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Types of directed triple systems

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Dedicated to the memory of Anne Penfold Street,
a dear friend and esteemed colleague.

Abstract

We introduce three types of directed triple systems. Two of these, Mendelsohn directed triple systems and Latin directed triple systems, have previously appeared in the literature but we prove further results about them. The third type, which we call skewed directed triple systems, is new and we determine the existence spectrum to be $v \equiv 1 \pmod 3$, $v \neq 7$, except possibly for $v = 22$, as well as giving enumeration results for small orders.

1 Introduction

In this paper we will be concerned with different types of directed triple systems. This concept and that of a Mendelsohn triple system are well known. Let V be a set of cardinality v and $\mathcal{B}$ a collection of ordered triples of distinct elements of

V. The pair $(V, \mathcal{B})$ is said to be a *directed triple system* DTS(v) or a *Mendelsohn triple system* MTS(v) if every ordered pair of distinct elements of V is contained in precisely one ordered triple; the two types of system being distinguished by the definition of containment. In a directed triple system containment is *transitive*; an ordered triple, denoted by $[x, y, z]$, contains the ordered pairs (x, y), (y, z) and (x, z). In a Mendelsohn triple system containment is *cyclic*; an ordered triple, denoted by (x, y, z) contains the ordered pairs (x, y), (y, z) and (z, x). A DTS(v) exists if and only if $v \equiv 0$ or 1 (mod 3) [14] and an MTS(v) exists if and only if $v \equiv 0$ or 1 (mod 3) $v \neq 6$, [19].

Let $D = (V, \mathcal{B})$ be a directed triple system of order v. Denote by $S_{a,b}$ the set of ordered pairs (x, y) in positions a and b respectively of the triples of $\mathcal{B}$. Then trivially the sets $S_{1,2}$, $S_{2,3}$ and $S_{1,3}$ are mutually disjoint and $S_{1,2} \cup S_{2,3} \cup S_{1,3} = \mathcal{U} = \{(x, y) : x \in V, y \in V, x \neq y\}$. Further, since reversing all the triples in a directed triple system gives a (not necessarily isomorphic) directed triple system, the sets $S_{2,1}$, $S_{3,2}$ and $S_{3,1}$ are also mutually disjoint and $S_{2,1} \cup S_{3,2} \cup S_{3,1} = \mathcal{U}$. The cardinalities of the sets $S_{a,b}$ are the same, namely $v(v-1)/3$ the number of triples in a DTS(v). The possibility exists therefore that the sets $S_{1,2}$, $S_{2,3}$ and $S_{1,3}$ may be equal to the sets $S_{2,1}$, $S_{3,2}$ and $S_{3,1}$ in some order. There appear to be six possibilities.

$$\begin{array}{ccccccccc}
S_{1,2} & = & S_{2,1} & S_{2,1} & S_{3,2} & S_{3,2} & S_{3,1} & S_{3,1} \\
S_{2,3} & = & S_{3,2} & S_{3,1} & S_{2,1} & S_{3,1} & S_{3,2} & S_{2,1} \\
S_{1,3} & = & S_{3,1} & S_{3,2} & S_{3,1} & S_{2,1} & S_{2,1} & S_{3,2} \\
 & & \#1 & \#2 & \#3 & \#4 & \#5 & \#6
\end{array}$$

However option #4 is not a viable possibility since $S_{1,2} = S_{3,2} \Rightarrow S_{2,3} = S_{2,1}$, and nor is option #6 since $S_{1,2} = S_{3,1} \Rightarrow S_{1,3} = S_{2,1}$. Now consider a directed triple system which satisfies the equations of option #2. If all the triples of the system are reversed then the equations of option #5 are satisfied. The converse is also true and thus these two options are effectively the same and we are left with just three possibilities, options #1, #2 and #3. The first and last of these have already appeared in the literature with the names *Mendelsohn directed triple systems*, MDTS(v) [12] and *Latin directed triple systems*, LDTS(v) [9, 6, 7, 8, 10, 15, 16] respectively. However directed triple systems satisfying option #2 appear to be new and it is the main purpose of this paper to discuss these systems. We will refer to them as *skewed directed triple systems*, SDTS(v). But before doing this we will briefly recap on some basic results about Mendelsohn and Latin directed triple systems, and in both cases prove new results about these systems too.

In particular, in Section 2 we extend known existence results on Mendelsohn directed triple systems by considering k-rotational systems. In this context we must have that $k \equiv 0$ (mod 3) and we prove that there exists a 3-rotational MDTS(v) for all $v \equiv 4$ or 10 (mod 18) except possibly for $v = 28$. Section 3 is concerned with Latin directed triple systems. Unlike the other two types of directed triple systems, LDTS(v) are not necessarily regular. In this section we deal with the situation where they are, a topic which has hitherto not been studied. We determine the existence spectrum with six possible exceptions. But the main results are in Section 4 which deals with the new skewed directed triple systems. We prove that these systems exist for all $v \equiv 1 \pmod 3$, $v \neq 7$, except possibly for $v = 22$ and also give enumeration results for small orders.

2 Mendelsohn directed triple systems

These systems were studied in [12] for general λ. They are so named because they are characterised by the property that any cyclic shift of all triples results again in a directed triple system, a property that they have in common with Mendelsohn triple systems. The systems are *regular*, i.e. if $c_i(x), i = 1, 2, 3$, denotes the number of times element $x \in V$ appears in position i of a triple of $\mathcal{B}$ then $c_1(x) = c_2(x) = c_3(x)$ for all $x \in V$. The latter concept was introduced by Colbourn and Colbourn [5] who proved that there exists a regular DTS(v) if and only if $v \equiv 1 \pmod 3$.

The construction of MDTS(v) for $v \equiv 1 \pmod 3$ is straightforward. A (v, K) *pairwise balanced design*, PBD(v, K), is a pair $(V, \mathcal{B})$ where V is a set of cardinality v and $\mathcal{B}$ is a collection of subsets of V, called *blocks*, with the property that the cardinality of every block is in the set K and every pair of elements of V is contained in precisely one block. Given a set K of positive integers the *PBD-closure* of K is the set $B(K) = \{v : \exists\ PBD(v, K)\}$. Since the PBD-closure of $\{4, 7\}$ is $\{v : v \equiv 1 \pmod 3,\ v \neq 10, 19\}$ [2] see also [1], the entire spectrum follows from the existence of MDTS(v) for $v = 4, 7, 10$ and 19. These are given below.

An MDTS(4) is given by the triples $[0, 2, 1]$, $[2, 0, 3]$, $[1, 3, 0]$, $[3, 1, 2]$. An MDTS(7) is given by the triples generated by $[0, 1, 3]$ and $[0, 6, 4]$ under the action of the mapping $i \mapsto i + 1 \pmod 7$ and an MDTS(19) by the triples generated by $[0, 1, 5]$, $[0, 18, 14]$, $[0, 2, 8]$, $[0, 17, 11]$, $[0, 3, 10]$ and $[0, 16, 9]$ under the action of the mapping $i \mapsto i + 1 \pmod{19}$. The triples, with brackets and commas removed for clarity, of an MDTS(10) are

021	054	087	347	593	836	274	952	628	203
506	809	158	671	914	385	763	439	130	460
790	269	482	725	196	841	517	312	645	978

We thus have

Theorem 2.1 (Grannell, Griggs & Quinn) *There exists an MDTS(v) if and only if* $v \equiv 1 \pmod 3$.

In [12] the same authors further consider MDTS(v) with a cyclic automorphism, i.e. an automorphism consisting of a v-cycle, and prove a further result.

Theorem 2.2 (Grannell, Griggs & Quinn) *There exists a cyclic MDTS(v) if and only if* $v \equiv 1 \pmod 6$.

Turning now to rotational automorphisms, recall that a triple system is k-rotational if it has an automorphism consisting of a single fixed point and k cycles all of the same length $(v-1)/k$. Easy arguments, see for example [3], give necessary conditions for a k-rotational DTS(v) to be either (i) $k \equiv 1$ or 2 (mod 3), $v \equiv 0$ (mod 3) and $v \equiv 1$ (mod k) or (ii) $k \equiv 0$ (mod 3) and $v \equiv 1$ (mod k). But from Theorem 2.1, for an MDTS(v) only the second option is possible. Now observe that if α is a 3-rotational automorphism, then α^m, where m divides $(v-1)/3$ is $3m$-rotational. Thus to show that the necessary condition for the existence of a k-rotational MDTS(v) is also sufficient, it is enough to construct 3-rotational MDTS(v) for $v \equiv 1$ (mod 3). We are not able to prove the existence of such systems for all values of v in this residue class but below we present a construction which in arithmetic set density terms deals with 1/3rd of the possible values.

Theorem 2.3 *There exists a 3-rotational MDTS(v) for all* $v \equiv 4$ *or* $10 \pmod{18}$*, except possibly for* $v = 28$*.*

Proof. Let $v = 18s+4$. Trivially the unique MDTS(4), given above, is 3-rotational. Now assume that $s \geq 1$. There exists a cyclic Steiner triple system of order u, STS(u), for all $u = 6s+1$ [20]. Suppose that the base set of such a system is $\mathbb{Z}_u$. The triples of the STS(u) can then be obtained from a set of starter blocks $\{0, a_i, b_i\},\ i = 1, 2, \dots, s$ (when $s = 0$, there are no such blocks) under the action of the mapping $i \mapsto i+1$ (mod u). Let $V = (\mathbb{Z}_u \times \mathbb{Z}_3) \cup \{\infty\}$.

Let $\mathbb{B}_u = \{[(0,j),(a_i,j),(b_i,j+1)], [(0,j),(b_i-a_i,j),(u-a_i,j+1)], [(0,j),(u-b_i,j),(u+a_i-b_i,j+1)], [(a_i,j),(0,j),(b_i,j+2)], [(b_i-a_i,j),(0,j),(u-a_i,j+2)], [(u-b_i,j),(0,j),(u+a_i-b_i,j+2)] :\ i = 1,2,\dots,s,\ j = 0,1,2\}$. Further let $\mathbb{B}_\infty = \{[\infty,(0,0),(0,1)], [(0,0),\infty,(0,2)], [(0,1),(0,2),\infty], [(0,2),(0,1),(0,0)]\}$.

Then the set of directed triples generated from the blocks of the set $\mathbb{B}_u \cup \mathbb{B}_\infty$, by the mapping $(i,j) \mapsto (i+1,j)$ (mod u), $\infty \mapsto \infty$ form a 3-rotational Mendelsohn directed triple system on the set V.

Let $v = 18s+10$. There exists a cyclic Steiner triple system of order u, STS(u), for all $u = 6s+3,\ s \neq 1$ [20]. As in the previous case, suppose that the base set of such a system is $\mathbb{Z}_u$. The triples of the STS(u) can then be obtained from a set of starter blocks $\{0, a_i, b_i\},\ i = 1, 2, \dots, s$ (when $s = 0$, there are no such blocks) and a further block $\{0, u/3, 2u/3\}$ under the action of the mapping $i \mapsto i+1$ (mod u). Let $V = (\mathbb{Z}_u \times \mathbb{Z}_3) \cup \{\infty\}$ and let $\mathbb{B}_u$ and $\mathbb{B}_\infty$ be as defined above. Further let $\mathbb{B}_{u/3} = \{[(0,j),(u/3,j),(2u/3,j+1)], [(u/3,j),(0,j),(2u/3,j+2)] :\ j = 0,1,2\}$. Then the set of directed triples generated from the blocks of the set $\mathbb{B}_u \cup \mathbb{B}_{u/3} \cup \mathbb{B}_\infty$, by the mapping $(i,j) \mapsto (i+1,j)$ (mod u), $\infty \mapsto \infty$ form a 3-rotational Mendelsohn directed triple system on the set V. □

3 Latin directed triple systems

These systems were introduced in [9]. Given a DTS(v), $(V, \mathcal{B})$, we can define an operation on the set V by the following rules. For $[x,y,z] \in \mathcal{B}$ let $x \cdot y = z$, $x \cdot z = y$ and $y \cdot z = x$. Further for all $x \in V$ let $x \cdot x = x$. However the algebraic structure thus obtained is not necessarily a quasigroup; if $[u,x,y]$ and $[y,v,x] \in \mathcal{B}$ then $u \cdot x =$

$v \cdot x = y$. The condition for $(V, \cdot)$ to be a quasigroup is $[x, y, z] \in \mathcal{B} \Rightarrow [w, y, x] \in \mathcal{B}$ for some $w \in V$. The system is then called a *Latin directed triple system*, LDTS(v), because the operation table forms a Latin square, and the quasigroup so formed is called a *DTS-quasigroup*.

Latin directed triple systems have a rich and interesting structure which is explored in further papers. One feature of particular interest is the following. Idempotent totally symmetric and idempotent semi-symmetric quasigroups obtained respectively from Steiner and Mendelsohn triple systems are in one-one correspondence with the systems from which they come. This is not the case for DTS-quasigroups and Latin directed triple systems. Non-isomorphic LDTS(v) can yield isomorphic, indeed actually identical, DTS-quasigroups [9, Example 2.4]. A directed triple system $(V, \mathcal{B})$ is said to be *pure* if $[x, y, z] \in \mathcal{B} \Rightarrow [z, y, x] \notin \mathcal{B}$. Pure LDTS($v$) give anti-commutative DTS-quasigroups and these are in one-one correspondence. At the other extreme commutative DTS-quasigroups correspond to idempotent totally symmetric quasigroups.

Another feature is concerned with the property of flexibility. Both idempotent totally symmetric quasigroups and idempotent semi-symmetric quasigroups satisfy the *flexible law*, i.e. $x \cdot (y \cdot x) = (x \cdot y) \cdot x$ for all $x, y \in V$. DTS-quasigroups need not. A necessary and sufficient condition for a DTS-quasigroup obtained from an LDTS(v), $(V, \mathcal{B})$, to be flexible is given in [9], $[x, y, z] \in \mathcal{B} \Rightarrow [x, z \cdot x, y \cdot x] \in \mathcal{B}$. The existence spectrum for non-flexible DTS-quasigroups is determined in [9] and for flexible DTS-quasigroups in [10].

Theorem 3.1 (Drápal, Kozlik & Griggs) *There exists a non-flexible LDTS(v) if and only if* $v \equiv 0$ *or* $1 \pmod 3$, $v \neq 3, 4, 6, 7, 10$.

Theorem 3.2 (Drápal, Kozlik & Griggs) *There exists a flexible LDTS(v) if and only if* $v \equiv 0$ *or* $1 \pmod 3$, $v \neq 4, 6, 10, 12$.

Unlike MDTS(v), LDTS(v) are not necessarily regular even when the necessary condition $v \equiv 1 \pmod 3$ is satisfied. A Latin directed triple system may be obtained by taking a Steiner triple system and replacing each triple $\{x, y, z\}$ by one of the pairs of directed triples $[x, y, z]$ and $[z, y, x]$, $[y, z, x]$ and $[x, z, y]$, or $[z, x, y]$ and $[y, x, z]$. Such systems are called *improper*. Cyclic Steiner triple systems of order v exist for all $v \equiv 1 \pmod 6$, $v \geq 7$ [20]. Hence it follows immediately that regular, but improper, LDTS(v) exist for the same values. However, in [16] a stronger result is proved.

Theorem 3.3 (Kozlik) *If* $v \equiv 1 \pmod 6$, $v \geq 13$, *then there exists a pure cyclic LDTS(v).*

This leaves the case $v \equiv 4 \pmod 6$ to be considered and we do this below in the proof of the next theorem, part of which uses a standard technique known as Wilson's fundamental construction. We assume that the reader is familiar with this construction but briefly the basic idea is as follows.

A *k-group divisible design*, k-GDD, is an ordered triple $(V,\mathcal{G},\mathcal{B})$ where V is a set of *points* of cardinality v, $\mathcal{G}$ is a partition of V into *groups* and $\mathcal{B}$ is a family of subsets of V, called *blocks*, each of cardinality k, such that every pair of distinct points is contained in either precisely one group or one block, but not both. If $v = a_1g_1 + a_2g_2 + \ldots + a_sg_s$ and if there are a_i groups of cardinality g_i, $i = 1, 2, \ldots, s$, then the k-GDD is said to be of *type* $g_1^{a_1}g_2^{a_2}\ldots g_s^{a_s}$. The construction proceeds as follows. Begin with a k-GDD of type g^u or g^um^1, usually called the *master GDD*. Each point is then assigned a weight, usually the same weight, say w. In effect, each point is replaced by w points. Each block of the master GDD is then replaced by a k-GDD of type w^k, called a *slave GDD*. We will only need to use the values $k = 3$ or $k = 4$ and $w = 2$ or $w = 3$, and instead of slave GDDs we will use partial regular Latin directed triple systems. To complete the construction we then "fill in" the groups of the expanded master GDD, sometimes adjoining an extra point, say ∞, to all of the groups. Thus we may need pure regular Latin directed triple systems of orders gw, mw, $gw + 1$ or $mw + 1$ as appropriate. For a more detailed explanation of this construction see, for example, the proof of Proposition 4.3 in [9].

Theorem 3.4 *There exists a regular LDTS(v) for all* $v \equiv 4 \pmod 6$, $v \geq 16$, *except possibly for* $v = 70, 82$ *and* 106.

Proof. We will use both 3-GDDs and 4-GDDs.

When working with the former, we will use the partial regular LDTS(9)whose blocks are

$[x, a, p], [p, a, x], [y, b, q], [q, b, y], [z, c, r], [r, c, z], [c, p, y], [y, p, c], [a, q, z],$
$[z, q, a], [b, r, x], [x, r, b], [q, x, c], [c, x, q], [r, y, a], [a, y, r], [p, z, b], [b, z, p],$

and the sets $\{a, b, c\}, \{p, q, r\}, \{x, y, z\}$ play the role of the groups.

When working with the latter, we will use the partial regular LDTS(12) whose blocks are

$[p, a, x], [s, a, p], [x, a, s], [q, b, y], [u, b, q], [y, b, u], [r, c, z], [t, c, r], [z, c, t], [c, p, u],$
$[u, p, y], [y, p, c], [a, q, t], [t, q, z], [z, q, a], [b, r, s], [s, r, x], [x, r, b], [c, s, y], [q, s, c],$
$[y, s, q], [b, t, x], [p, t, b], [x, t, p], [a, u, z], [r, u, a], [z, u, r], [c, x, q], [q, x, u], [u, x, c],$
$[a, y, r], [r, y, t], [t, y, a], [b, z, p], [p, z, s], [s, z, b],$

and the sets $\{a, b, c\}, \{p, q, r\}, \{s, t, u\}, \{x, y, z\}$ play the role of the groups.

Schema of the master GDDs and Latin directed triple systems needed to construct the regular LDTS(v) are given in the two tables below. In all cases we weight with 3 and adjoin the point ∞. Regular Latin directed triple systems for the orders needed come either from Theorem 3.3 or are given in the Appendix. Existence of the relevant group divisible designs can be verified by reference to [11].

Note that, since all the constituent parts used in Table 1 are pure, the systems constructed for the values $v \equiv 4$ or $10 \pmod{18}$ are also pure. Finally, pure regular LDTS(v) for the missing values $v = 46, 58, 64, 76$ and 112 in Table 1 and 88 in Table 2 are also given in the Appendix. □

Type of master 4-GDD	Orders of LDTS(v) needed	Residue classes covered modulo 36	Missing values
$4^{3s}7^1$, $s \geq 2$	13, 22	22	58
$4^{3s}13^1$, $s \geq 3$	13, 40	4	76, 112
6^s9^1, $s \geq 4$	19, 28	10, 28	46, 64, 82

Table 1: Schema for regular LDTS(v), $v \equiv 4$ or 10 (mod 18).

Type of master 3-GDD	Orders of LDTS(v) needed	Residue classes covered modulo 54	Missing values
$9^{2s}5^1$, $s \geq 2$	28, 16	16	70
$9^{2s}11^1$, $s \geq 2$	28, 34	34	88
$9^{2s}17^1$, $s \geq 2$	28, 52	52	106

Table 2: Schema for regular LDTS(v), $v \equiv 16$ (mod 18).

4 Skewed directed triple systems

First, observe that skewed directed triple systems, SDTS(v), are regular. Hence a necessary condition for their existence is $v \equiv 1 \pmod 3$. We show that with the exception of $v = 7$ and the possible exception of $v = 22$, this condition is also sufficient. But first it will be convenient to give enumeration results for small orders.

Proposition 4.1 *There is a unique SDTS(4).*

Proof. The triples are $[0, 1, 2]$, $[1, 0, 3]$, $[2, 3, 1]$, $[3, 2, 0]$. □

Proposition 4.2 *There is no SDTS(7).*

Proof. An SDTS(7) contains 14 directed triples and therefore, since it is regular, each point occurs precisely twice in each of the positions of the directed triples. Now suppose that any SDTS(7) contains a configuration consisting of a set of six directed triples which take the form

$$[x, y, \cdot],\ [y, x, \cdot],\ [y, z, \cdot],\ [z, y, \cdot],\ [z, x, \cdot],\ [x, z, \cdot].$$

Then to complete these triples we need an extra six distinct points, so nine points in total. Thus the above configuration is not possible. Moreover any SDTS(7) cannot contain a configuration consisting of a set of eight directed triples which cycle round as above, since if this were so then it must also contain the above configuration of six directed triples. Therefore on the base set $\mathbb{Z}_7$ and without loss of generality, any SDTS(7) must have the structure below.

Now consider the point 0. It cannot be placed in the final position of directed triples 0 to 3 or 10 to 13. So it must occur twice in this position in directed triples 4

Block number	Directed triple	Block number	Directed triple
0	$[0,1,\cdot]$	1	$[1,0,\cdot]$
2	$[1,2,\cdot]$	3	$[2,1,\cdot]$
4	$[2,3,\cdot]$	5	$[3,2,\cdot]$
6	$[3,4,\cdot]$	7	$[4,3,\cdot]$
8	$[4,5,\cdot]$	9	$[5,4,\cdot]$
10	$[5,6,\cdot]$	11	$[6,5,\cdot]$
12	$[6,0,\cdot]$	13	$[0,6,\cdot]$

to 9. But if we place 0 in triple 6 or 7 then it cannot be placed in any of the triples 4, 5, 8 or 9. Thus the point 0 must be placed once in directed triple 4 or 5 and once in directed triple 8 or 9. Using the same argument for each of the remaining points we see that point i, $0 \le i \le 6$, must be placed once in directed triples $2i+4$ or $2i+5$ and once in directed triples $2i+8$ or $2i+9$, arithmetic modulo 14, i.e. we have the following situation.

Point	Once in Block number	Once in Block number
0	4 or 5	8 or 9
1	6 or 7	10 or 11
2	8 or 9	12 or 13
3	10 or 11	0 or 1
4	12 or 13	2 or 3
5	0 or 1	4 or 5
6	2 or 3	6 or 7

So we have two choices for the block number 0, directed triples $[0,1,3]$ or $[0,1,5]$. Consider the former. Using the fact that $S_{1,3} = S_{3,2}$ and the information on the placement of the points given above, we have the sequence of implications
Block 0 is [0,1,3] $\Rightarrow$ Block 4 is [2,3,0] $\Rightarrow$ Block 12 is [6,0,2] $\Rightarrow$ Block 2 is [1,2,6] $\Rightarrow$ Block 10 is [5,6,1] i.e. the situation is as follows.

Block number	Directed triple	Block number	Directed triple
0	$[0,1,3]$	1	$[1,0,\cdot]$
2	$[1,2,6]$	3	$[2,1,\cdot]$
4	$[2,3,0]$	5	$[3,2,\cdot]$
6	$[3,4,\cdot]$	7	$[4,3,\cdot]$
8	$[4,5,\cdot]$	9	$[5,4,\cdot]$
10	$[5,6,1]$	11	$[6,5,\cdot]$
12	$[6,0,2]$	13	$[0,6,\cdot]$

But now since Block 10 is $[5,6,1]$, we must have that Block 3 is $[2,1,5]$ but this violates where point 5 can be placed and we have a contradiction.

Now consider the latter. Again using the fact that $S_{1,3} = S_{3,2}$ and the information on the placement of the points given above, we have the sequence of implications Block 0 is [0,1,5] ⇒ Block 8 is [4,5,0] ⇒ Block 12 is [6,0,4] ⇒ Block 6 is [3,4,6] ⇒ Block 10 is [5,6,3] ⇒ Block 4 is [2,3,5] i.e. the situation is as follows.

Block number	Directed triple	Block number	Directed triple
0	$[0, 1, 5]$	1	$[1, 0, \cdot]$
2	$[1, 2, \cdot]$	3	$[2, 1, \cdot]$
4	$[2, 3, 5]$	5	$[3, 2, \cdot]$
6	$[3, 4, 6]$	7	$[4, 3, \cdot]$
8	$[4, 5, 0]$	9	$[5, 4, \cdot]$
10	$[5, 6, 3]$	11	$[6, 5, \cdot]$
12	$[6, 0, 4]$	13	$[0, 6, \cdot]$

But now since Block 4 is $[2, 3, 5]$, we must have that Block 11 is $[6, 5, 2]$ but this again violates where point 2 can be placed and we have a contradiction. Thus there is no SDTS(7). □

Proposition 4.3 *There are precisely* 4 *pairwise non-isomorphic SDTS*(10)*s.*

Proof. This result was obtained by exhaustive computer search using the package Mace4 [18]. Two of the SDTS(10)s are 3-rotational. Let $V = (\mathbb{Z}_3 \times \mathbb{Z}_3) \cup \{\infty\}$. The systems are generated from the directed triples below by the mapping $(i, j) \mapsto (i + 1, j) \pmod 3$, $\infty \mapsto \infty$.

System #1.

$[\infty, (0, 0), (0, 1)]$, $[(1, 0), \infty, (0, 2)]$, $[(0, 2), (2, 1), \infty]$, $[(0, 0), (1, 0), (2, 1)]$, $[(0, 0), (2, 0), (0, 2)]$, $[(0, 1), (1, 1), (0, 2)]$, $[(0, 1), (2, 1), (0, 0)]$, $[(0, 2), (1, 2), (1, 1)]$, $[(0, 2), (2, 2), (2, 0)]$, $[(2, 1), (0, 2), (1, 0)]$.

System #2.

$[\infty, (0, 0), (0, 1)]$, $[(1, 0), \infty, (0, 2)]$, $[(0, 2), (2, 1), \infty]$, $[(1, 0), (0, 0), (2, 1)]$, $[(2, 0), (0, 0), (0, 2)]$, $[(0, 1), (1, 1), (0, 2)]$, $[(0, 1), (2, 1), (1, 0)]$, $[(0, 2), (1, 2), (1, 1)]$, $[(0, 2), (2, 2), (0, 0)]$, $[(2, 1), (0, 2), (2, 0)]$.

The other two systems are automorphism-free. The triples, with brackets and commas removed for clarity, are

System #3.

012	086	094	103	164	179	236	241	258	320
348	357	427	439	456	529	531	540	618	673
695	715	760	782	805	874	891	907	962	983

System #4.

012	076	098	103	174	196	236	241	287	320
348	359	425	437	460	531	562	580	649	657
683	705	718	793	829	854	861	904	915	972

□

The following enumeration results for cyclic skewed directed triple systems were also obtained using Mace4 [18].

Proposition 4.4 *There are precisely* 2 *pairwise non-isomorphic cyclic SDTS*(13)*s.*

Proof. Altogether, the computer found 24 realisations of cyclic SDTS(13). Testing for isomorphism using the Bays-Lambossy Theorem [17] (see also section 2.27 of [4]), gives the two pairwise non-isomorphic systems described below.

Let $V = \mathbb{Z}_{13}$. The systems are generated from the directed triples below by the action of the mapping $i \mapsto i + 1 \pmod{13}$.

System #1.
$[0, 3, 1]$, $[1, 9, 3]$, $[3, 0, 9]$, $[9, 1, 0]$.

System #2.
$[0, 3, 9]$, $[1, 9, 0]$, $[3, 0, 1]$, $[9, 1, 3]$. □

Proposition 4.5 *There are precisely* 7 *pairwise non-isomorphic cyclic SDTS*(16)*s.*

Proof. Altogether, the computer found 56 realisations of cyclic SDTS(16). Testing for multiplier isomorphisms reduces this to 7 systems listed below, each generated from the directed triples given by the action of the mapping $i \mapsto i + 1 \pmod{16}$.

System #1.
$[2, 0, 9]$, $[6, 0, 11]$, $[8, 0, 12]$, $[10, 0, 13]$, $[14, 0, 15]$.

System #2.
$[1, 0, 11]$, $[7, 0, 12]$, $[8, 0, 6]$, $[9, 0, 13]$, $[15, 0, 2]$.

System #3.
$[3, 0, 9]$, $[5, 0, 12]$, $[8, 0, 10]$, $[11, 0, 15]$, $[13, 0, 14]$.

System #4.
$[2, 0, 7]$, $[4, 0, 13]$, $[8, 0, 11]$, $[12, 0, 6]$, $[14, 0, 15]$.

System #5.
$[2, 0, 1]$, $[4, 0, 10]$, $[8, 0, 3]$, $[12, 0, 9]$, $[14, 0, 5]$.

System #6.
$[2, 0, 11]$, $[4, 0, 1]$, $[8, 0, 7]$, $[12, 0, 6]$, $[14, 0, 3]$.

System #7.
$[2, 0, 5]$, $[4, 0, 10]$, $[8, 0, 15]$, $[12, 0, 13]$, $[14, 0, 9]$.

In order to prove that the above seven systems are pairwise non-isomorphic we proceed as follows. By ignoring the order of the triples of an SDTS(v), we obtain a twofold triple system, TTS(v), $(V, \mathcal{B}')$. For each point $x \in V$, determine the neighbourhood $\mathcal{N}_x$, defined as the set of pairs $\{\{y, z\} : \{x, y, z\} \in \mathcal{B}'\}$. Clearly $\mathcal{N}_x$ consists of a union of cycles on the base set $V \setminus \{x\}$. In each of the above systems, because they are cyclic, the neighbourhoods about each point are isomorphic. For each point of the systems #1, #2 and #3 the neighbourhood consists of two 6-cycles and a 3-cycle. For each of the systems #4, #5, #6 and #7 the neighbourhood is a 15-cycle. This in itself proves that none of the systems #1, #2 or #3 can be isomorphic to any of the systems #4, #5, #6 or #7. It is now easy to check isomorphism between two systems. Because they are cyclic, if they are isomorphic then there exists an isomorphism which maps point 0 in one system to point 0 in the other system. Hence the neighbourhoods about point 0 must also map from one system to the other. This leaves a relatively small number of possibilities which it is easy to check by computer. We find that the above seven systems are pairwise non-isomorphic. □

Proposition 4.6 *There are precisely* 6 *pairwise non-isomorphic cyclic SDTS*(19)*s.*

Proof. Altogether, the computer found 84 realisations of cyclic SDTS(19). Testing for isomorphism using the Bays-Lambossy Theorem [17] (see also section 2.27 of [4]), gives the six pairwise non-isomorphic systems described below.

Let $V = \mathbb{Z}_{19}$. The systems are generated from the directed triples below by the action of the mapping $i \mapsto i + 1 \pmod{19}$.

System #1.
$[2, 0, 8]$, $[4, 0, 16]$, $[9, 0, 14]$, $[10, 0, 13]$, $[15, 0, 7]$, $[17, 0, 18]$.

System #2.
$[2, 0, 7]$, $[3, 0, 13]$, $[8, 0, 14]$, $[11, 0, 15]$, $[16, 0, 9]$, $[17, 0, 18]$.

System #3.
$[4, 0, 7]$, $[5, 0, 18]$, $[9, 0, 2]$, $[10, 0, 8]$, $[14, 0, 6]$, $[15, 0, 16]$.

System #4.
$[4, 0, 1]$, $[5, 0, 11]$, $[9, 0, 17]$, $[10, 0, 12]$, $[14, 0, 13]$, $[15, 0, 3]$.

System #5.
$[4, 0, 18]$, $[6, 0, 8]$, $[9, 0, 12]$, $[10, 0, 17]$, $[13, 0, 5]$, $[15, 0, 16]$.

System #6.
$[4, 0, 1]$, $[6, 0, 11]$, $[9, 0, 7]$, $[10, 0, 3]$, $[13, 0, 2]$, $[15, 0, 14]$.

Systems #1 to #4 each give rise to 18 realisations, while systems #5 and #6 each give rise to 6 realisations and are not only cyclic but 6-rotational with the mapping $i \mapsto 7i$ as an automorphism. □

We are now in a position to prove the main result of this section.

Theorem 4.7 *There exists an SDTS*(v) *for all* $v \equiv 1 \pmod 3$, $v \neq 7$, *except possibly* $v = 22$.

Proof. Let $v \equiv 1$ or $4 \pmod{12}$. There exists a Steiner system $S(2, 4, v)$ for all v in these residue classes [13]. Replace each block of the Steiner system by the SDTS(4) given in Proposition 4.1.

Let $v \equiv 7$ or 10 (mod 12). There exists a PBD($v, \{4, 10\}$) containing exactly one block of cardinality 10 for all v in these residue classes and $v \geq 31$ [21]. Replace each block of cardinality 4 of the pairwise balanced design by the SDTS(4) given in Proposition 4.1 and the block of cardinality 10 by one of the designs given in Proposition 4.3. Skewed directed triple systems of order 19 are given in Proposition 4.6. □

The existence of an SDTS(22) remains elusive. We have run lengthy computer searches assuming possible automorphisms but with no success. It would be good to find such a system to close the annoying gap in the above theorem.

Appendix

In the systems below for ease of reading, points (i, j) are denoted by i_j.

Pure regular LDTS(16).
$V = \mathbb{Z}_8 \times \mathbb{Z}_2$.
The triples are obtained from the following starter blocks under the action of the mappings $i_j \mapsto (i+1)_j$ and $i_j \mapsto i_{j+1}$.
$[2_0, 0_0, 6_1]$, $[6_1, 0_0, 3_1]$, $[3_1, 0_0, 7_1]$, $[7_1, 0_0, 7_0]$, $[7_0, 0_0, 2_0]$.

Pure regular LDTS(22).
$V = \mathbb{Z}_{11} \times \mathbb{Z}_2$.
The triples are obtained from the following starter blocks under the action of the mapping $i_j \mapsto (i+1)_j$.
$[1_0, 0_0, 5_0]$, $[5_0, 0_0, 10_1]$, $[10_1, 0_0, 6_1]$, $[6_1, 0_0, 7_1]$, $[7_1, 0_0, 0_1]$, $[0_1, 0_0, 3_0]$, $[3_0, 0_0, 1_0]$, $[2_0, 0_1, 9_0]$, $[9_0, 0_1, 6_1]$, $[6_1, 0_1, 2_0]$, $[8_0, 0_1, 10_0]$, $[10_0, 0_1, 3_1]$, $[3_1, 0_1, 2_1]$, $[2_1, 0_1, 8_0]$.

Pure regular LDTS(28).
$V = \mathbb{Z}_{14} \times \mathbb{Z}_2$.
The triples are obtained from the following starter blocks under the action of the mappings $i_j \mapsto (i+1)_j$ and $i_j \mapsto i_{j+1}$.
$[1_0, 0_0, 5_0]$, $[5_0, 0_0, 12_1]$, $[12_1, 0_0, 4_1]$, $[4_1, 0_0, 6_1]$, $[6_1, 0_0, 13_1]$, $[13_1, 0_0, 9_1]$, $[9_1, 0_0, 3_1]$, $[3_1, 0_0, 3_0]$, $[3_0, 0_0, 1_0]$.

Pure regular LDTS(34).
$V = \mathbb{Z}_{17} \times \mathbb{Z}_2$.
The triples are obtained from the following starter blocks under the action of the mapping $i_j \mapsto (i+1)_j$.
$[1_0, 0_0, 5_0]$, $[5_0, 0_0, 7_0]$, $[7_0, 0_0, 3_0]$, $[3_0, 0_0, 1_0]$, $[6_0, 0_0, 1_1]$, $[1_1, 0_0, 8_0]$, $[8_0, 0_0, 2_1]$, $[2_1, 0_0, 7_1]$, $[7_1, 0_0, 5_1]$, $[5_1, 0_0, 0_1]$, $[0_1, 0_0, 6_0]$, $[3_0, 1_1, 7_1]$, $[7_1, 1_1, 4_0]$, $[4_0, 1_1, 14_1]$, $[14_1, 1_1, 10_0]$, $[10_0, 1_1, 9_1]$, $[9_1, 1_1, 3_0]$, $[5_0, 1_1, 8_1]$, $[8_1, 1_1, 9_0]$, $[9_0, 1_1, 15_1]$, $[15_1, 1_1, 0_1]$, $[0_1, 1_1, 5_0]$.

Pure regular LDTS(40).

$V = \mathbb{Z}_{20} \times \mathbb{Z}_2$.

The triples are obtained from the following starter blocks under the action of the mappings $i_j \mapsto (i+1)_j$ and $i_j \mapsto i_{j+1}$.

$[1_0, 0_0, 5_0]$, $[5_0, 0_0, 1_1]$, $[1_1, 0_0, 11_0]$, $[11_0, 0_0, 18_1]$, $[18_1, 0_0, 8_1]$, $[8_1, 0_0, 12_0]$, $[12_0, 0_0, 3_1]$, $[3_1, 0_0, 5_1]$, $[5_1, 0_0, 14_0]$, $[14_0, 0_0, 14_1]$, $[14_1, 0_0, 7_0]$, $[7_0, 0_0, 3_0]$, $[3_0, 0_0, 1_0]$.

Pure regular LDTS(46).

$V = \mathbb{Z}_{23} \times \mathbb{Z}_2$.

The triples are obtained from the following starter blocks under the action of the mapping $i_j \mapsto (i+1)_j$.

$[1_0, 0_0, 5_0]$, $[5_0, 0_0, 7_0]$, $[7_0, 0_0, 3_0]$, $[3_0, 0_0, 1_0]$, $[6_0, 0_0, 3_1]$, $[3_1, 0_0, 12_1]$, $[12_1, 0_0, 19_1]$, $[19_1, 0_0, 15_1]$, $[15_1, 0_0, 15_0]$, $[15_0, 0_0, 6_0]$, $[14_0, 2_0, 16_1]$, $[16_1, 2_0, 15_0]$, $[15_0, 2_0, 15_1]$, $[15_1, 2_0, 19_1]$, $[19_1, 2_0, 14_0]$, $[6_0, 1_1, 13_1]$, $[13_1, 1_1, 4_1]$, $[4_1, 1_1, 16_0]$, $[16_0, 1_1, 22_1]$, $[22_1, 1_1, 15_0]$, $[15_0, 1_1, 16_1]$, $[16_1, 1_1, 14_0]$, $[14_0, 1_1, 19_1]$, $[19_1, 1_1, 20_0]$, $[20_0, 1_1, 6_0]$, $[6_0, 4_1, 17_1]$, $[17_1, 4_1, 11_0]$, $[11_0, 4_1, 10_1]$, $[10_1, 4_1, 3_1]$, $[3_1, 4_1, 6_0]$.

Pure regular LDTS(52).

$V = \mathbb{Z}_{26} \times \mathbb{Z}_2$.

The triples are obtained from the following starter blocks under the action of the mappings $i_j \mapsto (i+1)_j$ and $i_j \mapsto i_{j+1}$.

$[4_0, 0_0, 19_1]$, $[19_1, 0_0, 6_1]$, $[6_1, 0_0, 12_1]$, $[12_1, 0_0, 16_0]$, $[16_0, 0_0, 25_1]$, $[25_1, 0_0, 25_0]$, $[25_0, 0_0, 23_0]$, $[23_0, 0_0, 4_0]$, $[10_0, 2_0, 23_1]$, $[23_1, 2_0, 19_0]$, $[19_0, 2_0, 10_1]$, $[10_1, 2_0, 12_1]$, $[12_1, 2_0, 14_0]$, $[14_0, 2_0, 25_1]$, $[25_1, 2_0, 23_0]$, $[23_0, 2_0, 17_0]$, $[17_0, 2_0, 10_0]$.

Pure regular LDTS(58).

$V = \mathbb{Z}_{29} \times \mathbb{Z}_2$.

The triples are obtained from the following starter blocks under the action of the mapping $i_j \mapsto (i+1)_j$.

$[1_0, 0_0, 5_0]$, $[5_0, 0_0, 15_0]$, $[15_0, 0_0, 7_0]$, $[7_0, 0_0, 3_0]$, $[3_0, 0_0, 1_0]$, $[1_0, 24_0, 26_1]$, $[26_1, 24_0, 10_1]$, $[10_1, 24_0, 9_1]$, $[9_1, 24_0, 13_1]$, $[13_1, 24_0, 0_1]$, $[0_1, 24_0, 12_1]$, $[12_1, 24_0, 11_0]$, $[11_0, 24_0, 13_0]$, $[13_0, 24_0, 1_0]$, $[6_1, 28_0, 27_1]$, $[27_1, 28_0, 21_1]$, $[21_1, 28_0, 9_1]$, $[9_1, 28_0, 10_1]$, $[10_1, 28_0, 6_1]$, $[0_0, 0_1, 9_0]$, $[9_0, 0_1, 7_1]$, $[7_1, 0_1, 23_0]$, $[23_0, 0_1, 10_1]$, $[10_1, 0_1, 18_1]$, $[18_1, 0_1, 21_0]$, $[21_0, 0_1, 5_1]$, $[5_1, 0_1, 25_0]$, $[25_0, 0_1, 8_0]$, $[8_0, 0_1, 9_1]$, $[9_1, 0_1, 15_1]$, $[15_1, 0_1, 17_0]$, $[17_0, 0_1, 26_1]$, $[26_1, 0_1, 10_0]$, $[10_0, 0_1, 0_0]$, $[6_0, 1_1, 3_1]$, $[3_1, 1_1, 7_0]$, $[7_0, 1_1, 27_0]$, $[27_0, 1_1, 6_0]$.

Pure regular LDTS(64).

$V = \mathbb{Z}_{32} \times \mathbb{Z}_2$.

The triples are obtained from the following starter blocks under the action of the mappings $i_j \mapsto (i+1)_j$ and $i_j \mapsto i_{j+1}$.

$[8_0, 0_0, 25_1]$, $[25_1, 0_0, 6_1]$, $[6_1, 0_0, 22_0]$, $[22_0, 0_0, 9_0]$, $[9_0, 0_0, 5_1]$, $[5_1, 0_0, 20_0]$, $[20_0, 0_0, 31_1]$, $[31_1, 0_0, 31_0]$, $[31_0, 0_0, 29_0]$, $[29_0, 0_0, 25_0]$, $[25_0, 0_0, 8_0]$, $[3_0, 8_0, 29_0]$, $[29_0, 8_0, 18_1]$, $[18_1, 8_0, 20_1]$, $[20_1, 8_0, 22_0]$, $[22_0, 8_0, 26_1]$, $[26_1, 8_0, 0_1]$, $[0_1, 8_0, 17_1]$, $[17_1, 8_0, 21_1]$, $[21_1, 8_0, 5_1]$, $[5_1, 8_0, 3_0]$.

Pure regular LDTS(76).

$V = \mathbb{Z}_{38} \times \mathbb{Z}_2$.

The triples are obtained from the following starter blocks under the action of the mappings $i_j \mapsto (i+1)_j$ and $i_j \mapsto i_{j+1}$.

$[6_0, 0_0, 16_1]$, $[16_1, 0_0, 34_1]$, $[34_1, 0_0, 29_0]$, $[29_0, 0_0, 10_0]$, $[10_0, 0_0, 18_1]$, $[18_1, 0_0, 24_0]$, $[24_0, 0_0, 37_1]$, $[37_1, 0_0, 37_0]$, $[37_0, 0_0, 35_0]$, $[35_0, 0_0, 31_0]$, $[31_0, 0_0, 23_0]$, $[23_0, 0_0, 12_0]$, $[12_0, 0_0, 31_1]$, $[31_1, 0_0, 25_0]$, $[25_0, 0_0, 12_1]$, $[12_1, 0_0, 14_1]$, $[14_1, 0_0, 16_0]$, $[16_0, 0_0, 21_1]$, $[21_1, 0_0, 29_1]$, $[29_1, 0_0, 21_0]$, $[21_0, 0_0, 11_1]$, $[11_1, 0_0, 15_1]$, $[15_1, 0_0, 35_1]$, $[35_1, 0_0, 33_0]$, $[33_0, 0_0, 6_0]$.

Pure regular LDTS(88).

$V = \mathbb{Z}_{44} \times \mathbb{Z}_2$.

The triples are obtained from the following starter blocks under the action of the mappings $i_j \mapsto (i+1)_j$ and $i_j \mapsto i_{j+1}$.

$[6_0, 0_0, 18_1]$, $[18_1, 0_0, 40_1]$, $[40_1, 0_0, 5_1]$, $[5_1, 0_0, 25_1]$, $[25_1, 0_0, 33_1]$, $[33_1, 0_0, 23_0]$, $[23_0, 0_0, 12_0]$, $[12_0, 0_0, 35_1]$, $[35_1, 0_0, 8_1]$, $[8_1, 0_0, 30_0]$, $[30_0, 0_0, 13_0]$, $[13_0, 0_0, 7_1]$, $[7_1, 0_0, 28_0]$, $[28_0, 0_0, 43_1]$, $[43_1, 0_0, 43_0]$, $[43_0, 0_0, 41_0]$, $[41_0, 0_0, 37_0]$, $[37_0, 0_0, 29_0]$, $[29_0, 0_0, 14_1]$, $[14_1, 0_0, 16_1]$, $[16_1, 0_0, 18_0]$, $[18_0, 0_0, 24_1]$, $[24_1, 0_0, 34_0]$, $[34_0, 0_0, 25_0]$, $[25_0, 0_0, 13_1]$, $[13_1, 0_0, 17_1]$, $[17_1, 0_0, 41_1]$, $[41_1, 0_0, 39_0]$, $[39_0, 0_0, 6_0]$.

Pure regular LDTS(112).

$V = \mathbb{Z}_{56} \times \mathbb{Z}_2$.

The triples are obtained from the following starter blocks under the action of the mappings $i_j \mapsto (i+1)_j$ and $i_j \mapsto i_{j+1}$.

$[5_0, 0_0, 3_1]$, $[3_1, 0_0, 35_1]$, $[35_1, 0_0, 39_1]$, $[39_1, 0_0, 16_1]$, $[16_1, 0_0, 22_1]$, $[22_1, 0_0, 26_0]$, $[26_0, 0_0, 32_1]$, $[32_1, 0_0, 42_1]$, $[42_1, 0_0, 15_1]$, $[15_1, 0_0, 23_1]$, $[23_1, 0_0, 51_1]$, $[51_1, 0_0, 47_0]$, $[47_0, 0_0, 37_0]$, $[37_0, 0_0, 18_1]$, $[18_1, 0_0, 20_1]$, $[20_1, 0_0, 22_0]$, $[22_0, 0_0, 48_1]$, $[48_1, 0_0, 39_0]$, $[39_0, 0_0, 18_0]$, $[18_0, 0_0, 46_1]$, $[46_1, 0_0, 11_1]$, $[11_1, 0_0, 27_1]$, $[27_1, 0_0, 36_0]$, $[36_0, 0_0, 55_1]$, $[55_1, 0_0, 55_0]$, $[55_0, 0_0, 53_0]$, $[53_0, 0_0, 49_0]$, $[49_0, 0_0, 41_0]$, $[41_0, 0_0, 25_0]$, $[25_0, 0_0, 12_1]$, $[12_1, 0_0, 42_0]$, $[42_0, 0_0, 13_0]$, $[13_0, 0_0, 7_1]$, $[7_1, 0_0, 31_1]$, $[31_1, 0_0, 44_0]$, $[44_0, 0_0, 11_0]$, $[11_0, 0_0, 5_0]$.

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