(v)\}$ where $d_G(v, w)$ is the distance between v and w in G. A vertex v with $f(v) \neq 0$ is said to dominate all vertices in its f-neighbourhood. With this in mind, a broadcast f is dominating if each vertex in $V(G)$ is dominated by some vertex in $V(G)$. The set of all broadcasting vertices v (i.e, those with $f(v) \neq 0$) is denoted by V_f^+. A broadcast f is independent if there are no distinct vertices $v, w \in V_f^+$ such that $v \in N_f[w]$ (or $w \in N_f[v]$). A broadcast f is efficient if for any pair of distinct vertices $v, w \in V_f^+$ we have that $N_f[v] \cap N_f[w] = \emptyset$.

The primary concern in broadcast domination is finding minimum-cost dominating broadcasts, where the cost of a broadcast f is defined to be $c(f) = \sum_{v \in V(G)} f(v)$.

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Notice here that our modification of the definition of a broadcast has no impact on results pertaining to minimum-cost dominating broadcasts, as any minimum-cost broadcast f cannot have $f(v) > \text{diam}(G)$ for a vertex $v \in V(G)$. If G is a graph then the broadcast domination number $\gamma_b(G) = \min\{c(f) \mid f \text{ is a dominating broadcast on } G\}$. This notion of cost is rather inflexible as the cost for a vertex to broadcast distance x is assumed to be x. We can generalise broadcast domination by associating to a graph a cost function $k^*\colon \overline{\mathbb{N}} \to \overline{\mathbb{N}}$, where the cost to broadcast distance x is then $k^*(x)$. We believe this idea was first considered in [6]. We now consider broadcasts on the pair (G, k^*) of a graph G and a cost function k^*. The cost of a broadcast f on (G, k^*) is given by $c_{k^*}(f) = \sum_{v\in V(G)} k^*(f(v))$ and the cost domination number on (G, k^*) is $\gamma_c^{k^*}(G) = \min\{c_{k^*}(f) \mid f \text{ is a dominating broadcast on } (G, k^*)\}$. A broadcast f is optimal if it has cost equal to the cost domination number of (G, k^*).

2 Cost functions and adjoints

When studying broadcasts with a cost function we find that many results will only hold when k^* is superadditive (i.e. when $k^*(x + y) \geq k^*(x) + k^*(y)$ for all $x, y \in \overline{\mathbb{N}}$). Surprisingly, similar results will hold when the 'right adjoint' of k^* is subadditive. We investigate this below.

Let A and B be partially ordered sets and $k^*\colon A \to B$ an order-preserving map between them. Then the function $k_*\colon B \to A$ is called the right adjoint of k^* if $k_*(b)$ is the largest element in A such that $k^*(k_*(b)) \leq b$. In this case we also call k^* the left adjoint of k_*. Additionally it can be shown that k^* has the property that $k^*(b)$ is the least element such that $k_*(k^*(b)) \geq b$. As an example see the following.

Example 2.1 *Let $f^* : \overline{\mathbb{N}} \to \overline{\mathbb{N}}$ be given by $f^*(n) = 2n$. This has both adjoints. The right adjoint is $f_*(n) = \lfloor \frac{n}{2} \rfloor$ and the left adjoint is $f_!(n) = \lceil \frac{n}{2} \rceil$.*

For more information on adjunctions see [7].

General functions $k : \overline{\mathbb{N}} \to \overline{\mathbb{N}}$ need not have right adjoints. For this reason we define cost functions in the following way.

Definition 2.2 *A cost function is a function $k^*\colon \overline{\mathbb{N}} \to \overline{\mathbb{N}}$ such that*

1. $k^*(0) = 0$,
2. k^* *is order preserving,*
3. $k^*(\infty) = \sup\{k^*(x) \mid x \in \mathbb{N}\}$,
4. $k^*(x) = 0 \implies x = 0$.

The first two requirements are very natural constraints to put on a function which is modelling cost. With regard to the third requirement, note that any order preserving map $k\colon \mathbb{N} \to \mathbb{N}$ satisfying the other conditions can be extended uniquely to one of this form. The fourth condition is not necessary for the adjoint to exist, but we include it to exclude degenerate cases where it costs nothing to broadcast a non-zero distance. The first three conditions are the requisite conditions for the 'adjoint functor theorem for partially ordered sets' to apply which guarantees that a right adjoint exists.

Let k^* be a cost function and k_* its right adjoint. Given that k^* takes in a distance and returns a cost, we think of k_* as taking in a cost and returning a distance. From the definition of a right adjoint we see that $k_*(t)$ is the largest distance a vertex can broadcast for cost at most t. This intuition about the adjoint suggests a similar but different way for broadcasts to be generalized in the same vein. We consider a function $k_*\colon \overline{\mathbb{N}} \to \overline{\mathbb{N}}$ which takes in a cost t and returns the greatest distance which costs at most t to broadcast. These functions should have left adjoints and so we define them as follows.

Definition 2.3 *A scaling function is a function $k_*\colon \overline{\mathbb{N}} \to \overline{\mathbb{N}}$ such that*

1. $k_*(0) = 0$,
2. k_* *is order preserving,*
3. $k_*(\infty) = \infty$.

From well-known properties of adjoints it can be shown that if k_* is a scaling function then its left adjoint is a cost function and vice versa.

From this perspective we can consider graphs equipped with scaling functions, which we write as (G, k_*) for a graph G and a scaling function k_*, as the objects of study. For graphs equipped with scaling functions, broadcasts are no longer the fundamental objects of interest. This is because broadcasts return the distance associated with each vertex v and a scaling function must take in a cost as input. Instead we consider functions $h\colon V(G) \to \overline{\mathbb{N}}$, which we call S-casts (short for scaled broadcasts). We regard functions which associate to each vertex a distance as broadcasts and we regard functions which associate to each vertex a cost as S-casts. Given a graph G equipped with a scaling function k_* and an S-cast h, we compose h with the scaling function k_* to get a broadcast on G. An S-cast h is dominating when $k_* \circ h$ is a dominating broadcast. The cost of an S-cast h is given by $c(h) = \sum_{v \in V(G)} h(v)$ and the associated domination number is given by $\gamma_s^{k_*}(G) = \min\{c(h) \mid h \text{ is a dominating S-cast on } (G, k_*)\}$. A dominating S-cast h is optimal if its cost is equal to the S-cast domination number on (G, k_*). An S-cast h is independent when $k_* \circ h$ is independent, and h is efficient when $k_* \circ h$ is efficient. This perspective is the reason for the modification of the definition of a broadcast, as it is possible for there to exist vertices v such that $(k_* \circ h)(v) > \operatorname{diam}(G)$. Finally we define $V_h^+ = V_{k_* \circ h}^+$ and $N_h[v] = N_{k_* \circ h}[v]$.

It is worth emphasising the duality between the two approaches outlined above. Firstly, to every cost function we can associate a scaling function given by its right adjoint. Similarly, we can associate to each scaling function a cost function given by its left adjoint. Consider a graph equipped with a cost function, (G, k^*). There is a unique scaling function k_* associated to this pair. To each broadcast f on (G, k^*) we can associate the S-cast $k^* \circ f$ which is used when computing the cost of f. To each S-cast h we can associate the broadcast $k_* \circ h$ which is used when determining which vertices h dominates. In each approach the broadcast tells us the distances a vertex broadcast and the S-cast tells us the cost to broadcast a given distance.

The following theorem further connects the two approaches.

Theorem 2.4 *If G is a graph and k^* a cost function, then $\gamma_c^{k^*}(G) = \gamma_s^{k_*}(G)$.*

Proof. Let f be an optimal dominating broadcast on (G, k^*). It follows from the definition of a right adjoint that $k_*(k^*(f(x)) \geq f(x)$. Thus $k^* \circ f$ is a dominating S-cast. Its cost is equal to the cost of f and so $\gamma_s^{k_*}(G) \leq \gamma_c^{k^*}(G)$.

Let h be an optimal dominating S-cast on (G, k^*). From the definition of dominating S-cast we get that $k_* \circ h$ is a dominating broadcast. The cost of $k_* \circ h$ is given by summing $k^* \circ k_* \circ h$ over all vertices. We already know from the adjunction that $(k^* \circ k_* \circ h)(x) \leq h(x)$. As we showed above, when you compose a dominating broadcast with the cost function you get a dominating S-cast and so $k^* \circ k_* \circ h$ is a dominating S-cast. Thus if it were the case that $(k^* \circ k_* \circ h)(x) < h(x)$ at any point x, then this would contradict our assumption that h is optimal. Hence $(k^* \circ k_* \circ h)(x) = h(x)$ and so the cost of $k_* \circ h$ is equal to the cost of h. Therefore $\gamma_c^{k^*}(G) \leq \gamma_s^{k_*}(G)$. □

As mentioned above, superadditivity and subadditivity are important properties in the study of cost and scaling functions. We now relate these two properties.

Proposition 2.5 *Let k_* be a function with a left adjoint k^*. Then k_* is superadditive if and only if its left adjoint k^* is subadditive.*

Proof. Take $x, y \in \overline{\mathbb{N}}$. From the definition of adjoints we have that $k_*(k^*(x + y)) \geq x + y$ and that $k^*(x + y)$ is the least element for which this holds. Then note that by the superadditivity of k_* we get $k_*(k^*(x) + k^*(y)) \geq k_*(k^*(x)) + k_*(k^*(y)) \geq x + y$. Thus combining these two results we conclude that $k^*(x + y) \leq k^*(x) + k^*(y)$.

The other direction follows similarly. □

Notice that this means that a function k^* with a right adjoint k_* is subadditive if and only if k_* is superadditive.

It is not the case that if k^* is superadditive then k_* is subadditive or vice versa. For instance consider 2.1. Note that $f^*(n) = 2n$ is both superadditive and subadditive. Further note that its left adjoint $f_!(n) = \lceil \frac{n}{2} \rceil$ is not superadditive and that its right adjoint $f_*(n) = \lfloor \frac{n}{2} \rfloor$ is not subadditive.

Below we prove that when k_* is superadditive (or equivalently when k^* is subadditive) then there always exists an efficient optimal dominating S-cast on (G, k_*).

Lemma 2.6 *Let (G, k_*) be a graph equipped with a superadditive scaling function. Then there exists an independent optimal dominating S-cast on (G, k_*).*

Proof. Let h be an optimal dominating S-cast which is not independent. There there exist vertices $v, w \in V^+_{k^* \circ h}$ such that $v \in N_{k^* \circ h}(w)$. Construct a new S-cast h' identical to h except that $h'(v) = 0, h'(w) = h(v) + h(w)$. Since k_* is superadditive $k_*(h'(w)) \geq k_*(h(v)) + k_*(h(w))$, thus $N_h(v) \cup N_h(w) \subseteq N_{h'}(w)$ and so h' is dominating. Furthermore $c(h') = c(h)$ and so h' is optimal. This process can be repeated until we have an independent dominating broadcast satisfying the conditions of the lemma. □

Lemma 2.7 *Let (G, k_*) be a graph equipped with a superadditive scaling function. Then there exists an efficient optimal dominating S-cast on (G, k_*).*

Proof. By Lemma 2.6 we have that there exists an optimal independent dominating S-cast h on (G, k_*). Assume h is not efficient. Then there are vertices $v, w \in V^+_h$ with $v \neq w$ and a vertex u such that $u \in N_h[v] \cap N_h[w]$. Since h is independent, $u \neq v, w$. It follows that there exists a path P from v to w passing through u with length less or equal to $k_*(h(v)) + k_*(h(w))$. Let x be a vertex in P distance $k_*(h(w))$ from v.

Consider a new S-cast h' identical to h except that $h'(v) = 0 = h'(w)$ and $h'(x) = h(v) + h(w)$. Note that $c(h') = c(h)$ and $|V^+_{h'}| < |V^+_h|$. Since $k_*(h(v) + h(w)) \geq k_*(h(v)) + k_*(h(w))$, it follows immediately that $N_h[v] \cup N_h[w] \subseteq N_{h'}[x]$ which implies that h' is dominating.

It is possible that h' is not efficient. Since V^+_h is finite and $|V^+_{h'}| < |V^+_h|$, we can repeat the procedure mentioned in this proof and the process will eventually terminate with an optimal efficient dominating broadcast. □

We can go further and show that if k_* is not superadditive then there exists a graph G such that (G, k_*) has no efficient optimal dominating broadcast.

Theorem 2.8 *Let k_* be a scaling function. The following are equivalent.*

(i) For every graph G, (G, k_*) has an efficient $\gamma_s^{k_*}(G)$ dominating S-cast.

(ii) k_* is superadditive.

Proof. We have shown that if k_* is superadditive then there is always an optimal efficient dominating S-cast.

Assume k_* is not superadditive. Since k_* is not superadditive there exist natural numbers m and n such that $k_*(m + n) < k_*(m) + k_*(n)$. Pick x to be the smallest m for which there is some n such that the above inequality holds. Then pick y to be the smallest number such that $k_*(x + y) < k_*(x) + k_*(y)$. Note that this means that $k_*(x - 1) < k_*(x)$, as if $k_*(x - 1) = k_*(x)$ then we would have $k_*((x - 1) + y) \leq$

$k_*(x+y) < k_*(x) + k_*(y) = k_*(x-1) + k_*(y)$ contradicting that fact that x was chosen to be minimal. A similar argument gives that $k_*(y-1) < k_*(y)$.

We construct a graph G in the following way. Glue $x+y+1$ copies of $P_{k_*(x)+1}$ together at an end vertex, i.e. pick an end vertex in each of the $x+y+1$ paths and identify them all. Call this graph G_x. Construct G_y in the same way except with $x+y+1$ copies of $P_{k_*(y)+1}$. Now construct G by gluing G_x and G_y together at an end vertex of each.

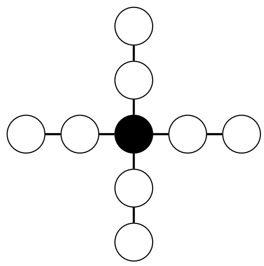

Figure 1: G_y when $y=2$ and $x=1$ and $k_*(y)=2$.

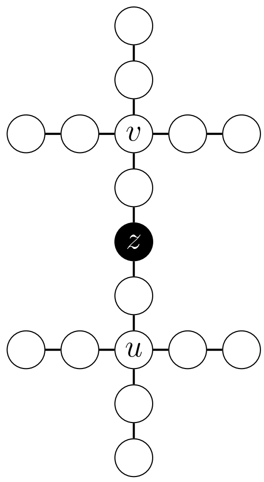

Figure 2: G when $x=1$ and $y=2$ and $k_*(x)=2=k_*(y)$.

Let v be the centre of G_x and u the centre of G_y and z the vertex that attaches G_x and G_y. We define an S-cast h by

$$h(s) = \begin{cases} 0 & \text{if } s \notin \{u,v\} \\ x & \text{if } s = v \\ y & \text{if } s = u. \end{cases}$$

It is clear that h is dominating. Furthermore $z \in N_h(v)$ and $z \in N_h(u)$, so h is not efficient. We now show that $\gamma_s^{k_*}(G) = x+y$ and then that h is the only optimal dominating broadcast.

Let h' be an optimal dominating S-cast. Then $c(h') \leq c(h) = x+y$. Let $u_1, \ldots, u_{x+y}$ refer to the end vertices closest to u and $v_1, \ldots, v_{x+y}$ the end vertices closest to v. Note that the diameter of G is $d_G(u_i, v_j) = 2k_*(x) + 2k_*(y)$ where $1 \leq i, j \leq x+y$. Consider a $u_i - v_j$ path P and a vertex w at distance $k_*(x) + k_*(y)$ from v_j. Then $d_G(w, u_i) = k_*(x) + k_*(y)$ since G is a tree. We can conclude that w is a central vertex and that $\mathrm{rad}(G) = k_*(x) + k_*(y)$. Furthermore the distance from w to any other leaf vertex is $k_*(x) + k_*(y)$ by symmetry.

Since $k_*(x+y) < k_*(x) + k_*(y)$ it is not possible for a single vertex to dominate a u_i and a v_j in particular this gives us that $|V_{h'}^+| \neq 1$. We now show that $|V_{h'}^+| = 2$. To do so we show that any vertex $p \in V_{h'}^+$ dominates either all of $v_1, \ldots, v_{x+y}$ or all of $u_1, \ldots, u_{x+y}$.

It is not possible for each of the end vertices $v_1, \ldots, v_{x+y}$ to be dominated by distinct vertices as there are $x+y$ of these end vertices and $c(h') \leq x+y$. This would leave no vertices to dominate the other set of end vertices without the cost of h' exceeding $x+y$. Now suppose that $s \in V_{h'}^+$, $v_i, v_j \in N_{h'}[s]$, that $t \in V_{h'}^+$ and that $u_{i'}, u_{j'} \in N_{h'}[t]$. Let $d_G(s, v) = l$. Then $(k_* \circ h')(s) \geq k_*(x) + l$ as either the $s - v_i$ path or the $s - v_j$ path passes through v. This implies that $\{v, v_1, \ldots, v_{x+y}\} \subseteq N_{h'}[s]$. A similar argument gives that $\{u, u_1, \ldots, u_{x+y}\} \subseteq N_{h'}[t]$. It follows that $V(G) \subseteq N_{h'}[s] \cup N_{h'}[t]$ and so $V_{h'}^+ = \{s, t\}$.

Assume $l = d_G(s, v) \neq 0$. Then $h'(s) > x$ as $(k_* \circ h')(s) > k_*(x)$. Thus since $c(h') = x+y$ we must have that $h'(t) < y$. Then $(k_* \circ h')(t) < k_*(y)$ (since $k_*(y-1) < k_*(y)$) which contradicts the fact that t dominates the leaves $u_1, \ldots, u_{x+y}$. Thus $l = 0$ and so $s = v$. A symmetric argument gives $t = u$. If $h'(v) < x$ then the leaves $v_1, \ldots, v_{x+y}$ are not dominated by v and similarly if $h'(u) < y$ then the leaves $u_1, \ldots, u_{x+y}$ are not dominated by u. Hence $h' = h$ and so the only optimal dominating S-cast on (G, k_*) is inefficient. □

3 Generalising graph product bounds

In this section we show that when k^* is a superadditive cost function, then the bounds found in [1] generalise for finite products of trees and when k^* is linear the bounds generalise for finite products of graphs. In addition we show that if k_* is a subadditive scaling function then the results generalise too for finite products of trees and further that if k_* is a linear scaling function the results generalise for finite products of graphs.

Definition 3.1 *Let G and H be graphs. The Cartesian product $G \,\square\, H$ has vertex set $V(G \,\square\, H) = V(G) \times V(H)$. The vertex (u, u') is adjacent to (v, v') if $u = v$ and u' is adjacent to v' in H or $u' = v'$ and u is adjacent to v in G.*

Definition 3.2 *Let G and H be graphs. The strong product $G \boxtimes H$ has vertex set $V(G \boxtimes H) = V(G) \times V(H)$. The vertex (u, u') is adjacent to (v, v') if $u = v$ and u' is adjacent to v' in H or $u' = v'$ and u is adjacent to v in G or finally if u is adjacent to v in G and u' is adjacent to v' in H.*

It is shown in [1] that given two graphs G and H

$$\gamma_b(G \,\square\, H) \leq \tfrac{3}{2}(\gamma_b(G) + \gamma_b(H)),$$

and that

$$\gamma_b(G \boxtimes H) \leq \tfrac{3}{2}\max\{\gamma_b(G), \gamma_b(H)\}.$$

The results all stem from a lemma in [1] showing that $\gamma_b(G) \geq \left\lceil \frac{2\mathrm{rad}(G)}{3} \right\rceil$. We prove a more general version of this lemma which we then apply to the above mentioned cases of cost and scaling functions.

Lemma 3.3 *Let G be a graph and k_* a superadditive scaling function. Then there exists a spanning tree T of G such that $\gamma_s^{k_*}(T) = \gamma_s^{k_*}(G)$.*

Proof. A spanning tree H of G can be thought of as G with potentially some edges deleted. With this in mind it is clear that if h is a dominating broadcast on (H, k_*) then it is also dominating on (G, k_*). Thus $\gamma_s^{k_*}(G) \leq \gamma_s^{k_*}(H)$ for any spanning tree H. That k_* is superadditive will allow us to find a spanning tree such that the above inequality holds in the other direction. By Lemma 2.7 there exists an efficient optimal dominating S-cast h on (G, k_*). Thus the neighbourhoods $N_h[v]$ for $v \in V_h^+$ are all pairwise disjoint. For each $N_h[v]$ consider the subgraph it generates and a spanning tree T_v such that $d_{T_v}(v, x) = d_G(v, x)$ for all $x \in N_h[v]$. To construct such a T_v we systematically consider geodesics from v to every vertex x and in each case select one which does not result in a cycle. The trees T_v can be connected with edges in such a way that the result is a spanning tree T of G. By the construction h is a dominating broadcast on (T, k_*) and so we get that $\gamma_s^{k_*}(T) = \gamma_s^{k_*}(G)$ as desired. □

Note that by Theorem 2.4 and Proposition 2.5 we get that the above lemma holds when k^* is a subadditive cost function.

Lemma 3.4 *Let (P_n, k_*) be such that there exists an optimal dominating S-cast h with $h(v) \leq k^*(1)$ for each $v \in V(P_n)$. Then*

$$\gamma_s^{k_*}(P_n) = \left\lceil \frac{n}{2k_*(k^*(1)) + 1} \right\rceil k^*(1).$$

Proof. By the adjunction we get that $k_*(x) = 0$ for all $x < k^*(1)$ which gives us that $h(v) = 0$ or $k^*(1)$, as h is optimal. Each broadcasting vertex dominates itself and $k_*(k^*(1))$ vertices on either side of it, which is $2k_*(k^*(1)) + 1$ vertices in total. Since h is optimal and on a path we can assume it is efficient. Thus there are $\lceil \frac{n}{2k_*(k^*(1))+1} \rceil$ broadcasting vertices and so we conclude that

$$c(h) = \left\lceil \frac{n}{2k_*(k^*(1)) + 1} \right\rceil k^*(1).$$

□

This result restricts to $\gamma_b(P_n) = \lceil \frac{n}{3} \rceil$ in the broadcast setting.

Lemma 3.5 *Let k^* and k_* be adjoint cost and scaling functions. Then if k_* is subadditive or k^* is superadditive, we have that for every path P_n there exists an optimal dominating broadcast h on (P_n, k_*) with $h(v) \leq k^*(1)$ for each $v \in V(P_n)$.*

Proof. First assume k_* is subadditive. We begin by showing that $k^*(1) = 1$. Assume to the contrary that $k^*(1) = t > 1$. Then $k_*(t) \geq 1 > 0 = k_*(t-1) + k_*(1)$ and so is not subadditive. Let h be a dominating S-cast on (P_n, k^*, k_*) and assume there exists a $v \in V(P_n)$ with $h(v) > 1$. The neighbourhood $N_h[v]$ contains $2k_*(h(v)) + 1$ vertices. Since k_* is subadditive we get $k_*(h(v)) \leq k_*(h(v) - 1) + k_*(1)$ which gives that $2k_*(h(v)-1)+2k_*(1)+2 > 2k_*(h(v))+1$. Thus there exists vertices $u, w \in N_h[v]$ such that it would be more effective to have an S-cast h' identical to h except that $h'(v) = 0$, $h'(u) = 1$ and $h'(w) = h(v) - 1$. Applying this argument repeatedly to an optimal dominating broadcast gives that an optimal dominating broadcast exists satisfying the conditions in Lemma 3.4.

Next assume k^* is superadditive. Let f be an optimal dominating broadcast on (P_n, k^*, k_*) with $f(v) > 1$ for some $v \in V(P_n)$. As before note that $N_f[v]$ contains $2f(v) + 1$ vertices. Thus there exists vertices $u, w \in N_h[v]$ such that a broadcast f' identical to f except that $f'(v) = 0$, $f'(u) = 1$ and $f'(w) = f(v) - 1$ with $N_f[v] \subseteq N_{f'}[u] \cup N_{f'}[w]$ exists. Also since k^* is superadditive we get that $k^*(f(v)) \geq k^*(f(1)) + k^*(f(v) - 1)$. Thus it is no more expensive to use the alternative f'. Applying this argument repeatedly gives us that there is an optimal dominating broadcast f with $f(v) \leq 1$ for each $v \in V(P_n)$. The associated S-cast $k^* \circ f$ satisfies $(k^* \circ g)(v) \leq k^*(1)$. Theorem 2.4 tells us that this S-cast is optimal and so we are done. □

Corollary 3.6 *Consider (P_n, k_*) where k_* is subadditive. Then*

$$\gamma_s^{k_*}(P_n) = \left\lceil \frac{n}{2k_*(1) + 1} \right\rceil.$$

Proof. As discussed in the previous proof $k^*(1) = 1$ and so the result follows. □

Corollary 3.7 *Consider (P_n, k_*) where k^* is superadditive. Then*

$$\gamma_s^{k_*}(P_n) = \left\lceil \frac{n}{3} \right\rceil k^*(1).$$

Proof. This follows from the fact that $k_*(k^*(1)) = 1$. Assume $k_*(k^*(1)) = t > 1$. Then $k^*(t) = k^*(1)$ which gives $k^*(t) < k^*(1) + k^*(t-1)$, violating the superadditivity of k^*. □

Lemma 3.8 *Let (T, k^*, k_*) be a tree equipped with a scaling function k_* such that there exists an optimal dominating S-cast h on (P_n, k_*) with $h(v) \leq k^*(1)$ for each $v \in V(P_n)$. Then*

$$\gamma_s^{k_*}(T) \geq \left\lceil \frac{2\,\mathrm{rad}(T)}{2k_*(k^*(1)) + 1} \right\rceil k^*(1).$$

Proof. Let T be a tree. We know that $\mathrm{diam}(T) \geq 2\,\mathrm{rad}(T) - 1$ and so there exists a path P in T containing $2\mathrm{rad}(T)$ vertices. Let f be an optimal dominating broadcast on T. For each vertex $x \in V(T)$ denote by x' the unique closest vertex to x on P. Consider the broadcast f' on P given by $f'(x) = \max\{f(y) \mid y' = x\}$. Because f is dominating it follows that f' is dominating on P. Furthermore applying Lemma 3.4 we get the following inequality

$$c(f) \geq c(f') \geq \gamma_s^{k_*}(P_{2\mathrm{rad}(T)}) = \left\lceil \frac{2\,\mathrm{rad}(T)}{2k_*(k^*(1)) + 1} \right\rceil k^*(1).$$

$\square$

Corollary 3.9 *Let (T, k_*) be a tree equipped with a subadditive scaling function k_*. Then*

$$\gamma_s^{k_*}(T) \geq \left\lceil \frac{2\,\mathrm{rad}(T)}{2k_*(1) + 1} \right\rceil.$$

Corollary 3.10 *Let (T, k_*) be a tree equipped with a superadditive cost function k^*. Then*

$$\gamma_s^{k_*}(T) \geq \left\lceil \frac{2\,\mathrm{rad}(T)}{3} \right\rceil k^*(1).$$

Corollary 3.11 *Let (G, k_*) be a graph equipped with a linear scaling function k_*. Then*

$$\gamma_s^{k_*}(G) \geq \left\lceil \frac{2\,\mathrm{rad}(G)}{2k_*(1) + 1} \right\rceil.$$

Proof. That k_* is linear means that it is superadditive and subadditive. Thus we can invoke Lemma 3.3 to find a spanning tree T of G with $\gamma_s^{k_*}(G) = \gamma_s^{k_*}(T) \geq \lceil \frac{2\mathrm{rad}(T)}{2k_*(1)+1} \rceil \geq \lceil \frac{2\,\mathrm{rad}(G)}{2k_*(1)+1} \rceil$. $\square$

Corollary 3.12 *Let (G, k_*) be a graph equipped with a linear cost function k^*. Then*

$$\gamma_s^{k_*}(G) \geq \left\lceil \frac{2\mathrm{rad}(G)}{3} \right\rceil k^*(1).$$

We can now apply these corollaries to the products below.

3.1 Cartesian product

Given two graphs G and H it is known that $\mathrm{rad}(G \square H) = \mathrm{rad}(G) + \mathrm{rad}(H)$. Furthermore it is clear that $\gamma_s^{k_*}(G) \leq k^*(\mathrm{rad}(G))$, as any S-cast h with $h(u) = k^*(\mathrm{rad}(G))$ for a central vertex u is dominating. Combining these facts we get the following results. Let $\square_{i=1}^{n} G_i = G_1 \square G_2 \square \cdots \square G_n$.

Proposition 3.13 *Let $G_1, G_2, \ldots, G_n$ be a finite family of graphs and k_* a scaling function. If k_* is linear, or if each G_i is a tree and k_* is subadditive we have the following bound*

$$\gamma_s^{k_*}\left(\mathop{\Box}_{i=1}^{n} G_i\right) \leq k^*\left(\frac{2k_*(1)+1}{2}\left(\sum_{i=1}^{n}\gamma_s^{k_*}(G_i)\right)\right).$$

Proof. The bound in Corollary 3.9 can be manipulated to give

$$\operatorname{rad}(G) \leq \frac{2k_*(1)+1}{2}\gamma_s^{k_*}(G).$$

The conditions on the graphs and scaling function allow us to apply Corollary 3.9 and Corollary 3.11 respectively to give

$$\begin{aligned}\gamma_s^{k_*}\left(\mathop{\Box}_{i=1}^{n} G_i\right) \leq k^*\left(\operatorname{rad}\left(\mathop{\Box}_{i=1}^{n} G_i\right)\right) &\leq k^*\left(\sum_{i=1}^{n}\operatorname{rad}(G_i)\right)\\ &\leq k^*\left(\frac{2k_*(1)+1}{2}(\sum_{i=1}^{n}\gamma_s^{k_*}(G_i))\right).\end{aligned}$$

□

Similarly we get the following proposition.

Proposition 3.14 *Let $G_1, G_2, \ldots, G_n$ be a finite family of graphs and k^* a cost function. If k^* is linear, or if each G_i is a tree and k^* is superadditive we have the following bound*

$$\gamma_s^{k_*}\left(\mathop{\Box}_{i=1}^{n} G_i\right) \leq k^*\left(\frac{3}{2k^*(1)}\left(\sum_{i=1}^{n}\gamma_s^{k_*}(G_i)\right)\right).$$

This is proved identically.

3.2 Strong product

Given two graphs G and H it is known that $\operatorname{rad}(G \boxtimes H) = \max\{\operatorname{rad}(G), \operatorname{rad}(H)\}$. Furthermore we define $\boxtimes_{i=1}^{n} G_i = G_1 \boxtimes G_2 \boxtimes \ldots \boxtimes G_n$. Applying essentially the same proof techniques as used in Proposition 3.13 we get the following two results.

Proposition 3.15 *Let $G_1, G_2, \ldots, G_n$ be a finite family of graphs and k_* a scaling function. If k_* is linear, or if each G_i is a tree and k_* is subadditive we have the following bound*

$$\gamma_s^{k_*}\left(\mathop{\boxtimes}_{i=1}^{n} G_i\right) \leq k^*\left(\frac{2k_*(1)+1}{2}\max\{\gamma_s^{k_*}(G_i) \mid 1 \leq i \leq n\}\right).$$

Proposition 3.16 *Let $G_1, G_2, \ldots, G_n$ be a finite family of graphs and k^* a cost function. If k^* is linear, or if each G_i is a tree and k^* is subadditive we have the following bound*

$$\gamma_s^{k_*}\left(\mathop{\boxtimes}_{i=1}^{n} G_i\right) \leq k^*\left(\frac{3}{2k^*(1)}\max\{\gamma_s^{k_*}(G_i) \mid 1 \leq i \leq n\}\right).$$

We omit bounds on the tensor product here but they can be found in [4] where we further develop the theory. In [1] the bounds were all shown to be sharp. A potential avenue for further research would be to determine whether there are classes of graphs (which would necessarily depend on the scaling function) which achieve these bounds.

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