Sesqui-arrays, a generalisation of triple arrays

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Abstract

A triple array is a rectangular array containing letters, each letter occurring equally often with no repeats in rows or columns, such that the number of letters common to two rows, two columns, or a row and a column are (possibly different) non-zero constants. Deleting the condition on the letters common to a row and a column gives a double array. We propose the term *sesqui-array* for such an array when only the condition on pairs of columns is deleted. Thus all triple arrays are sesqui-arrays.

In this paper we give three constructions for sesqui-arrays. The first gives $(n+1) \times n^2$ arrays on $n(n+1)$ letters for $n \geq 2$. (Such an array for $n = 2$ was found by Bagchi.) This construction uses Latin squares. The second uses the *Sylvester graph*, a subgraph of the Hoffman–Singleton graph, to build a good block design for 36 treatments in 42 blocks of size 6, and then uses this in a 7×36 sesqui-array for 42 letters.

We also give a construction for $K \times (K-1)(K-2)/2$ sesqui-arrays on $K(K-1)/2$ letters. This construction uses biplanes. It starts with a block of a biplane and produces an array which satisfies the requirements for a sesqui-array except possibly that of having no repeated letters in a row or column. We show that this condition holds if and only if the *Hussain chains* for the selected block contain no 4-cycles. A sufficient condition for the construction to give a triple array is that each Hussain chain is a union of 3-cycles; but this condition is not necessary, and we give a few further examples.

We also discuss the question of which of these arrays provide good designs for experiments.

Dedication

This paper is dedicated to the memory of Anne Penfold Street. Throughout her career, her research focussed on various combinatorial designs. However, she also linked these ideas to experimental design, as the book [40] shows. Indeed, some of her combinatorial research on neighbour designs [22, 28, 39, 42] was inspired by problems in the design of agricultural experiments. We hope that this paper also manages to bridge the two areas.

The second author had the privilege of being taught by Anne at the University of Queensland in the 1960s. In those far-off days, there was no such subject as combinatorics or design theory; Anne's lectures were on measure theory. The first author remembers with gratitude Anne's hospitality (both mathematical and social) during various research visits to the University of Queensland.

1 Introduction

1.1 Definitions and Notation

Suppose that we have an $r \times c$ array Δ in which each of the rc cells contains one letter from a set of size v. In order to exclude Latin squares and Youden squares, we assume that $v > \max\{r, c\}$. Figures 1–3 show three such arrays.

A	F	C	D	H	J
B	A	I	J	E	H
C	H	G	B	I	D
D	G	A	I	F	E
E	B	J	F	C	G

Figure 1: A triple array with $r = 5$, $c = 6$ and $v = 10$

A	B	C	D
F	A	B	E
C	D	E	F

Figure 2: A double array with $r = 3$, $c = 4$ and $v = 6$

A	H	B	G	C	F
B	G	F	C	E	D
C	F	E	D	A	H
D	E	A	H	G	B

Figure 3: A sesqui-array with $r = 4$, $c = 6$ and $v = 8$

Such an array gives rise to six incidence matrices. The incidence matrix N_{LR} is the $v \times r$ matrix whose (i,j)-entry is the number of times that letter i occurs in row j; while $N_{RL} = N_{LR}^\top$. The matrices N_{LC} and N_{CL} are defined similarly. Likewise, N_{RC} is the $r \times c$ matrix whose (i,j)-entry is the number of letters in the unique cell where row i meets column j: the assumption at the start of this section is that every entry in N_{RC} is equal to 1. Also $N_{CR} = N_{RC}^\top$.

The array Δ also defines various component designs. In the row component design $\Delta_{R(L)}$ we consider rows as points and letters as blocks. Thus $\Delta_{R(L)}$ is balanced if $N_{RL}N_{LR}$ is completely symmetric, which means that it is a linear combination of the identity matrix and the all-1 matrix. The column component design $\Delta_{C(L)}$ is analogous. The duals of these designs are $\Delta_{L(R)}$ and $\Delta_{L(C)}$. The component design $\Delta_{L(R,C)}$ is really what we have described, by considering which letter occurs in which cell. Its dual is $\Delta_{R,C(L)}$, which is concerned with which row-column combinations are allocated to each letter. We shall return to these last two components in Section 5.

Here are some conditions that the array Δ may satisfy, with names often used in the statistics literature.

(A0) No letter occurs more than once in any row or any column. (The component designs $\Delta_{R(L)}$ and $\Delta_{C(L)}$ are both *binary*; equivalently, all the entries in N_{LR} and N_{LC} are in $\{0,1\}$.)

(A1) Each letter occurs a constant number k of times, where $vk = rc$. (The array is *equireplicate*.)

(A2) The number of letters common to any two rows is a non-zero constant λ_{rr}. (The component design $\Delta_{R(L)}$ is *balanced*.)

(A3) The number of letters common to any two columns is a non-zero constant λ_{cc}. (The component design $\Delta_{C(L)}$ is *balanced*.)

(A4) The number of letters common to any row and any column is a constant λ_{rc}. (In the context that each cell contains exactly one letter, this means that rows and columns have *adjusted orthogonality* with respect to letters.)

Condition (A0) implies that the diagonal entries of $N_{RL}N_{LR}$ are all equal to c; and those of $N_{CL}N_{LC}$ are all equal to r. Conditions (A0) and (A1) imply that the diagonal entries of $N_{LR}N_{RL}$ and $N_{LC}N_{CL}$ are all equal to k.

Condition (A3) states that the off-diagonal entries of $N_{CL}N_{LC}$ are all equal. In general, we shall call these entries the *column-intersection numbers*. In the context of the component design $\Delta_{C(L)}$, they are usually called *concurrences*.

Condition (A4) states that every entry of $N_{RL}N_{LC}$ is equal to λ_{rc}.

In the literature of the 21st century, an array which has $v > \max\{r, c\}$ and which satisfies all five conditions (A0)–(A4) is called a *triple array*, while one satisfying conditions (A0)–(A3) is called a *double array*. We propose the term *sesqui-array* for an array satisfying (A0)–(A2) and (A4): the prefix "sesqui" means "one-and-a-half", and, of the last three conditions, we require adjusted orthogonality and half

of the two balance conditions required for a triple array. For these three types of array we use the notation $\text{TA}(v, k, \lambda_{rr}, \lambda_{cc}, \lambda_{rc} : r \times c)$, $\text{DA}(v, k, \lambda_{rr}, \lambda_{cc} : r \times c)$ and $\text{SA}(v, k, \lambda_{rr}, \Gamma, \lambda_{rc} : r \times c)$ respectively, where Γ denotes the set of intersection numbers for pairs of distinct columns. The array in Figure 1, given in [31], is a $\text{TA}(10, 3, 3, 2, 3 : 5 \times 6)$; the array in Figure 2 is a $\text{DA}(6, 2, 2, 1 : 3 \times 4)$; while that in Figure 3 is a $\text{SA}(8, 3, 4, \{0, 2\}, 3 : 4 \times 6)$.

1.2   An important inequality

If $\Delta_{R(L)}$ is balanced then N_{RL} has rank r. This gives the shortest proof of Fisher's Inequality, that balance implies $r \leq v$. Adjusted orthogonality also leads to some useful inequalities, as we now show.

Theorem 1.1 *If there is exactly one letter in each cell of Δ, and Δ has adjusted orthogonality, then* $\text{rank}(N_{RL}) + \text{rank}(N_{LC}) \leq v + 1$.

PROOF: Condition (A4) implies that $N_{RL}N_{LC}$ has rank 1. The row-space of N_{RL} has dimension $\text{rank}(N_{RL})$, and this is mapped by N_{LC} onto a space of dimension at most 1. Hence $v - \text{rank}(N_{LC}) \geq \text{rank}(N_{RL}) - 1$.   □

Corollary 1.2 *If Δ is a sesqui-array then* $v \geq r + \text{rank}(N_{LC}) - 1$.

Corollary 1.3 *If Δ is a triple array then*

$$v \geq r + c - 1. \tag{1.1}$$

Bagchi proved Theorem 1.1 and Corollary 1.3 in [3]. Corollary 1.3 was also proved in [10, 24].

The triple array in Figure 1 satisfies inequality (1.1). The double array in Figure 2 does not satisfy condition (A4) but it does satisfy inequality (1.1), as do all known double arrays. The sesqui-array in Figure 3 satisfies the inequality in Corollary 1.2 but not (1.1) because $v - r + 1 = 5 > 4 = \text{rank}(N_{LC})$ but $c = 6$.

There are various statistical optimality criteria for designs, which we discuss in Section 5. If $r < v$ and the component design $\Delta_{R(L)}$ is good by these criteria, then it is often true that N_{RL} has rank r. Likewise, if $c < v$ and $\Delta_{C(L)}$ has good statistical properties then usually N_{LC} has rank c. Thus arrays which satisfy conditions (A0), (A1) and (A4) and which are useful in experimental design often satisfy inequality (1.1) even if they do not satisfy both of (A2) and (A3). Most of the sesqui-arrays in this paper satisfy inequality (1.1).

However, Corollary 1.2 does exclude some block designs from being the column component of a sesqui-array.

Example 1 Consider a block design for six points in eight blocks of size three. The average concurrence is 8/5. Up to isomorphism, the only block design with all concurrences in $\{1, 2\}$ is the one made by developing the blocks $\{1, 2, 5\}$ and $\{1, 3, 5\}$ modulo 6. The incidence matrix of this block design has rank 6, and so Corollary 1.2 shows that it is impossible to have a 4×6 sesqui-array with this as its column component.

1.3   The construction problem

There are already some sesqui-arrays in the literature, but without this name. Bagchi [2] constructed an infinite family of $(n + 1) \times 2n$ sesqui-arrays with $n^2 + n$ letters, partly by using mutually orthogonal Latin squares. The set of column-intersection numbers is $\{0, 1, 2\}$ and component $\Delta_{C(L)}$ is partially balanced with respect to the rectangular association scheme R$(2, n)$. (We refer the reader to [40, Chapter 11] and [7] for information about association schemes and partially balanced designs.) Figure 4 gives an example with $n = 4$. Bagchi and van Berkum [4] gave an infinite family of arrays which, when transposed, are sesqui-arrays with $r = s$, $c = st$ and $v = s^2$, where s is a prime power and t is the size of a difference set in GF(s). The component $\Delta_{L(C)}$ is a square lattice design (see Section 3.1). Some of these were also given by Eccleston and Street in [17].

A	F	G	H	E	P	R	K
I	B	K	L	S	J	H	M
M	N	C	P	L	Q	O	F
Q	R	S	D	N	G	I	T
E	J	O	T	A	B	C	D

Figure 4: A sesqui-array with $r = 5$, $c = 8$ and $v = 20$

These papers all used one or more direct constructions, in the sense of finding a rule specifying the letter allocated to the cell in row i and column j and then proving that the rule produces arrays with the properties desired.

Nilson and Öhman [27] constructed double arrays from projective planes; Nilson and Cameron [26] constructed them from difference sets in finite groups.

Some triple arrays have also been given with a direct construction. Preece used cyclic constructions in [32, 33, 34]; Seberry [36], Street [41], Bagchi [3] and Preece, Wallis and Yucas [35] used finite fields; Bailey and Heidtmann [10] used properties of the groups A_5 and S_6; Yucas [44] used projective geometry; Nilson and Cameron [26] used difference sets with multiplier -1 in finite Abelian groups.

A different approach, for both sesqui- and triple arrays, is to specify the component designs $\Delta_{L(R)}$ and $\Delta_{L(C)}$ in such a way that the desired conditions are satisfied. Thus $\Delta_{L(R)}$ is specified by the set $R(i)$ of letters in row i, for $i = 1, \ldots, r$, and $\Delta_{L(C)}$ is specified by the set $C(j)$ of letters in column j, for $j = 1, \ldots, c$. Condition (A4)

states that $|R(i) \cap C(j)| = \lambda_{rc}$ for all i and j. Given these sets, can we put one letter of $R(i) \cap C(j)$ into cell (i, j), for all i and j, in such a way that the set of letters in row i is $R(i)$ and the set of letters in column j is $C(j)$?

Agrawal took this approach in [1] for the extremal case that $v = r + c - 1$. He observed:

Proposition 1.4 *The existence of an $r \times c$ triple array with v letters, where $v = r + c - 1$, implies the existence of a symmetric balanced incomplete-block design for $v + 1$ points in blocks of size r.*

(A proof was subsequently given in [24].) Agrawal used this to give canonical sets $R(i)$ and $C(j)$ from the same set of v letters. For all the cases that he tried with $k > 2$, he was able to choose a representative element of $R(i) \cap C(j)$ to go in cell (i, j) in such a way that the representatives in each row were all distinct, as were the representatives in each column, so that the outcome was a triple array. He conjectured that this is always possible if $k > 2$. So far, this conjecture has been neither proved nor refuted.

There are two further issues with Agrawal's approach. The first is that, unless $v = r + c - 1$, there may not be a canonical way of labelling the blocks of $\Delta_{R(L)}$ and $\Delta_{C(L)}$ by the same set of letters. Thus it may be possible to permute the names of the letters in one component while still satisfying (A4). Is it possible that an acceptable set of distinct representatives can be chosen for one labelling but not the other?

The second issue is that the general problem of finding an array of distinct representatives, given the sets $R(i)$ and $C(j)$, has been shown to be NP-complete by Fon-Der-Flaass in [18].

A third approach is exhaustive computer search. Some of the arrays given in [23, 24, 38] were found like this.

In the next three sections of this paper we give some direct constructions of sesqui-arrays, some of which turn out to be triple arrays. On the way, we find an incomplete-block design which seems to be a good practical substitute for the non-existent affine plane of order 6. The following section assesses whether these arrays have statistical properties desirable for experimental designs, while the final section poses some suggestions for further work.

2 Sesqui-arrays from Latin squares

Here we give a method of constructing a sesqui-array for $n(n+1)$ letters in a rectangle with $n + 1$ rows and n^2 columns. It works for every integer n with $n \geq 2$. The array has $v = n^2 + n = c + r - 1$ and so it satisfies inequality (1.1). As usual, we interpret "letter" to mean any symbol.

The method has three ingredients. One is a Latin square Φ_1 of order n on a set Λ_1 of letters. The second is an $n \times n$ array Φ_2 with n^2 distinct letters allocated to its cells; these letters make a set Λ_2 disjoint from Λ_1. The third is a Latin square Φ_3 of order $n + 1$ with letters 1, 2, ..., n and ∞.

Here is the algorithm for constructing an $(n+1) \times n^2$ array Δ whose set of letters is $\Lambda_1 \cup \Lambda_2$.

1. Start with the array Φ_3.

2. Remove the column in which ∞ occurs in the last row.

3. For $i = 1, \ldots, n$, replace the occurrence of ∞ in row i by the i-th row of Φ_1.

4. For $i = 1, \ldots, n$, replace every occurrence of the symbol i by the i-th row of Φ_2.

Now, the final row of Δ contains every letter of Λ_2, while row i replaces those in row j of Φ_2 by Λ_1, where j is the symbol in Φ_3 which is in row i of the column which has been removed. Hence every pair of rows have $n(n-1)$ letters in common.

Moreover, each column of Δ contains an entire column of Φ_2 and one letter of Λ_1. Thus it has n letters in common with every row.

Finally, consider any column of Δ. It has n letters of Λ_2 in common with every other column derived from the same column of Φ_2. It has one letter of Λ_1 in common with each of $n-1$ other columns, and no letters in common with any other column.

Hence Δ is a SA $(n(n+1), n, n(n-1), \{0, 1, n\}, n : (n+1) \times n^2)$.

In fact, the columns of Δ can be labelled by ordered pairs whose first elements are columns of Φ_2 and whose second elements are letters in Λ_1. This labelling gives another $n \times n$ array, which gives the rectangular association scheme $R(n, n)$ on the set of columns.

We have therefore proved the following.

Theorem 2.1 *If n is an integer with $n \geq 2$ then there is an $(n+1) \times n^2$ sesqui-array with $n(n+1)$ letters whose column component design is partially balanced with respect to the rectangular association scheme $R(n, n)$. Each column has intersection number 0 with $(n-1)^2$ other columns, 1 with $n-1$ other columns, and n with $n-1$ other columns.*

Example 2 When $n = 2$ we may take

$$\Phi_1 = \begin{array}{|c|c|} \hline A & B \\ \hline B & A \\ \hline \end{array}, \quad \Phi_2 = \begin{array}{|c|c|} \hline C & D \\ \hline E & F \\ \hline \end{array} \quad \text{and} \quad \Phi_3 = \begin{array}{|c|c|c|} \hline 1 & 2 & \infty \\ \hline 2 & \infty & 1 \\ \hline \infty & 1 & 2 \\ \hline \end{array}.$$

Thus $\Lambda_1 = \{A, B\}$ and $\Lambda_2 = \{C, D, E, F\}$. Removing the first column of Φ_3, replacing each ∞ by the appropriate row of Φ_1, replacing each 1 by (C, D) and each 2

E	F	A	B
B	A	C	D
C	D	E	F

Figure 5: A sesqui-array with $r = 3$, $c = 4$ and $v = 6$: double vertical lines indicate the method of construction

by (E, F) gives the sesqui-array in Figure 5. Up to relabelling of the letters, this is identical to an example given by Bagchi in [2].

Example 3 For $n = 4$, put

$$\Phi_1 = \begin{array}{|c|c|c|c|} \hline A & B & C & D \\ \hline D & A & B & C \\ \hline C & D & A & B \\ \hline B & C & D & A \\ \hline \end{array}, \quad \Phi_2 = \begin{array}{|c|c|c|c|} \hline E & F & G & H \\ \hline I & J & K & L \\ \hline M & N & O & P \\ \hline Q & R & S & T \\ \hline \end{array} \quad \text{and}$$

$$\Phi_3 = \begin{array}{|c|c|c|c|c|} \hline \infty & 1 & 2 & 3 & 4 \\ \hline 4 & \infty & 1 & 2 & 3 \\ \hline 3 & 4 & \infty & 1 & 2 \\ \hline 2 & 3 & 4 & \infty & 1 \\ \hline 1 & 2 & 3 & 4 & \infty \\ \hline \end{array}.$$

Removing the last column of Φ_3 gives the sesqui-array in Figure 6.

A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P
Q	R	S	T	D	A	B	C	E	F	G	H	I	J	K	L
M	N	O	P	Q	R	S	T	C	D	A	B	E	F	G	H
I	J	K	L	M	N	O	P	Q	R	S	T	B	C	D	A
E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T

Figure 6: A sesqui-array with $r = 5$, $c = 16$ and $v = 20$: double vertical lines indicate the method of construction

This construction has the advantage of being very flexible, because it can be used for all integers n greater than 1 and because there are no constraints on the isotopism classes of the Latin squares Φ_1 and Φ_3. For most values of n, there are no triple arrays with these parameters. However, the column component $\Delta_{C(L)}$ has very unequal concurrences, particularly as n increases, and this makes the design inefficient in the sense discussed in Section 5.

When $n = 2$, there is no triple array for this parameter set. That is why the arrays in Figures 2 and 5 are not isomorphic. The first satisfies condition (A3) and the second satisfies (A4) but neither satisfies both.

D	H	F	L	E	K	I	G	J
A	K	I	B	J	G	C	L	H
J	A	L	D	B	F	K	E	C
G	E	A	H	I	B	D	C	F

Figure 7: A triple array with $r = 4$, $c = 9$ and $v = 12$

For $n \in \{3, 4, 5\}$, Sterling and Wormald gave triple arrays for these parameter sets in [38]. Figure 7 shows that for $n = 3$. Agrawal also gave triple arrays for these in [1]. McSorley at al. gave one for $n = 7$ in [24], and also constructed some for $n \in \{8, 11, 13\}$, which can be found at [23].

For $n = 6$ the column component cannot be balanced, as there is no affine plane of order 6. The next section gives an efficient new design for 36 points in 42 blocks of size 6, and then a sesqui-array which has this as its column component.

3   A design for the case $n = 6$

3.1   Square lattice designs

If n is a power of a prime then there is an affine plane of order n. It has n^2 points, and its $n(n + 1)$ lines form the blocks of a balanced incomplete-block design. If $2 \leq r \leq n + 1$, then the blocks of any r parallel classes give a block design known as a *square lattice design* [43] or *net* [13]. Even if there is no affine plane of order n, if there are $r - 2$ mutually orthogonal Latin squares of order n then there is a square lattice design for n^2 points in rn blocks of size n. Such designs are known to be optimal in the sense discussed in Section 5. All concurrences are in $\{0, 1\}$.

When $n = 6$ then there is no pair of orthogonal Latin squares and so there are square lattice designs for $r = 2$ and $r = 3$ but for no larger values of r. This lack does not prevent the need for efficient block designs for 36 points in $6r$ blocks of size 6 for larger values of r. A heuristic for a good block design (in the sense explained in Section 5) is that all concurrences are in $\{0, 1, 2\}$, since they cannot all be equal and their average value is $r/(n + 1)$, which is at most 1.

Patterson and Williams gave a good block design for $n = 6$ and $r = 4$ in [29]; it has all concurrences in $\{0, 1, 2\}$. We are not aware of similar block designs for $r = 5$ or $r = 6$. In the following subsections we construct such block designs for $r \in \{4, 5, 6, 7\}$ and then construct a sesqui-array which has the one with $r = 7$ as its column component.

3.2   Construction of the block designs

Our construction uses a property of the number 6: the symmetric group S_6 admits an outer automorphism, and 6 is the only cardinal number for which this is true.

See [14, Chapter 6] for the use of this outer automorphism in various constructions, including the *Hoffman–Singleton graph* [20]: this is a graph on 50 vertices with valency 7, diameter 2 and girth 5. We require no knowledge of the outer automorphism of S_6, but write the construction just in terms of the Hoffman–Singleton graph HS, and its subgraph the Sylvester graph. Details of these graphs can be found at [11].

Let a_0 and b_0 be adjacent vertices of the graph HS; let A be the set of neighbours of a_0 excluding b_0, and B the set of neighbours of b_0 excluding a_0. Since there are no triangles, $A \cap B = \emptyset$. This accounts for $2 + 6 + 6 = 14$ of the 50 vertices. If x is one of the remaining 36 vertices, then x lies at distance 2 from both a_0 and b_0, and hence x is joined to a unique vertex a in A and a unique vertex in b in B. (Uniqueness holds because there are no quadrilaterals in the graph HS.) So we can label x with the ordered pair (a, b), and identify the set of vertices non-adjacent to a_0 and b_0 with the Cartesian product $A \times B$.

We are particularly concerned with the induced subgraph of the graph HS on the set $A \times B$. This graph is known as the *Sylvester graph* [12, Theorem 13.1.2]. We denote it by Σ. Each vertex (a, b) is joined in HS to a, b, and five vertices in $A \times B$. Of the six vertices in $A \times B$ in the closed neighbourhood of (a, b), no two have the same first or second coordinate (since this would create a short cycle in HS).

The elements of $A \times B$ form the points of our new design Θ. The 42 blocks are labelled by the elements of $B \cup (A \times B)$. The block labelled b contains the six points (a, b) for a in A; the block labelled (a, b) contains the point (a, b) and all its neighbours in the graph Σ. Thus every block contains six points and the design is binary.

Now we check the concurrences of pairs of distinct points (a_1, b_1) and (a_2, b_2). If $b_1 = b_2$ then the unique block in which they concur is block b_1. If $a_1 = a_2$ then no block contains both. If points (a_1, b_1) and (a_2, b_2) are adjacent in Σ then they concur in blocks (a_1, b_1) and (a_2, b_2) but no others. Otherwise, there is a unique point (a_3, b_3) adjacent to both in Σ, and so they concur in block (a_3, b_3) but no other. Hence the set of concurrences is $\{0, 1, 2\}$.

Note that this design Θ is partially balanced with respect to the four-class association scheme whose relations are "same first component", "same second component", "adjacent in Σ", and "none of the foregoing".

The design is Θ also resolvable. The blocks labelled by the elements of B form one replicate. There are two ways of forming the remaining replicates: either group together all blocks labelled (a, b) for a fixed a, or do the same for a fixed b. Removing one or more replicates gives a resolvable design with $r < 7$.

3.3   Construction of the sesqui-array

Now we construct the 7×36 array Δ, whose column design is Θ. Our strategy is as follows: first we build an array Δ_0 with the correct column design; however, Δ_0 fails spectacularly to be a sesqui-array, having many repeats of letters in rows. Then we

fix the problem by permuting letters in columns to obtain Δ.

The rows of Δ_0 are labelled with elements of $\{*\} \cup A$, where $*$ is a new symbol, and the columns by $A \times B$, identified with the points of Θ. The letters are labelled by $B \cup (A \times B)$, identified with the blocks of Θ. The rule for placing letters in the array is as follows:

- The letter labelled (a, b) goes in row a, column (a', b'), for every (a', b') adjacent to (a, b) in the graph Σ (5 occurrences), and also in row $*$, column (a, b) (1 occurrence).

- The letter labelled b goes in row a, column (a, b) for all $a \in A$ (6 occurrences).

We see at once that the column component $\Delta_{C(L)}$ is the design Θ. Hence columns are binary, the design is equireplicate, and the set of column concurrences is $\{0, 1, 2\}$. However, for each a in A, row a contains letters from B each once, and letters of the form (a, b) each five times, for each $b \in B$, and no letters (a', b) for $a' \neq a$. The following modification, which simply permutes entries in columns, does not change the column design, which thus remains binary and equireplicate, so that (A1) and half of (A0) are satisfied.

Consider the six permutations, where we regard the entries and labels as taken from the set $A = \{1, \ldots, 6\}$:

$$
\begin{aligned}
\sigma_1 &= (1)(6, 5, 4, 3, 2) \\
\sigma_2 &= (2)(5, 6, 4, 1, 3) \\
\sigma_3 &= (3)(6, 2, 5, 1, 4) \\
\sigma_4 &= (4)(2, 3, 6, 1, 5) \\
\sigma_5 &= (5)(3, 4, 2, 1, 6) \\
\sigma_6 &= (6)(4, 5, 3, 1, 2)
\end{aligned}
$$

(These correspond to the blocks of a neighbour-balanced design for six treatments in six circular blocks of size five given in [6].) It is readily checked that, for each ordered pair $(a_1, a_2) \in A \times A$, there is a unique permutation in the set which maps a_1 to a_2: the set is *sharply transitive*.

Now take the array Δ_0. Consider column (a, b), which contains the letters (a, b) (in row $*$), b (in row a), and (a', b') (in row a'), where (a', b') is joined to (a, b) in the graph Σ. We permute the elements of this column, fixing the entry in row $*$, so that (a', b') is placed in row $a' \sigma_a$. Since σ_a fixes a, the entry in row a is not changed; moreover, the set of elements in column (a, b) is not changed. Perform this operation on every column. Let Δ be the resulting array.

We claim first that, in the resulting array, the letter (a, b) is never contained in row a. For in column (a, b), the entry in row a is b, while in column (a', b'), the pair with first element a has been moved to row $a\sigma_{a'}$, which is not equal to a.

Next we claim that, in Δ, no letter is repeated in a row. For suppose that letter (a, b) occurs twice in row a', say in columns (a_1, b_1) and (a_2, b_2). Then $a\sigma_{a_1} =$

$a' = a\sigma_{a_2}$. By the sharp transitivity of our set of permutations, this implies that $a_1 = a_2$. Now the property of the Sylvester graph (that two points with the same first coordinate have no common neighbour) implies that $b_1 = b_2$. This contradicts the assumption that the columns (a_1, b_1) and (a_2, b_2) are distinct.

This completes the verification of (A0).

In Δ, the letters in row a are those in B together with all (a', b') with $a' \neq a$ (while those in row $*$ are all ordered pairs in $A \times B$). Thus any two rows have 30 common letters, so the row design $\Delta_{R(L)}$ is balanced; that is, (A2) holds.

Finally, we check condition (A4).

- Row $*$ and column (a, b) have letters (a, b) and its five neighbours in common $(1 + 5 = 6$ of these).

- Row a and column (a, b) have in common the letters b and (a', b') for the five neighbours (a', b') of the vertex (a, b) in Σ $(1 + 5 = 6$ of these).

- Row a and column (a', b) with $a' \neq a$ have in common the letters b, (a', b), and the four neighbours (a'', b') of the vertex (a', b) with $a'' \neq a$ $(1 + 1 + 4 = 6$ of these).

Thus Δ is a sesqui-array $\mathrm{SA}(42, 6, 30, \Gamma, 6 : 7 \times 36)$, where $\Gamma = \{0, 1, 2\}$. It is shown (transposed) in Figure 8. In the figure, we have labelled the elements of both A and B as $1, \ldots, 6$; the position in the array determines whether a digit belongs to A or B. (In an ordered pair, the first member is in A and the second in B; the column labels other than $*$ are elements of A, and the single entries in table cells are in B.)

4   Sesqui-arrays from biplanes

We now construct some sesqui-arrays (and triple arrays) from biplanes.

4.1   Biplanes and Hussain chains

Let Ξ be a biplane (a symmetric 2-$(V, K, 2)$ design with $V = 1 + \binom{K}{2}$). (Lower-case v and k are normally used, but these would conflict with the notation used for our arrays.) There are only finitely many biplanes known, with $K = 3, 4, 5, 6, 9, 11, 13$; the numbers up to isomorphism are 1, 1, 1, 3, 4, 5, 2 (there is no classification for $K = 13$ as yet). All known biplanes are described in detail in Section 15.8 of Marshall Hall's *Combinatorial Theory*, 2nd edition [19], the standard reference on biplanes.

We caution the reader that there are several misprints in [19] in the details on the biplanes:

- In the list of blocks for the biplane B_1 with $k = 6$, the entry 16 in block A_{10} should be 15 (p. 322).

	$*$	1	2	3	4	5	6
$(1,1)$	$(1,1)$	1	$(3,6)$	$(4,5)$	$(5,4)$	$(6,3)$	$(2,2)$
$(1,2)$	$(1,2)$	2	$(3,4)$	$(4,3)$	$(5,5)$	$(6,6)$	$(2,1)$
$(1,3)$	$(1,3)$	3	$(3,5)$	$(4,2)$	$(5,6)$	$(6,1)$	$(2,4)$
$(1,4)$	$(1,4)$	4	$(3,2)$	$(4,6)$	$(5,1)$	$(6,5)$	$(2,3)$
$(1,5)$	$(1,5)$	5	$(3,3)$	$(4,1)$	$(5,2)$	$(6,4)$	$(2,6)$
$(1,6)$	$(1,6)$	6	$(3,1)$	$(4,4)$	$(5,3)$	$(6,2)$	$(2,5)$
$(2,1)$	$(2,1)$	$(4,4)$	1	$(1,2)$	$(6,5)$	$(3,3)$	$(5,6)$
$(2,2)$	$(2,2)$	$(4,6)$	2	$(1,1)$	$(6,4)$	$(3,5)$	$(5,3)$
$(2,3)$	$(2,3)$	$(4,5)$	3	$(1,4)$	$(6,6)$	$(3,1)$	$(5,2)$
$(2,4)$	$(2,4)$	$(4,1)$	4	$(1,3)$	$(6,2)$	$(3,6)$	$(5,5)$
$(2,5)$	$(2,5)$	$(4,3)$	5	$(1,6)$	$(6,1)$	$(3,2)$	$(5,4)$
$(2,6)$	$(2,6)$	$(4,2)$	6	$(1,5)$	$(6,3)$	$(3,4)$	$(5,1)$
$(3,1)$	$(3,1)$	$(5,5)$	$(6,4)$	1	$(1,6)$	$(2,3)$	$(4,2)$
$(3,2)$	$(3,2)$	$(5,6)$	$(6,3)$	2	$(1,4)$	$(2,5)$	$(4,1)$
$(3,3)$	$(3,3)$	$(5,4)$	$(6,2)$	3	$(1,5)$	$(2,1)$	$(4,6)$
$(3,4)$	$(3,4)$	$(5,3)$	$(6,1)$	4	$(1,2)$	$(2,6)$	$(4,5)$
$(3,5)$	$(3,5)$	$(5,1)$	$(6,6)$	5	$(1,3)$	$(2,2)$	$(4,4)$
$(3,6)$	$(3,6)$	$(5,2)$	$(6,5)$	6	$(1,1)$	$(2,4)$	$(4,3)$
$(4,1)$	$(4,1)$	$(6,6)$	$(5,3)$	$(2,4)$	1	$(1,5)$	$(3,2)$
$(4,2)$	$(4,2)$	$(6,5)$	$(5,4)$	$(2,6)$	2	$(1,3)$	$(3,1)$
$(4,3)$	$(4,3)$	$(6,4)$	$(5,1)$	$(2,5)$	3	$(1,2)$	$(3,6)$
$(4,4)$	$(4,4)$	$(6,3)$	$(5,2)$	$(2,1)$	4	$(1,6)$	$(3,5)$
$(4,5)$	$(4,5)$	$(6,2)$	$(5,6)$	$(2,3)$	5	$(1,1)$	$(3,4)$
$(4,6)$	$(4,6)$	$(6,1)$	$(5,5)$	$(2,2)$	6	$(1,4)$	$(3,3)$
$(5,1)$	$(5,1)$	$(2,6)$	$(4,3)$	$(6,2)$	$(3,5)$	1	$(1,4)$
$(5,2)$	$(5,2)$	$(2,3)$	$(4,4)$	$(6,1)$	$(3,6)$	2	$(1,5)$
$(5,3)$	$(5,3)$	$(2,2)$	$(4,1)$	$(6,5)$	$(3,4)$	3	$(1,6)$
$(5,4)$	$(5,4)$	$(2,5)$	$(4,2)$	$(6,6)$	$(3,3)$	4	$(1,1)$
$(5,5)$	$(5,5)$	$(2,4)$	$(4,6)$	$(6,3)$	$(3,1)$	5	$(1,2)$
$(5,6)$	$(5,6)$	$(2,1)$	$(4,5)$	$(6,4)$	$(3,2)$	6	$(1,3)$
$(6,1)$	$(6,1)$	$(3,4)$	$(1,3)$	$(5,2)$	$(2,5)$	$(4,6)$	1
$(6,2)$	$(6,2)$	$(3,3)$	$(1,6)$	$(5,1)$	$(2,4)$	$(4,5)$	2
$(6,3)$	$(6,3)$	$(3,2)$	$(1,1)$	$(5,5)$	$(2,6)$	$(4,4)$	3
$(6,4)$	$(6,4)$	$(3,1)$	$(1,5)$	$(5,6)$	$(2,2)$	$(4,3)$	4
$(6,5)$	$(6,5)$	$(3,6)$	$(1,4)$	$(5,3)$	$(2,1)$	$(4,2)$	5
$(6,6)$	$(6,6)$	$(3,5)$	$(1,2)$	$(5,4)$	$(2,3)$	$(4,1)$	6

Figure 8: A sesqui-array with $r = 7$, $c = 36$, $v = 42$ (transposed)

- In the description of the biplane $B_L(9)$, the second occurrence of 31 in the permutation ψ on p. 324 should be 34, and the element 32 in the second base block should be 22 (pp. 324, 326).

- There are two misprints in the list of blocks of $B_H(9)$ on p. 325: in the block beginning 2, 8, the entry 38 should be 34; and in the block beginning 7, 8, the entry 10 should be 16.

- In the table of chain-lengths on p. 333, the chain written as 5-5-3 should be 5-3-3.

- There are two misprints in the generators of the automorphism group of $B(13)$ on p. 334: the generator x should have 69 inserted before 70 in the last cycle; and in z, the cycle $(15, 25)$ should be $(15, 24)$.

We do not claim to have spotted all the misprints. In addition we have computed the Hussain chain lengths for the Aschbacher biplane and its dual (see below); these are not given by Hall.

Let B be a block of Ξ. Any pair of points of B lie in one further block of Ξ, and any other block B' of Ξ not equal to B meets B in two points. So we can label the remaining blocks by the 2-subsets of B.

Following Hall, we define, for any point $q \notin B$, a graph of valency 2 on B called a *Hussain chain*. The edges of the graph are the intersections with B of the blocks containing q. Thus there are K edges, and any point of B lies on two edges; so the graph is a union of cycles. We call this graph $H(q)$. The lists of cycle lengths in Hussain chains for all blocks of all the known biplanes with $K < 13$ are given by Hall.

The collection of graphs $H(q)$ for $q \notin B$ has the following properties.

(H1) Any two intersecting pairs of points of B are both edges in $H(q)$ for a unique q.

(H2) Any two disjoint pairs of points of B are both edges in $H(q)$ for exactly two values of q.

(H3) Any two Hussain chains share two (disjoint) edges.

The proofs are straightforward. For every pair of points in B, there is a unique block B' meeting B in just that pair; if two such blocks contain q, they can have at most one point of B in common.

The collection of Hussain chains determines the biplane:

- the points of Ξ are the points of B and the Hussain chains;

- the blocks of Ξ are a symbol $*$ and the pairs of points of B;

- $*$ is incident with every point of B, every point of B with every pair containing it, and each pair with every Hussain chain of which it is an edge.

4.2   The construction

Let Ξ be a biplane with V points and block size K, and let B be a block of Ξ. We exclude the case $K = 3$. We form an array Δ of size $K \times (V - K)$, whose rows are indexed by the points of B and columns by the points outside B. The letters in the array are indexed by the 2-element subsets of B. In row p and column q, we put $\{p_1, p_2\}$, if p_1 and p_2 are the two neighbours of p in the Hussain chain $H(q)$; in other words, the blocks containing p and q meet B again in the points p_1 and p_2.

We now check the conditions for a triple array.

(A0) A letter cannot occur more than once in a row, by property (H1). In fact, the letters in row p are the 2-subsets of $B \setminus \{p\}$. On the other hand, $\{p_1, p_2\}$ could occur in column q and rows p and p'; indeed this happens if and only if $\{p, p_1, p', p_2\}$ is a 4-cycle in $H(q)$. We conclude that the array Δ is binary if and only if no Hussain chain $H(q)$ contains a 4-cycle.

(A1) For any letter $\{p_1, p_2\}$, for each $p \notin \{p_1, p_2\}$, there is a unique q such that $H(q)$ has edges $\{p, p_1\}$ and $\{p, p_2\}$, by property (H1). Thus each letter occurs $K - 2$ times in the array Δ, which is therefore equireplicate.

(A2) Rows p and p' share all letters $\{p_1, p_2\}$ for which $\{p_1, p_2\} \cap \{p, p'\} = \emptyset$. So any two rows have $\binom{K-2}{2}$ common letters. If $K > 3$ this number is non-zero and so the component design $\Delta_{R(L)}$ is balanced.

(A3) It is not true in general that the component design $\Delta_{C(L)}$ is balanced. We return to this point later.

(A4) Each row has $K - 2$ common letters with each column. For if the row and column indices are p and q, then the common letters are the "short diagonals" (pairs of vertices joined by a path of length 2) of $H(q)$ which do not contain p; there are $K - 2$ of these. (The presence of 4-cycles does not affect this count, since the "short diagonals" of a 4-cycle are counted twice.) In other words, adjusted orthogonality always holds.

Thus, if we choose a biplane and a block such that no Hussain chain contains a 4-cycle, then we obtain a binary array Δ satisfying (A0)–(A2) and (A4) but not necessarily (A3). As stated earlier, such an array is a sesqui-array. So we have proved the first two parts of the following result.

Theorem 4.1 *Suppose that B is a block of a biplane Ξ with block size at least 4.*

1. *The array Δ constructed above satisfies (A1), (A2) and (A4).*

2. *It is binary if and only if none of the Hussain chains on the block B contains a 4-cycle.*

3. *It is a triple array if every Hussain chain on B is a union of 3-cycles.*

PROOF: Only the third statement remains to be proved. So suppose that all the Hussain chains on B are unions of 3-cycles. Then the array is binary. If $\{p, p_1\}$ and $\{p, p_2\}$ are edges of $H(q)$, then so is $\{p_1, p_2\}$: thus the symbols in column q are the edges of $H(q)$. Property (H3) shows that any two Hussain chains $H(q)$ and $H(q')$ share precisely two edges. □

There are just two known biplanes which have this property. The $(16, 6, 2)$ biplane B_1 in Hall [19] has this property for any choice of block. The $(37, 9, 2)$ biplane $B_H(9)$ has a unique block for which all the Hussain chains consist of three triangles. These give a 6×10 triple array with 15 symbols, which is shown in this form in [8, Figure 32], and a 9×28 triple array with 36 symbols.

The tranpose of the latter is shown in Figure 9. The nine points of B are identified with the projective line $\mathrm{PG}(1, 8)$. The elements of $\mathrm{GF}(8)$ are denoted 0, 1, a, b, c, d, e and f, where $b = a^2$, $f = a^3 = a + 1$, $c = a^4 = a + b$, $e = a^5 = a + b + 1$ and $d = a^6 = b + 1$. The rows are labelled by the Hussain chains, and the letters by pairs of points of B. The group $\mathrm{PGL}(2, 8)$ has 28 subgroups of order 3, each having three orbits: these give the 28 Hussain chains. The automorphism group of this array is $\mathrm{P\Gamma L}(2, 8)$, which has order 1512.

Even if the conditions are not all satisfied, interesting arrays can be obtained. Consider the unique $(7, 4, 2)$ biplane. For any block B, the three Hussain chains on B are the three possible 4-cycles on this set. Thus, the array Δ is not binary, but it is easy to see that it satisfies all the other conditions for a triple array: every pair of points is the short diagonal of a unique 4-cycle, but occurs twice in the corresponding column.

Another example comes from the unique $(11, 5, 2)$ biplane. In this case, all Hussain chains are pentagons, and they form one orbit of the alternating group A_5 on pentagons. The other orbit consists of the pentagons formed by the diagonals of those in the first orbit, which form another biplane isomorphic to the first. So our construction, for which the pairs in column q are the diagonals in $H(q)$, can be represented by the edges of the image of the biplane under an odd permutation of B, and so (just as in the above proof) the columns are balanced. So we do indeed obtain a triple array, which is displayed in this way in [8, Figure 31]. This example shows that the condition in the third statement of the theorem is sufficient but not necessary.

This phenomenon is even more widespread: all biplanes with $K = 6$ give triple arrays. There is a reason for this, and a surprising further fact:

Proposition 4.2 *Let Ξ be any biplane with $K = 6$, and B any block of Ξ. Then the sesqui-array obtained from it by our construction is a triple array. Moreover, applying Proposition 1.4 to this triple array gives the same biplane in all three cases (the one denoted by B_1 in Hall [19]).*

PROOF: The letters in the array are 2-subsets of the distinguished block B. The column corresponding to a point $q \notin B$ contains the edges of the "two-step graph"

	0	1	a	b	f	c	e	d	∞
$(0,1,\infty)(a,c,b)(d,e,f)$	1 ∞	0 ∞	b c	a c	d e	a b	d f	e f	0 1
$(0,a,\infty)(b,e,f)(1,d,c)$	a ∞	c d	0 ∞	e f	b e	1 d	b f	1 c	0 a
$(0,b,\infty)(f,d,c)(a,1,e)$	b ∞	a e	1 e	0 ∞	c d	d f	1 a	c f	0 b
$(0,f,\infty)(c,1,e)(b,a,d)$	f ∞	c e	b d	a d	0 ∞	1 e	1 c	a b	0 f
$(0,c,\infty)(e,a,d)(f,b,1)$	c ∞	b f	d e	1 f	1 b	0 ∞	a d	a e	0 c
$(0,e,\infty)(d,b,1)(c,f,a)$	e ∞	b d	c f	1 d	a c	a f	0 ∞	1 b	0 e
$(0,d,\infty)(1,f,a)(e,c,b)$	d ∞	a f	1 f	c e	1 a	b e	b c	0 ∞	0 d
$(a,f,\infty)(0,b,c)(e,d,1)$	b c	d e	f ∞	0 c	a ∞	0 b	1 d	1 e	a f
$(b,c,\infty)(0,f,e)(d,1,a)$	e f	a d	1 d	c ∞	0 e	b ∞	0 f	1 a	b c
$(f,e,\infty)(0,c,d)(1,a,b)$	c d	a b	1 b	1 a	e ∞	0 d	f ∞	0 c	e f
$(c,d,\infty)(0,e,1)(a,b,f)$	1 e	0 e	b f	a f	a b	d ∞	0 1	c ∞	c d
$(e,1,\infty)(0,d,a)(b,f,c)$	a d	e ∞	0 d	c f	b c	b f	1 ∞	0 a	1 e
$(d,a,\infty)(0,1,b)(f,c,e)$	1 b	0 b	d ∞	0 1	c e	e f	c f	a ∞	a d
$(1,b,\infty)(0,a,f)(c,e,d)$	a f	b ∞	0 f	1 ∞	0 a	d e	c d	c e	1 b
$(b,d,\infty)(c,a,0)(1,f,e)$	a c	e f	0 c	d ∞	1 e	0 a	1 f	b ∞	b d
$(f,1,\infty)(e,b,0)(a,c,d)$	b e	f ∞	c d	0 e	1 ∞	a d	0 b	a c	1 f
$(c,a,\infty)(d,f,0)(b,e,1)$	d f	b e	c ∞	1 e	0 d	a ∞	1 b	0 f	a c
$(e,b,\infty)(1,c,0)(f,d,a)$	1 c	0 c	d f	e ∞	a d	0 1	b ∞	a f	b e
$(d,f,\infty)(a,e,0)(c,1,b)$	a e	b c	0 e	1 c	d ∞	1 b	0 a	f ∞	d f
$(1,c,\infty)(b,d,0)(e,a,f)$	b d	c ∞	e f	0 d	a e	1 ∞	a f	0 b	1 c
$(a,e,\infty)(f,1,0)(d,b,c)$	1 f	0 f	e ∞	c d	0 1	b d	a ∞	b c	a e
$(c,e,\infty)(b,0,a)(f,1,d)$	a b	d f	0 b	0 a	1 d	e ∞	c ∞	1 f	c e
$(e,d,\infty)(f,0,b)(c,a,1)$	b f	a c	1 c	0 f	0 b	1 a	d ∞	e ∞	d e
$(d,1,\infty)(c,0,f)(e,b,a)$	c f	d ∞	b e	a e	0 c	0 f	a b	1 ∞	1 d
$(1,a,\infty)(e,0,c)(d,f,b)$	c e	a ∞	1 ∞	d f	b d	0 e	0 c	b f	1 a
$(a,b,\infty)(d,0,e)(1,c,f)$	d e	c f	b ∞	a ∞	1 c	1 f	0 d	0 e	a b
$(b,f,\infty)(1,0,d)(a,e,c)$	1 d	0 d	c e	f ∞	b ∞	a e	a c	0 1	b f
$(f,c,\infty)(a,0,1)(b,d,e)$	1 a	0 a	0 1	d e	c ∞	f ∞	b d	b e	c f

Figure 9: Triple array with $r = 28$, $c = 9$ and $v = 36$

of $H(q)$, that is, two points are joined if they have a common neighbour in $H(q)$. We denote this graph by $H^*(q)$. We note that, because $K = 6$, each Hussain chain $H(q)$ is either a pair of triangles or a hexagon, and so $H^*(q)$ is always a pair of triangles (and is equal to $H(q)$ if $H(q)$ is a pair of triangles).

We show that all the graphs $H^*(q)$ are distinct; it follows, since there are ten of them, that they must consist of all possible pairs of triangles.

Suppose that $H^*(q_1) = H^*(q_2)$. There are three cases:

- $H(q_1)$ and $H(q_2)$ are both double triangles. Then

$$H(q_1) = H^*(q_1) = H^*(q_2) = H(q_2),$$

  so $q_1 = q_2$.
- $H(q_1)$ is a double triangle (so $H(q_1) = H^*(q_1)$) and $H(q_2)$ is a hexagon. In this case, $H(q_1) = H^*(q_2)$ consists of the short diagonals of the hexagon, and has no edges in common with $H(q_2)$, contradicting property (H3).
- $H(q_1)$ and $H(q_2)$ are both hexagons, and they have the same two-step graph. Assume that the vertices of $H(q_1)$ are $1, 2, \ldots, 6$ in order round the hexagon. By (H3), without loss of generality, $H(q_2)$ contains the edge $\{1, 2\}$. Then the other neighbour of 2 in $H(q_2)$ must be either 3 or 5; the first is impossible since then $H(q_1)$ and $H(q_2)$ would share adjacent edges $\{1, 2\}$ and $\{2, 3\}$, contradicting (H1). So $\{2, 5\}$ is an edge of $H(q_2)$. By the same argument, $\{1, 4\}$ is also an edge. Now the only way to complete this to a hexagon with the correct two-step graph is to have also the edges $\{5, 6\}$, $\{6, 3\}$ and $\{3, 4\}$. But then $H(q_1)$ and $H(q_2)$ share three edges, a contradiction.

Now the letters in the array are 2-subsets of $\{1, \ldots, 6\}$. By construction, the letters in row p are all the 2-sets not containing p; and the letters in column q are the edges of $H^*(q)$, which form a pair of disjoint triangles, and every such pair occurs in some column. This is exactly a description of the biplane B_1.   □

Using the list of cycle lengths in the Hussain chains of the known biplanes given by Hall, we see that we obtain (binary) sesqui-arrays from the biplanes with block size 5, 6 (all three biplanes, and indeed all of these are triple arrays), and 9 ($B_H(9)$ and $B'_H(9)$, for any block – the first of these gives a triple array for one chosen block, as shown in Figure 9, but we verified by computer that none of the other blocks give triple arrays – and $B_L(9)$, for 10 of the 37 blocks). All of the biplanes with block size 11 have 4-cycles for any choice of block.

Hall does not compute the chain lengths for the Aschbacher biplanes with block size 13, so we have done this computation; the results are as follows (using Hall's notation for the cycle lengths in Hussain chains).

For $B(13)$, there are

- one block with chain structure 10-3 (11 points) and 13 (55 points);

- one block with chain structure 5-5-3 (11 points) and 13 (55 points);

- 22 blocks with chain structure 7-3-3 (5 points), 5-5-3 (5 points), 10-3 (16 points), 9-4 (5 points), 13 (35 points);

- 55 blocks with chain structure 7-3-3 (1 point), 6-4-3 (6 points), 5-5-3 (1 point), 10-3 (7 points), 9-4 (3 points), 8-5 (6 points), 7-6 (9 points), 13 (33 points).

For its dual $B'(13)$, there are

- two blocks with chain structure 6-4-3 (55 points) and 10-3 (11 points);

- 11 blocks with chain structure 5-5-3 (1 point), 8-5 (10 points), 7-6 (10 points) and 13 (45 points);

- 11 blocks with chain structure 7-3-3 (5 points), 10-3 (1 point), 8-5 (10 points), 7-6 (10 points), and 13 (40 points);

- 55 blocks with chain structure 7-3-3 (2 points), 6-4-3 (4 points), 5-5-3 (3 points), 10-3 (13 points), 9-4 (5 points), 8-5 (2 points), 7-6 (5 points), 13 (32 points).

Thus we obtain (binary) sesqui-arrays from the first two blocks of $B(13)$ and from the 22 blocks of the second and third types in $B'(13)$.

5 Optimality

5.1 Optimality criteria

The statistical quality of the block design $\Delta_{C(L)}$ is determined by the eigenvalues of its *scaled information matrix*, which is $I_c - (rk)^{-1}N_{CL}N_{LC}$, where I_c is the identity matrix of order c. The constant vectors have eigenvalue 0. If this eigenvalue has multiplicity more than one then some differences between columns cannot be estimated and the design is said to be *disconnected*. Otherwise, let the remaining eigenvalues be μ_1, ..., μ_{c-1}, with $0 < \mu_1 \leq \mu_2 \leq \cdots \leq \mu_{c-1} \leq 1$. These eigenvalues are called *canonical efficiency factors*. For a good design they should all be as large as possible, but they are constrained by the equation

$$\sum_{i=1}^{c-1} \mu_i = \frac{c(k-1)}{k}.$$

The design is said to be A-optimal if it maximizes the harmonic mean μ_A of μ_1, ..., μ_{c-1}; to be D-optimal if it maximizes the geometric mean μ_D of μ_1, ..., μ_{c-1}; and to be E-optimal if it maximizes μ_1. See [9, 37] and [40, Section 1.7].

The canonical efficiency factors of the dual design $\Delta_{L(C)}$ are μ_1, ..., μ_{c-1} and $v - c$ others equal to 1. Thus $\Delta_{L(C)}$ is optimal, under any of the three criteria, if and only if $\Delta_{C(L)}$ is.

If $\Delta_{C(L)}$ is balanced then $\mu_1 = \cdots = \mu_{c-1} = c(k-1)/[(c-1)k]$. This design is A-optimal, D-optimal and E-optimal. When such a design exists, no non-balanced design with these parameters is optimal on any of these criteria. If there is no balanced design available for given values of c, r, v and k, then we usually try to find a design whose optimality criteria are not too far short of $c(k-1)/[(c-1)k]$.

For the particular case that $c = n^2$, $k = n$ and $v = rn$ with $r \leq n+1$, it is known that a square lattice design, if it exists, is A-, D- and E-optimal: see [15]. It has $r(n-1)$ canonical efficiency factors equal to $(r-1)/r$ and $(n+1-r)(n-1)$ equal to 1. Thus

$$\mu_1 = \frac{r-1}{r} \quad \text{and} \quad \mu_A = \frac{rn - n + r - 1}{rn - n + 2r - 1}.$$

From now on, we concentrate on the criteria μ_A and μ_1.

5.2   Optimality properties of the column component designs

In a sesqui-array, the row component design is balanced. The column component may not be, and so statistical properties of the whole array depend on it. Here we give the efficiency factors for the non-balanced column components used in this paper.

The column component of the sesqui-array in Figure 3 is partially balanced with respect to the group divisible association scheme GD(3, 2). Its canonical efficiency factors are 2/3 (three times) and 1 (twice), so that $\mu_1 = 2/3$ and $\mu_A = 10/13 \approx 0.769$. The alternative column design mentioned in Example 1 has canonical efficiency factors 2/3, 3/4 (twice) and 11/12 (twice), giving $\mu_1 = 2/3$ and $\mu_A = 330/419 \approx 0.788$. Thus these two designs are equally good on the E-criterion. The second is slightly better on the A-criterion, but cannot be incorporated into a sesqui-array.

Bagchi showed that the column components of the sesqui-arrays constructed in [2] are E-optimal when $n \geq 5$. The column components of the sesqui-arrays in [4, 17] are the duals of square lattice designs, and hence optimal on all three criteria.

For the sesqui-arrays constructed in Section 2, the column component has canonical efficiency factors $1/(n+1)$ and $n/(n+1)$ (both with multiplicity $n-1$) and 1 (with multiplicity $(n-1)^2$). Thus

$$\mu_1 = \frac{1}{n+1} \quad \text{and} \quad \mu_A = \frac{n(n+1)}{2n^2 + n + 1}.$$

On the other hand, a balanced square lattice design for these parameters, if it exists, has $\mu_1 = \mu_A = n/(n+1)$. Thus the column components of the sesqui-arrays in Section 2 are far from optimal, and become worse as n increases.

Now we consider the block design for 36 points in 42 blocks of size 6 constructed in Section 3.2. Using properties of the association scheme described there, we can show that the canonical efficiency factors are 11/14, 6/7, 19/21 and 1 with multiplicities 16, 5, 9 and 5 respectively. These give $\mu_1 \approx 0.786$ and $\mu_A \approx 0.851$. The non-achievable upper bounds for these given by the non-existent affine plane are both

equal to 6/7, which is approximately 0.857, so this design seems to be very good, and may indeed be optimal.

For $r = 4$, the unachievable upper bound on μ_A is 0.840; the design given in [29] has $\mu_A \approx 0.836$.

5.3 Optimality properties of the rectangular arrays

Rectangular arrays are used for designed experiments in two different contexts. In the first, described by Preece [32, 33] and others, the design is $\Delta_{R,C(L)}$. There are v blocks of size k, one set of r treatments and another set of c treatments. One treatment from each set is applied to each unit in each block, and a response is measured on that unit. The aim is to estimate the effects of each set of treatments, under the assumption that there is no interaction, which means that they do not affect each other. In order to remove the effects of any differences between blocks, the data have to be projected onto the orthogonal complement of the v-dimensional space defined by the blocks. If R and C have adjusted orthogonality with respect to L then no further adjustment is needed in order to estimate the effects of R and C. Thus if there is an array Δ satisfying (A0), (A1) and (A4) for which $\Delta_{R(L)}$ and $\Delta_{C(L)}$ are both optimal then Δ is optimal for the combined experiment.

In the other use for designed experiments, the experimental units form an $r \times c$ rectangle and there are v treatments, which must be allocated to those units. Thus the design is $\Delta_{L(R,C)}$. The scaled information matrix for letters in this design is $I_v - (rk)^{-1}N_{LC}N_{CL} - (ck)^{-1}N_{LR}N_{RL} + (vk)^{-1}J_v$, where J_v is the $v \times v$ matrix with all entries equal to 1. The optimality criteria for $\Delta_{L(R,C)}$ are based on the non-trivial eigenvalues of this matrix.

Bagchi and Shah proved in [5] that if Δ is a triple array then $\Delta_{L(R,C)}$ is optimal with respect to all the standard optimality criteria among the class of equireplicate designs for v treatments in a $r \times c$ rectangle.

The design $\Delta_{L(R,C)}$ is said to have *general balance* in the sense of Nelder [25] if the matrices $N_{LR}N_{RL}$ and $N_{LC}N_{CL}$ commute with each other. Note that adjusted orthogonality implies general balance.

Shah and Sinha state and prove the following as Theorem 4.4.1 of [37]. If the array Δ has adjusted orthogonality and each of the two component designs $\Delta_{L(R)}$ and $\Delta_{L(C)}$ is optimal, then the whole design $\Delta_{L(R,C)}$ is optimal among equireplicate designs with general balance. Again, this applies to all the standard optimality criteria.

Theorem 4.4.2 of [37] is the following. If the array Δ is equireplicate and has adjusted orthogonality and one component is E-optimal and the other component has E-criterion bigger than the first, then the design $\Delta_{L(R,C)}$ is E-optimal (without restriction to equal replication or general balance).

Pages 81–82 of [37] give an example from John and Eccleston [21] with $r = 4$, $c = 6$ and $v = 12$. Two equireplicate arrays are compared. One has adjusted

orthogonality; the other does not even have general balance but it is better for $\Delta_{L(R,C)}$ on the A-optimality criterion.

Denote by μ_{AR}, μ_{AC} and μ_{ARC} the value of the A-criterion for the component designs $\Delta_{L(R)}$, $\Delta_{L(C)}$ and $\Delta_{L(R,C)}$ respectively. Eccleston and McGilchrist proved in [16] that, for equireplicate designs,

$$\frac{1}{\mu_{ARC}} \geq \frac{1}{\mu_{AR}} + \frac{1}{\mu_{AC}} - 1$$

with equality if and only if the array has adjusted orthogonality. This shows immediately that, among designs with adjusted orthogonality, the best thing to do is, if possible, make sure that the component designs $\Delta_{L(R)}$ and $\Delta_{L(C)}$ are both A-optimal. This result led several authors to conjecture that an array with adjusted orthogonality in which each component design is A-optimal is either A-optimal overall or not far from A-optimal overall. The counter-example above does not really destroy that. However, Shah and Sinha pointed out that Eccleston and McGilchrist's proof assumes general balance. Hence the restriction in their own theorem.

Nonetheless, it does appear that, to find an array that is good for either $\Delta_{R,C(L)}$ or $\Delta_{L(R,C)}$, a good strategy is to find a sesqui-array Δ whose column component performs well on the relevant optimality criterion.

6 Further directions

We conclude with some open problems indicating further directions.

- Our sesqui-arrays have $k = r - 1$ (Sections 2–3) or $k = r - 2$ (Section 4). Those in [2] have $k = 2$; while those in [4] have k equal to $r - 1$, $(r \pm 1)/2$, $(r + 3)/4$, $(r - 1)/4$ and $(r \pm \sqrt{r})/2$, for suitable values of r; and those in [26] have $r = 2k \pm \sqrt{k}$. What other values of k are possible?

- We have seen sesqui-arrays that satisfy inequality (1.1) and others that do not. Is there a weaker inequality, other than the one in Corollary 1.2, that is satisfied by all sesqui-arrays?

- Do all double arrays satisfy inequality (1.1)?

- Apart from Corollary 1.2, what other constraints must the column component design $\Delta_{C(L)}$ satisfy if it is to be incorporated into a sesqui-array?

- There are also questions about isomorphism of sesqui-arrays. Phillips, Preece and Wallis [30] enumerated the 5×6 triple arrays: there are seven isomorphism classes of these. Can such classifications be extended to sesqui-arrays? More modestly, do non-isomorphic biplanes give rise (by our construction) to non-isomorphic triple arrays? This is particularly interesting for the 6×10 arrays with 15 letters, where, as we have seen, they all give rise to the same biplane.

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