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On the packing numbers in graphs

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In memory of Anne Penfold Street

Abstract

In this paper, we find upper bounds on the open packing and k-limited packing numbers with emphasis on the cases $k = 1$ and $k = 2$. We solve the problem of characterizing all connected graphs on n vertices with $\rho_o(G) = n/\delta(G)$ which was raised in 2015 by Hamid and Saravanakumar. Also, by establishing a relationship between the k-limited packing number and double domination number we improve two upper bounds given by Chellali and Haynes in 2005.

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1 Introduction

Throughout this paper, let G be a finite graph with vertex set $V = V(G)$, edge set $E = E(G)$, minimum degree $\delta = \delta(G)$ and maximum degree $\Delta = \Delta(G)$. We use [10] for any terminology and notation not defined here. For any vertex $v \in V(G)$, $N(v) = \{u \in V(G) \mid uv \in E(G)\}$ denotes the *open neighborhood* of v of G, and $N[v] = N(v) \cup \{v\}$ denotes its *closed neighborhood.*

A subset $B \subseteq V(G)$ is a *packing* (an *open packing*) in G if for every two distinct vertices $u, v \in B$, $N[u] \cap N[v] = \emptyset$ $(N(u) \cap N(v) = \emptyset)$. The *packing number* $\rho(G)$ (*open packing number* $\rho_o(G)$) is the maximum cardinality of a packing (an open packing) in G. These concepts have been studied in [7, 8], and elsewhere.

In [5], Harary and Haynes introduced the concept of tuple domination numbers. Let $1 \leq k \leq \delta(G) + 1$. A set $D \subseteq V(G)$ is a *k-tuple dominating set* in G if $|N[v] \cap D| \geq k$, for all $v \in V(G)$. The *k-tuple domination number*, denoted $\gamma_{\times k}(G)$, is the smallest number of vertices in a k-tuple dominating set. In fact, the authors showed that every graph G with $\delta \geq k - 1$ has a k-tuple dominating set and hence a k-tuple domination number. When $k = 2$, $\gamma_{\times 2}(G)$ is called the *double domination number* of G. For the special case $k = 1$, $\gamma_{\times 1}(G) = \gamma(G)$ is the well-known domination number (see [6]). The concept of tuple domination has been studied by several authors including [3, 9]. In general, the reader can find comprehensive information on various domination parameters in [1] and [6].

Gallant et al. [3] introduced the concept of k-limited packing in graphs and exhibited some real-world applications of it to network security, market saturation and codes. A set of vertices $B \subseteq V$ is called a *k-limited packing set* in G if $|N[v] \cap B| \leq k$ for all $v \in V$, where $k \geq 1$. The *k-limited packing number*, $L_k(G)$, is the largest number of vertices in a k-limited packing set. When $k = 1$, we have $L_1(G) = \rho(G)$.

In this paper, we find upper bounds on the k-limited packing numbers. In Section 2, we prove that $2(n-\ell+s\delta^*)/(1+\delta^*)$ is a sharp upper bound on $L_2(G)$ for a connected graph G on $n \geq 3$ vertices, where ℓ, s and $\delta^* = \delta^*(G)$ are the number of end-vertices, the number of support vertices and $\min\{\deg(v) \mid v \text{ is not an end-vertex}\}$, respectively. Also, we give an upper bound on $L_k(G)$ (with characterization of all graphs attaining it) in terms of the order, size and k. In Section 3, we exhibit a solution to the problem of characterizing all connected graphs of order $n \geq 2$ with $\rho_o(G) = n/\delta(G)$ posed in [4]. Moreover, we prove that $\gamma_{\times 2}(G) + \rho(G) \leq n - \delta(G) + 2$ when $\delta(G) \geq 2$. This improves two results in [2] given by Chellali and Haynes, simultaneously.

2 Main results

The 2-limited packing number of G has been bounded from above by $2n/(\delta(G) + 1)$ (see [9]). We present the following upper bound which works better for all graphs with end-vertices, especially trees. First, we recall that a support vertex is called a *weak support vertex* if it is adjacent to just one end-vertex.

Theorem 2.1. *Let G be a connected graph of order $n \geq 3$ with s support vertices and ℓ end-vertices. Then,*

$$L_2(G) \leq \frac{2(n - \ell + s\delta^*(G))}{1 + \delta^*(G)}$$

and this bound is sharp. Here $\delta^*(G)$ is the minimum degree taken over all vertices which are not end-vertices.

Proof. Let $\{u_1, \ldots, u_{s_1}\}$ be the set of weak support vertices in G. Let G' be the graph of order n' formed from G by adding new vertices $v_1, \ldots, v_{s_1}$ and edges $u_1v_1, \ldots, u_{s_1}v_{s_1}$ to G (we note that $G = G'$ if G has no weak support vertex). Clearly

$$s' = s, n' = n + s_1 \text{ and } \ell' = \ell + s_1 \tag{1}$$

in which s' and ℓ' are the number of support vertives and end-vertices of G', respectively. Let $v \in V(G')$ be a vertex of degree $\deg_{G'}(v) = \delta^*(G') \geq 2$. Since G is connected and $n \geq 3$, $\deg_G(v) \geq 2$. Therefore,

$$\delta^*(G') = \deg_{G'}(v) \geq \deg_G(v) \geq \delta^*(G). \tag{2}$$

Let B' be a maximum 2-limited packing in G'. First we show that $|N[u] \cap B'| = 2$ for each support vertex u. Suppose to the contrary that there exists a support vertex u in G' for which $|N[u] \cap B'| \leq 1$. Thus, there exists an end-vertex $v \notin B'$ adjacent to u. It is easy to see that $B' \cup \{v\}$ is a 2-limited packing in G' which contradicts the maximality of B'. So, we may always assume that B' contains two end-vertices at each support vertex. This implies that all support vertices and the other $\ell_u - 2$ end-vertices for each support vertex u belong to $V(G') \setminus B'$, in which ℓ_u is the number of end-vertices adjacent to u. Moreover, these end-vertices have no neighbors in B'. Therefore,

$$|[B', V(G') \setminus B']| \leq 2(n' - |B'| - \ell' + 2s'). \tag{3}$$

On the other hand, each end-vertex in B' has exactly one neighbor in $V(G') \setminus B'$ and each of the other vertices in B' has at least $\delta^*(G') - 1$ neighbors in $V(G') \setminus B'$. Therefore,

$$(|B'| - 2s')(\delta^*(G') - 1) + 2s' \leq |[B', V(G') \setminus B']|. \tag{4}$$

Together inequalities (3) and (4) imply that

$$|B'| \leq \frac{2(n' - \ell' + s'\delta^*(G'))}{1 + \delta^*(G')}. \tag{5}$$

We now let B be a maximum 2-limited packing in G. Clearly, B is a 2-limited packing in G', as well. Thus, $|B| \leq |B'|$. By (1) and (5) we have

$$L_2(G) = |B| \leq |B'| \leq \frac{2(n - \ell + s\delta^*(G'))}{1 + \delta^*(G')}. \tag{6}$$

On the other hand, $f(x) = \frac{2(n-\ell+sx)}{1+x}$ is a decreasing function. So,

$$L_2(G) \leq \frac{2(n-\ell+s\delta^*(G'))}{1+\delta^*(G')} = f(\delta^*(G')) \leq f(\delta^*(G)) = \frac{2(n-\ell+s\delta^*(G))}{1+\delta^*(G)},$$

by (2) and (6).

To show that the upper bound is sharp, let $C_k = v_1v_2\cdots v_kv_1$be a cycle of length $k \geq 3$. Now let H be the graph consisting of C_k such that each v_i is adjacent to $p \geq 2$ end-vertices. Then $L_2(H) = 2k$, $n(H) = k(p+1)$, $\ell(H) = kp$, $s(H) = k$ and $\delta^*(H) = p+2$. This implies that

$$\frac{2(n(H)-\ell(H)+s(H)\delta^*(H))}{1+\delta^*(H)} = 2k.$$

This completes the proof. □

It is easy to see that $L_k(G) = n$ if and only if $k \geq \Delta(G)+1$. So, in what follows we may always assume that $k \leq \Delta(G)$ when we deal with $L_k(G)$. In Theorem 2.2 below, we provide an upper bound on $L_k(G)$ of a graph G in terms of its order, size and k. Also, we bound $\rho_o(G)$ from above just in terms of the order and size. First, we define Ω and Σ to be the families of all graphs G having the following properties, respectively.

(p_1) There exists a clique S such that $G[V(G)\setminus S]$ is $(k-1)$-regular and every vertex in S has exactly k neighbors in $V(G)\setminus S$.

(p_2) There exists a clique S such that $G[V(G)\setminus S]$ is a disjoint union of copies of K_2 and every vertex in S has exactly one neighbor in $V(G)\setminus S$.

Theorem 2.2. *Let G be a graph of order n and size m. If $k \leq 2(n-\sqrt{n^2-n-2m})$ or $\delta(G) \geq k-1$, then*

$$L_k(G) \leq n + k/2 - \sqrt{k^2/4+(1-k)n+2m}$$

with equality if and only if $G \in \Omega$.

Furthermore, $\rho_o(G) \leq n-\sqrt{2m-n}$ for any graph G with no isolated vertex. The bound holds with equality if and only if $G \in \Sigma$.

Proof. Let L be a maximum k-limited packing set in G and let $E(G[L])$ and $E(G[V\setminus L])$ be the edge set of the subgraphs of G induced by L and $V\setminus L$, respectively. Clearly,

$$m = |E(G[L])| + |[L, V(G)\setminus L]| + |E(G[V\setminus L])|. \tag{7}$$

Therefore,

$$2m \leq (k-1)|L| + 2k(n-|L|) + (n-|L|)(n-|L|-1). \tag{8}$$

Solving the above inequality for $|L|$ we obtain

$$L_k(G) = |L| \leq \frac{2n + k - \sqrt{k^2 + 4(1-k)n + 8m}}{2},$$

as desired (note that $k \leq 2(n - \sqrt{n^2 - n - 2m})$ or $\delta(G) \geq k - 1$ implies that $k^2/4 + (1-k)n + 2m \geq 0$).

We now suppose that the equality in the upper bound holds. Therefore

$$2|E(G[L])| = (k-1)|L|, \;\; |[L, V(G) \setminus L]| = k(n - |L|)$$

and $2|E(G[V(G) \setminus L])| = (n - |L|)(n - |L| - 1)$, by (8). This shows that $V(G) \setminus L$ is a clique satisfying the property (p_1). Thus, $G \in \Omega$. Conversely, suppose that $G \in \Omega$. Let S be a clique of the minimum size among all cliques having the property (p_1). Then, it is easy to see that $L = V(G) \setminus S$ is a k-limited packing for which the upper bound holds with equality.

The proof of the second result is similar to the proof of the first one when $k = 1$. □

3 The special case $k = 1$

Hamid and Saravanakumar [4] proved that

$$\rho_o(G) \leq \frac{n}{\delta(G)} \tag{9}$$

for any connected graph G of order $n \geq 2$. Moreover, the authors characterized all the regular graphs which attain the above bound. In general, they posed the problem of characterizing all connected graphs of order $n \geq 2$ with equality in (9). We solve this problem in this section. For this purpose, we define the family Γ containing all graphs G constructed as follows. Let H be the disjoint union of $t \geq 1$ copies of K_2. Join every vertex u of H to k new vertices as its private neighbors lying outside $V(H)$. Let $V = V(H) \cup (\cup_{u \in V(H)} pn(u))$, in which $pn(u)$ is the set of neighbors (private neighbors) of u which lies outside $V(H)$. Add new edges among the vertices in $\cup_{u \in V(H)} pn(u)$ to construct a connected graph G on the set of vertices in $V = V(G)$ with $\deg(v) \geq k + 1$, for all $v \in \cup_{u \in V(H)} pn(u)$. Clearly, every vertex in $V(H)$ has the minimum degree $\delta(G) = k + 1$ and every vertex in $\cup_{u \in V(H)} pn(u)$ has exactly one neighbor in $V(H)$.

We are now in a position to present the following theorem.

Theorem 3.1. *Let G be a connected graph of order $n \geq 2$. Then, $\rho_o(G) = \frac{n}{\delta(G)}$ if and only if $G \in \Gamma$.*

Proof. We first state a proof for (9). Let B be a maximum open packing in G. Every vertex in $V(G)$ has at most one neighbor in B and hence every vertex in B has at least $\delta(G) - 1$ neighbors in $V(G) \setminus B$, by the definition of an open packing. Thus,

$$(\delta(G) - 1)|B| \leq |[B, V(G) \setminus B]| \leq n - |B|. \tag{10}$$

Therefore, $\rho_o(G) = |B| \leq \frac{n}{\delta(G)}$.

Suppose now that the equality in (9) holds. Then both the inequalities in (10) hold with equality, necessarily. Since every vertex in B has at least $\delta(G)-1$ neighbors in $V(G)\setminus B$ and $(\delta(G)-1)|B| = |[B, V(G)\setminus B]|$, every vertex in B has exactly $\delta(G)-1$ neighbors in $V(G) \setminus B$ and one neighbor in $V(H) = B$, necessarily. Therefore, H is a disjoint union of $t = |B|/2$ copies of K_2 and each vertex in B has the minimum degree $\delta(G)$. Moreover, $|[B, V(G)\setminus B]| = n - |B|$ shows that every vertex in $V(G)\setminus B$ has exactly one neighbor in B. So, each vertex in B has $\delta(G) - 1$ private neighbors lying outside B. This implies that, $G \in \Gamma$.

Now let $G \in \Gamma$. Then $B = V(H)$ is an open packing in G, for which the inequalities in (10) hold with equality, by the construction of G. So, $\rho_o(G) \geq |B| = \frac{n}{\delta(G)}$. This completes the proof. □

Remark 3.2. Similar to the proof of Theorem 3.1 we have $\rho(G) \leq n/(\delta(G)+1)$, for each connected graph G of order n. Furthermore, the characterization of graphs G attaining this bound can be obtained in a similar fashion by making some changes in Γ. It is sufficient to consider H as a subgraph of G with no edges in which every vertex has exactly $\delta(G)$ private neighbors lying outside $V(H)$.

In [2], Chellali and Haynes proved that for any graph G of order n with $\delta(G) \geq 2$,

$$\gamma_{\times 2}(G) + \rho(G) \leq n.$$

Also, they proved that

$$\gamma_{\times 2}(G) \leq n - \delta(G) + 1$$

for any graph G with no isolated vertices.

We note that the second upper bound is trivial for $\delta(G) = 1$. So, we may assume that $\delta(G) \geq 2$. In the following theorem, using the concepts of double domination and k-limited packing, we improve these two upper bounds, simultaneously.

Theorem 3.3. *Let G be a graph of order n. If $\delta(G) \geq 2$, then*

$$\gamma_{\times 2}(G) + \rho(G) \leq n - \delta(G) + 2.$$

Furthermore, this bound is sharp.

Proof. Let B be a maximum $(\delta(G) - 1)$-limited packing set in G. Every vertex in B has at most $\delta(G) - 2$ neighbors in B. Therefore it has at least two neighbors in $V(G) \setminus B$. On the other hand, every vertex in $V(G) \setminus B$ has at most $\delta(G) - 1$ neighbors in B, hence it has at least one neighbor in $V(G) \setminus B$. This implies that $V(G) \setminus B$ is a double dominating set in G. Therefore,

$$\gamma_{\times 2}(G) + L_{\delta(G)-1}(G) \leq n. \tag{11}$$

Now let $1 \leq k \leq \Delta(G)$ and let B be a maximum k-limited packing set in G. Then $|N[v] \cap B| \leq k$, for all $v \in V(G)$. We claim that $B \neq V(G)$. If $B = V(G)$ and $u \in V(G)$ such that $\deg(u) = \Delta(G)$, then $\Delta(G) + 1 = |N[u] \cap B| \leq k \leq \Delta(G)$, a contradiction. Now let $u \in V(G) \backslash B$. It is easy to check that $|N[v] \cap (B \cup \{u\})| \leq k+1$, for all $v \in V(G)$. Therefore $B \cup \{u\}$ is a $(k+1)$-limited packing set in G. Hence

$$L_{k+1}(G) \geq |B \cup \{u\}| = |B| + 1 = L_k(G) + 1,$$

for $k = 1, \ldots, \Delta(G)$. Applying this inequality repeatedly leads to

$$L_{\delta-1}(G) \geq L_1(G) + \delta(G) - 2 = \rho(G) + \delta(G) - 2.$$

Hence, $\gamma_{\times 2}(G) + \rho(G) \leq n - \delta(G) + 2$ by (11). Finally, the upper bound is sharp for the complete graph K_n with $n \geq 3$ □

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