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4-Cycle decompositions of complete 3-uniform hypergraphs

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Abstract

A 3-uniform complete hypergraph of order n has vertex set $\{1, 2, \ldots, n\}$ and, as its edge set, the set of all possible subsets of size 3. A 4-cycle in this hypergraph is $v_1, e_1, v_2, e_2, v_3, e_3, v_4, e_4, v_1$ where $\{v_1, v_2, v_3, v_4\}$ are distinct vertices and $\{e_1, e_2, e_3, e_4\}$ are distinct 3-edges such that $v_i, v_{i+1} \in e_i$ for $i = 1, 2, 3$ and $v_4, v_1 \in e_4$ (also known as a Berge cycle). A decomposition of a hypergraph is a partition of its edge set into edge-disjoint subsets. In this paper, we give necessary and sufficient conditions for a decomposition of the complete 3-uniform hypergraph of order n into 4-cycles.

1 Introduction

Problems concerning decompositions of graphs into edge-disjoint subgraphs have been well-studied; see for example the survey in [6]. A *decomposition* of a graph G is a set $\{F_1, F_2, \ldots, F_k\}$ of subgraphs of G such that $E(F_1) \cup E(F_2) \cup \cdots \cup E(F_k) = E(G)$ and $E(F_i) \cap E(F_j) = \emptyset$ for all $1 \leq i < j \leq k$. If F is a fixed graph and $\mathcal{F} = \{F_1, F_2, \ldots, F_k\}$ is a decomposition such that $F_1 \cong F_2 \cong \ldots \cong F_k \cong F$, then $\mathcal{F}$ is called an *F-decomposition*. The problem of determining all values of n for which there is an F-decomposition of the complete graph K_n of order n has attracted a lot of interest for various graphs F (see the survey [1]).

The notion of decompositions of graphs naturally extends to hypergraphs. A *hypergraph* H consists of a finite nonempty set V of *vertices* and a set $E = \{e_1, e_2, \ldots, e_m\}$ of *hyperedges* where each $e_i \subseteq E$ with $|e_i| > 0$ for $1 \le i \le m$. If $|e_i| = h$, then we call e_i an *h-edge*. If every edge of H is an h-edge for some h, then we say that H is *h-uniform*. The *complete h-uniform hypergraph* $K_n^{(h)}$ is the hypergraph with vertex set V, where $|V| = n$, in which every h-subset of V determines an h-edge. It then follows that $K_n^{(h)}$ has $\binom{n}{h}$ hyperedges. When $h = 2$, then $K_n^{(2)} = K_n$, the complete graph on n vertices. We will use the notation $K_n - I$ to denote the complete graph of order n with the edges of a 1-factor I removed.

As in the case of graphs, a *decomposition* of a hypergraph H is a partition of its edge set into subsets. A *decomposition* of a hypergraph H is a set $\{F_1, F_2, \ldots, F_k\}$ of subhypergraphs of H such that $E(F_1) \cup E(F_2) \cup \cdots \cup E(F_k) = E(H)$ and $E(F_i) \cap E(F_j) = \emptyset$ for all $1 \le i < j \le k$. If F is a fixed hypergraph and $\mathcal{F} = \{F_1, F_2, \ldots, F_k\}$ is a decomposition such that $F_1 \cong F_2 \cong \ldots \cong F_k \cong F$, then $\mathcal{F}$ is called an *F-decomposition*. In [7], necessary and sufficient conditions are given for an F-decompostion of $K_n^{(3)}$ for all 3-uniform hypergraphs F with at most three edges and at most six vertices.

A *cycle* of length k in a hypergraph H with vertex set $V(H) = \{v_1, v_2, \ldots, v_n\}$ and hyperedge set $E(H) = \{e_1, e_2, \ldots, e_m\}$ is a sequence of the form

$$v_1, e_1, v_2, e_2, \ldots, v_k, e_k, v_1$$

where $\{v_1, v_2, \ldots, v_k\}$ are distinct vertices and $\{e_1, e_2, \ldots, e_k\}$ are distinct hyperedges satisfying $v_i, v_{i+1} \in e_i$ for $1 \le i \le k-1$ and $v_k, v_1 \in e_k$. This cycle is known as a Berge cycle, having been introduced by Berge in [3]. Decompositions of the complete 3-uniform hypergraph into hamiltonian cycles were considered in [4, 5] and the completion of the proof of their existence was completed in [15]. Decompositions of the complete k-uniform hypergraph into hamiltonian cycles were considered in [11, 13], where a complete solution was given in [11] for $k \ge 4$ and $n \ge 30$ and cyclic decompositions were considered in [13]. In [10], a different type of cycle in a hypergraph was introduced: a *tight* ℓ-cycle in a k-uniform hypergraph is a cyclic ordering of ℓ vertices, $\ell > k$, such that each consecutive k-tuple of vertices is a hyperedge. Tight hamiltonian cycles of 3-uniform hypergraphs were investigated in [2, 9, 12], and no complete resolution of the problem is known. As a consequence of the results in [2, 9, 12], decompositions of $K_n^{(3)}$ into tight hamiltonian cycles are known for all admissible $n \le 46$. Tight (not necessarily hamiltonian) cycles are briefly considered in [12] where it is remarked that a decomposition of $K_n^{(3)}$ into tight 4-cycles exists if and only if $n \equiv 2, 4 \pmod 6$ due to a classical result of Hanani [8] regarding the existence of balanced incomplete block designs of order 4.

Thus, in this paper, we are interested in (Berge) 4-cycle decompositions of complete 3-uniform hypergraphs. We seek to partition the edge set of $K_n^{(3)}$ into subsets of four hyperedges each such that each subset gives rise to a 4-cycle in $K_n^{(3)}$. For convenience, we will often write the 3-edge $\{a, b, c\}$ as abc and cycles of length k in a 3-uniform hypergraph as

$$(x_1y_1x_2, x_2y_2x_3, \ldots, x_{k-1}y_{k-1}x_k, x_ky_kx_1),$$

where $x_iy_ix_{i+1}$ is a 3-edge for $1 \leq i \leq k$ (addition modulo k), $\{x_1, x_2, \ldots, x_k\}$ are distinct vertices, and all 3-edges in the cycle are different.

A necessary condition for the existence of a 4-cycle decomposition of $K_n^{(3)}$ is that 4 must divide the number of hyperedges in $K_n^{(3)}$, that is, $4 \mid \binom{n}{3}$. Clearly, if n is even, then $4 \mid \binom{n}{3}$ and if n is odd and $4 \mid \binom{n}{3}$, then $n \equiv 1 \pmod 8$. Hence, we have the following lemma.

Lemma 1.1 *For $n \geq 4$, if there exists a 4-cycle decomposition of $K_n^{(3)}$, then either n is even or $n \equiv 1 \pmod 8$.*

For n even, we handle the case in which $n \equiv 4, 0, 2 \pmod 6$ in Sections 2, 3, and 4 respectively. The case in which $n \equiv 1 \pmod 8$ is handled in Section 5.

2 The $n \equiv 4 \pmod 6$ case

In this section, we consider the case when $n \equiv 4 \pmod 6$. In this case, since $4 \mid \binom{n}{3}$ and $n \equiv 4 \pmod 6$, we know that $4 \mid [n(n-2)/2]$ and $3 \mid (n-1)$. Thus, since $K_n - I$ has $n(n-2)/2$ edges, we may use a decomposition of $K_n - I$ into 4-cycles, and then blow up each 4-cycle of $K_n - I$ exactly $(n-1)/3$ times to obtain a 4-cycle decomposition of $K_n^{(3)}$. For the rest of this section, we will assume the vertex set of $K_n^{(3)}$ (or K_n) is $\mathbb{Z}_n$, the integers modulo n. Without loss of generality, we consider a specific 1-factor of K_n, namely,

$$I = \{\{0, n/2\}, \{1, n/2+1\}, \ldots, \{n/2-1, n-1\}\}.$$

Note that $K_n^{(3)}$ has $n(n-1)(n-2)/6$ hyperedges and $K_n - I$ has $n(n-2)/2$ edges. Now, as mentioned previously, if we have a decomposition of $K_n - I$ into 4-cycles, we seek a procedure by which we can build each 4-cycle of $K_n - I$ into $(n-1)/3$ 4-cycles in $K_n^{(3)}$. Thus, following [15], we define a *choice design* on a given 3-uniform hypergraph H to be a choice of one vertex from each 3-edge of H to represent that 3-edge. Given two vertices a and b, we define $ab*$ to be the set of all 3-edges containing both a and b for which neither a nor b is the representative.

The following grouping of the elements of the vertex set $V = \mathbb{Z}_n$ of either $K_n^{(3)}$ or K_n will be used in the construction of a suitable choice design. Group the elements of V into $n/2$ groups $G_i = \{i, n/2+i\}$ for $0 \leq i \leq n/2-1$. The notation $G(a)$ will denote the subscript of the group containing element a, that is, $G(a) = i$ if $a \in G_i$. Let $\binom{V}{3}$ denote the set of all 3-edges of $K_n^{(3)}$ and define two types of 3-edges in $\binom{V}{3}$:

Type 1: 3-edges abc in which a and b are in the same group and c is in a different group; and

Type 2: 3-edges abc in which a, b, and c are all in different groups.

The following lemma describes a choice design on $K_n^{(3)}$ in which given b and c in different groups, the set $bc*$ contains $(n-1)/3$ elements.

Lemma 2.1 *For every positive integer* $n \equiv 4 \pmod 6$*, there exists a choice design on* $K_n^{(3)}$ *with vertex set* $V = \mathbb{Z}_n$ *grouped into sets* $G_i = \{i, i+n/2\}$ *for* $i = 0, 1, \ldots, n/2-1$ *such that*

1. *if* $abc \in \binom{V}{3}$ *and* a *and* b *are in the same group, then* c *is not chosen as the representative of this* 3*-edge; and*
2. *given* b *and* c *in different groups, the set* $bc*$ *contains* $(n-1)/3$ *elements.*

PROOF: Let $n \equiv 4 \pmod 6$ be a positive integer, say $n = 6k+4$ for some positive integer k. We construct a choice design on $K_n^{(3)}$ and then show it satisfies the two conditions given above.

Let $V = \mathbb{Z}_n$ be the vertex set of $K_n^{(3)}$ and let $G_i = \{i, i+n/2\}$ for $i = 0, 1, \ldots, n/2-1$.

Choosing representatives for 3*-edges of Type 1:* Order the 3-edge abc of Type 1 as $a, a+n/2, b$ so that $a, a+n/2 \in G_i$ for some i with $0 \le i \le n/2-1$. Then, choose the representative for this 3-edge as follows:

- if $b < a$, choose $a + n/2$;
- if $a < b < a + n/2$, choose a; and
- if $b > a + n/2$ choose $a + n/2$.

Choosing representatives for 3*-edges of Type 2:* Order the 3-edge abc so that $G(a) < G(b) < G(c)$. Then, choose the representative for this 3-edge as follows:

- if $a + b + c \equiv 0 \pmod 3$, choose a;
- if $a + b + c \equiv 1 \pmod 3$, choose b; and
- if $a + b + c \equiv 2 \pmod 3$, choose c.

We must now prove that this is indeed the desired choice design. Clearly, Condition (1) follows immediately by the choice of representatives for Type 1 edges. We now wish to show Condition (2) holds. Let b and c belong to different groups and without loss of generality assume $b < c$. We wish to show that $bc*$ contains $(n-1)/3$ elements. Consider first the Type 1 edges containing b and c. There are only two: $bc(b+n/2)$ or $bc(c+n/2)$ where all arithmetic is done modulo n. If $c < c+n/2$, then $c+n/2$ represents $bc(c+n/2)$ and b represents $bc(b+n/2)$. If $c > c+n/2$, then rewrite $bc(c+n/2)$ as $(c-n/2)bc$. If $c-n/2 < b < c$, then $c-n/2$ represents this edge and b represents $bc(b+n/2)$. On the other hand, if $b < c-n/2$, then c represents $(c-n/2)bc$ and $b+n/2$ represents $bc(b+n/2)$. In all cases, we conclude that if b and c are in different groups, then exactly one representative is added to $bc*$ from the Type 1 edges.

Now suppose abc is a Type 2 edge. With b and c fixed, the 3-edges abc of Type 2 are created by allowing a to run through each of the two elements in the remaining $3k$ groups, giving $6k$ possible choices for a. Thus, exactly $2k$ times a will be chosen as the representative, $2k$ times b will be chosen at the representative, and $2k$ times c will be chosen as the representative. Hence, $bc*$ will contain $2k + 1 = (n - 1)/3$ elements. □

We now show that $K_n^{(3)}$ decomposes into 4-cycles when $n \equiv 4 \pmod 6$.

Theorem 2.2 *For each positive integer $n \geq 10$ with $n \equiv 4 \pmod 6$, the complete 3-uniform hypergraph $K_n^{(3)}$ decomposes into 4-cycles.*

Proof: Let $n \geq 10$ be a positive integer with $n \equiv 4 \pmod 6$. Then, $4 \mid [n(n-2)/2]$ and it is well-known that $K_n - I$ decomposes into 4-cycles. Hence let $V(K_n) = \mathbb{Z}_n$ and decompose $K_n - I$ into 4-cycles. Consider the choice design on $K_n^{(3)}$ given by Lemma 2.1. Let (x_1, x_2, x_3, x_4) be a 4-cycle in the decomposition of $K_n - I$, and let y_j^i represent each of the $(n - 1)/3$ representatives in $x_j x_{j+1}*$, that is, $x_j x_{j+1}* = \{y_j^1, y_j^2, \ldots, y_j^{(n-1)/3}\}$, for $j = 1, 2, 3, 4$ and where all arithmetic is done modulo 4. Then, for $i = 1, 2, \ldots, (n - 1)/3$, the 4-cycle (x_1, x_2, x_3, x_4) in the decomposition of $K_n - I$ will give rise to $(n-1)/3$ edge-disjoint 4-cycles $(x_1 y_1^i x_2, x_2 y_2^i x_3, x_3 y_3^i x_4, x_4 y_4^i x_1)$ in $K_n^{(3)}$. Thus, the $n(n - 2)/8$ edge-disjoint 4-cycles in the decomposition of $K_n - I$ will give rise to $n(n - 1)(n - 2)/24$ edge-disjoint 4-cycles in $K_n^{(3)}$. □

3 The $n \equiv 0 \pmod 6$ case

In this section, we consider the case when $n \equiv 0 \pmod 6$. We begin with a few special cases.

Lemma 3.1 *The hypergraph $K_6^{(3)}$ decomposes into 4-cycles.*

Proof: A decomposition of $K_6^{(3)}$ into 4-cycles can be found in the Appendix. □

Define the 3-uniform hypergraph H_m of order $2m$ as follows: Let $V(H_m) = \{0, 1, \ldots, 2m - 1\}$ grouped as $G_0 = \{0, 2, \ldots 2m - 2\}$ and $G_1 = \{1, 3, \ldots, 2m - 1\}$. Let $E(H_m)$ be the set of all 3-edges abc such that a, b, and c are not all from the same group, that is, at least one of a, b, c is an element of G_0 and at least one of a, b, c is an element of G_1. Note that $|E(H_m)| = m^2(m - 1)$.

We now require a decomposition of H_6 of order 12 into 4-cycles.

Lemma 3.2 *The 3-uniform hypergraph H_6, as defined above, decomposes into 4-cycles.*

PROOF: Note that H_6 is the 3-uniform hypergraph with $V(H_6) = \{0, 1, \ldots, 11\}$ groups as $G_0 = \{0, 2, 4, 6, 8, 10\}$ and $G_1 = \{1, 3, 5, 7, 9, 11\}$, every 3-edge abc has at least one element of G_0 and at least one element of G_1. Note also that $|E(H_6)| = 180$. First, $K_{6,6}$ decomposes into 9 edge-disjoint 4-cycles and we seek a decomposition of H_6 into 45 edge-disjoint 4-cycles. Thus, we want to define a choice design on H_6 so that $bc*$ is empty if b and c are in the same group or $bc*$ has 5 elements if b and c are in different groups. Such a choice design is given in the Appendix.

As in the proof of Theorem 2.2, each 4-cycle in the decomposition of $K_{6,6}$ with partite sets $\{0, 2, \ldots, 10\}$ and $\{1, 3, \ldots, 11\}$ will give rise to five edge-disjoint 4-cycles in H_6. Thus, the desired conclusion follows. □

Next, define the 3-uniform hypergraph H'_m of order $3m$ as follows: Let $V(H'_m) = \{0, 1, \ldots, 3m-1\}$ and let $E(H'_m)$ be the set of all 3-edges abc such that $a \in \{0, 1, \ldots, m-1\}$, $b \in \{m, m+1, \ldots, 2m-1\}$, and $c \in \{2m, 2m+1, \ldots, 3m-1\}$. Note that $|E(H'_m)| = m^3$. We now show that H'_m decomposes into 4-cycles when m is even.

Lemma 3.3 *For each positive integer $k \geq 1$, the 3-uniform hypergraph H'_{2k}, as defined above, decomposes into 4-cycles.*

PROOF: Note that $V(H'_{2k}) = \{0, 1, \ldots, 6k-1\}$ and that $E(H'_{2k})$ is the set of all 3-edges abc such that $a \in \{0, 1, \ldots, 2k-1\}$, $b \in \{2k, 2k+1, \ldots, 4k-1\}$, and $c \in \{4k, 4k+1, \ldots, 6k-1\}$. Note that $|E(H'_{2k})| = 8k^3$ and thus we seek to decompose H'_{2k} into $2k^3$ edge-disjoint 4-cycles. Recall that $K_{2k,2k}$, with partite sets $\{0, 1, \ldots, 2k-1\}$ and $\{2k, 2k+1, \ldots, 4k-1\}$, decomposes into 4-cycles by [14]. For each 4-cycle (x_1, x_2, x_3, x_4) of $K_{2k,2k}$, construct $2k$ edge-disjoint 4-cycles $(x_1(4k+i)x_2, x_2(4k+i)x_3, x_3(4k+i)x_4, x_4(4k+i)x_1)$ of H'_{2k} where $0 \leq i \leq 2k-1$. Thus, the k^2 edge-disjoint 4-cycles in $K_{2k,2k}$ will give rise to $2k^3$ edge-disjoint 4-cycles in H'_{2k}. □

We now have all the tools necessary to show that the complete 3-uniform hypergraph $K_n^{(3)}$ decomposes into 4-cycles when $n \equiv 0 \pmod 6$ with $n \geq 6$.

Theorem 3.4 *For each positive integer $n \geq 6$ with $n \equiv 0 \pmod 6$, the complete 3-uniform hypergraph $K_n^{(3)}$ decomposes into 4-cycles.*

PROOF: Let $n \geq 6$ with $n \equiv 0 \pmod 6$, say $n = 6k$ for some positive integer k. The case $k = 1$ is given in Lemma 3.1, and thus we may assume $k > 1$. Now, we may think of $K_n^{(3)}$ as k copies of $K_6^{(3)}$ with a copy of H_6 between any two of these copies of $K_6^{(3)}$, giving $k(k-1)/2$ copies of H_6, and a copy of H'_6 between any three of these copies of $K_6^{(3)}$, giving $k(k-1)(k-2)/6$ copies of H'_6. Since H'_6, H_6 and $K_6^{(3)}$ all decompose into 4-cycles, the desired result follows. □

4 The $n \equiv 2 \pmod 6$ case

In this section, we consider the case when $n \equiv 2 \pmod 6$. We begin with a special case.

Lemma 4.1 *The hypergraph $K_8^{(3)}$ decomposes into 4-cycles.*

PROOF: A decomposition of $K_8^{(3)}$ into 4-cycles can be found in the Appendix. □

When $n \equiv 2 \pmod 6$, say $n = 6k + 2$, it is helpful to think of the vertex set $V(K_n^{(3)})$ of $K_n^{(3)}$ as

$$\{\infty_1, \infty_2\} \cup \left(\bigcup_{0 \le i \le k-1} \{6i, 6i+1, \ldots, 6i+5\} \right).$$

Then, a 3-edge has one of the following forms:

1. $\infty_1\infty_2 c$ where $c \in \{6\ell, 6\ell+1, \ldots, 6\ell+5\}$ for some $0 \le \ell \le k-1$;
2. $\infty_j bc$ where $j \in \{1, 2\}$ and $b, c \in \{6\ell, 6\ell+1, \ldots, 6\ell+5\}$ for some $0 \le \ell \le k-1$;
3. $\infty_j bc$ where $j \in \{1, 2\}$, $b \in \{6\ell_1, 6\ell_1+1, \ldots, 6\ell_1+5\}$ and $c \in \{6\ell_2, 6\ell_2+1, \ldots, 6\ell_2+5\}$ where $0 \le \ell_1 < \ell_2 \le k-1$;
4. abc where $a, b, c \in \{6\ell, 6\ell+1, \ldots, 6\ell+5\}$ for some $0 \le \ell \le k-1$;
5. abc where $a, b \in \{6\ell_1, 6\ell_1+1, \ldots, 6\ell_1+5\}$ and $c \in \{6\ell_2, 6\ell_2+1, \ldots, 6\ell_2+5\}$ for some $0 \le \ell_1, \ell_2 \le k-1$ with $\ell_1 \ne \ell_2$; and
6. abc where $a \in \{6\ell_1, 6\ell_1+1, \ldots, 6\ell_1+5\}$, $b \in \{6\ell_2, 6\ell_2+1, \ldots, 6\ell_2+5\}$ and $c \in \{6\ell_3, 6\ell_3+1, \ldots, 6\ell_3+5\}$ where $0 \le \ell_1 < \ell_2 < \ell_3 \le k-1$.

Note that, for a fixed value of ℓ, the hypergraph with edges of types (1), (2), and (4) above is isomorphic to $K_8^{(3)}$ which decomposes into 4-cycles by Lemma 4.1. Next, the hypergraph with edges of type (5) for fixed values of ℓ_1 and ℓ_2 is isomorphic to the hypergraph H_6 given in Section 3 which decomposes into 4-cycles by Lemma 3.2, and the hypergraph with edges of type (6) for fixed values of ℓ_1, ℓ_2 and ℓ_3 is the hypergraph H_6' given in Section 3 which decomposes into 4-cycles by Lemma 3.3. Thus, it remains to show that the hypergraph with edges of type (3) for fixed values of ℓ_1 and ℓ_2 decomposes into 4-cycles.

Define the hypergraph H_m'' of order $2m+1$ as follows: let $V(H_m'') = \{\infty, 0, 1, \ldots, 2m-1\}$ and let $E(H_m'')$ be the set of all 3-edges ∞ab where $a \in \{0, 1, \ldots, m-1\}$ and $b \in \{m, m+1, \ldots, 2m-1\}$. Note that $|E(H_m'')| = m^2$ and that for fixed values of ℓ_1 and ℓ_2, the hypergraph with edges of type (3) above is isomorphic to H_6''. We now show that H_m'' decomposes into 4-cycles when m is even.

Lemma 4.2 *For each positive integer* $k \geq 1$*, the* 3*-uniform hypergraph* H''_{2k}*, as defined above, decomposes into* 4*-cycles.*

PROOF: Let H''_{2k} be the hypergraph with $V(H''_{2k}) = \{\infty, 0, 1, \ldots, 4k-1\}$ and $E(H''_{2k})$ is the set of all 3-edges ∞ab where $a \in \{0, 1, \ldots, 2k-1\}$ and $b \in \{2k, 2k+1, \ldots, 4k-1\}$. Note that $|E(H''_{2k})| = 4k^2$. Now $K_{2k,2k}$ has $4k^2$ edges and decomposes into k^2 edge-disjoint 4-cycles by [14], say (x_1, x_2, x_3, x_4) is one such 4-cycle where the partite sets of $K_{2k,2k}$ are $\{0, 1, \ldots, 2k-1\}$ and $\{2k, 2k+1, \ldots, 4k-1\}$. Thus, for each 4-cycle of $K_{2k,2k}$, construct the 4-cycle $(x_1\infty x_2, x_2\infty x_3, x_3\infty x_4, x_4\infty x_1)$ of H''_{2k}. □

We now have all the tools necessary to show that the complete 3-uniform hypergraph $K_n^{(3)}$ decomposes into 4-cycles when $n \equiv 2 \pmod 6$ with $n \geq 8$.

Theorem 4.3 *For each positive integer* $n \geq 8$ *with* $n \equiv 2 \pmod 6$*, the complete* 3*-uniform hypergraph* $K_n^{(3)}$ *decomposes into* 4*-cycles.*

PROOF: Let $n \geq 8$ with $n \equiv 2 \pmod 6$, say $n = 6k + 2$ for some positive integer k. The case $k = 1$ is given in Lemma 4.1, and thus we may assume that $k > 1$. Now, we may think of $K_n^{(3)}$ as k copies of $K_8^{(3)}$, $k(k-1)/2$ copies of the hypergraph H_6 given in Section 3, $k(k-1)$ copies of the hypergraph H''_6 given above, and $k(k-1)(k-2)/6$ copies of the hypergraph H'_6 given in Section 3. Since $K_8^{(3)}, H_6, H'_6$ and H''_6 all decompose into 4-cycles by Lemmas 4.1, 3.2, 3.3, and 4.2, the desired result follows. □

5 The $n \equiv 1 \pmod 8$ case

In this section, we consider the case when $n \equiv 1 \pmod 8$. We begin with a special case.

Lemma 5.1 *The hypergraph* $K_9^{(3)}$ *decomposes into* 4*-cycles.*

PROOF: A decomposition of $K_9^{(3)}$ into 4-cycles can be found in the Appendix. □

When $n \equiv 1 \pmod 8$, say $n = 8k + 1$, it is helpful to think of the vertex set $V(K_n^{(3)})$ of $K_n^{(3)}$ as

$$\{\infty\} \cup \left(\bigcup_{0 \leq i \leq k-1} \{8i, 8i+1, \ldots, 8i+7\} \right).$$

Then, a 3-edge abc has one of the following forms:

1. ∞bc where $b, c \in \{8\ell, 8\ell+1, \ldots, 8\ell+7\}$ for some $0 \leq \ell \leq k-1$;

2. ∞bc where $b \in \{8\ell_1, 8\ell_1+1, \ldots, 8\ell_1+7\}$ and $c \in \{8\ell_2, 8\ell_2+1, \ldots, 8\ell_2+7\}$ where $0 \leq \ell_1 < \ell_2 \leq k-1$;

3. abc where $a, b, c \in \{8\ell, 8\ell+1, \ldots, 8\ell+7\}$ for some $0 \leq \ell \leq k-1$;

4. abc where $a, b \in \{8\ell_1, 8\ell_1+1, \ldots, 8\ell_1+7\}$ and $c \in \{8\ell_2, 8\ell_2+1, \ldots, 8\ell_2+7\}$ for some $0 \leq \ell_1, \ell_2 \leq k-1$ with $\ell_1 \neq \ell_2$; and

5. abc where $a \in \{8\ell_1, 8\ell_1+1, \ldots, 8\ell_1+7\}$, $b \in \{8\ell_2, 8\ell_2+1, \ldots, 8\ell_2+7\}$ and $c \in \{8\ell_3, 8\ell_3+1, \ldots, 8\ell_3+7\}$ where $0 \leq \ell_1 < \ell_2 < \ell_3 \leq k-1$.

Note that, for a fixed value of ℓ, the hypergraph with edges of types (1)and (3) above is isomorphic to $K_9^{(3)}$ which decomposes into 4-cycles by Lemma 5.1. Next, the hypergraph with edges of type (5) for fixed values of ℓ_1, ℓ_2 and ℓ_3 is the hypergraph H_8', given in Section 3 which decomposes into 4-cycles by Lemma 3.3 and the hypergraph with edges of type (2) for fixed values of ℓ_1 and ℓ_2 is the hypergraph H_8'' given in Section 4 which decomposes into 4-cycles by Lemma 4.2. The hypergraph with edges of type (4) for fixed values of ℓ_1 and ℓ_2 is the hypergraph H_8 defined in Section 3, and it remains to show that this hypergraph decomposes into 4-cycles.

Lemma 5.2 *The 3-uniform hypergraph H_8 decomposes into 4-cycles.*

PROOF: Note that H_8 is the 3-uniform hypergraph with $V(H_8) = \{0, 1, \ldots, 15\}$ groups as $G_0 = \{0, 2, 4, 6, 8, 10, 12, 14\}$ and $G_1 = \{1, 3, 5, 7, 9, 11, 13, 15\}$, every 3-edge abc has at least one element of G_0 and at least one element of G_1. Note also that $|E(H_8)| = 448$. First, $K_{8,8}$ decomposes into 16 edge-disjoint 4-cycles and we seek a decomposition of H_8 into 112 edge-disjoint 4-cycles. Thus, we want to define a choice design on H_8 so that $bc*$ is empty if b and c are in the same group or $bc*$ has 7 elements if b and c are in different groups. Such a choice design is given in the Appendix.

As in the proof of Theorem 2.2, each 4-cycle in the decomposition of $K_{8,8}$ with partite sets $\{0, 2, \ldots, 10, 12, 14\}$ and $\{1, 3, \ldots, 11, 13, 15\}$ will give rise to 7 edge-disjoint 4-cycles in H_8. Thus, the desired conclusion follows. □

We now have all the tools necessary to show that the complete 3-uniform hypergraph $K_n^{(3)}$ decomposes into 4-cycles when $n \equiv 1 \pmod 8$ with $n \geq 9$.

Theorem 5.3 *For each positive integer $n \geq 9$ with $n \equiv 1 \pmod 8$, the complete 3-uniform hypergraph $K_n^{(3)}$ decomposes into 4-cycles.*

PROOF: Let $n \geq 8$ with $n \equiv 1 \pmod 8$, say $n = 8k+1$ for some positive integer k. The case $k = 1$ is given in Lemma 5.1, and thus we may assume that $k > 1$. Now, we may think of $K_{8k+1}^{(3)}$ as k copies of $K_9^{(3)}$, $k(k-1)/2$ copies of the hypergraph H_8, $k(k-1)/2$ copies of the hypergraph H_8'', and $k(k-1)(k-2)/6$ copies of the hypergraph H_8'. Since $K_9^{(3)}, H_8, H_8'$ and H_8'' all decompose into 4-cycles by Lemmas 5.1, 5.2, 3.3, and 4.2, the desired result follows. □

6 Appendix

Let $V(K_6^{(3)})$ be $\{0,1,2,3,4,5\}$. Then the following five 4-cycles decompose $K_6^{(3)}$:

(132, 243, 354, 421), (143, 325, 530, 041), (125, 502, 230, 051), (130, 024, 415, 531), (210, 034, 405, 542)

Let $V(K_8^{(3)})$ be $\{0,1,2,3,4,5,6,7\}$. Then the following 14 4-cycles decompose $K_8^{(3)}$:

(041, 162, 203, 340), (051, 102, 213, 370), (072, 214, 416, 640), (062, 204, 426, 630), (045, 502, 217, 740), (065, 512, 237, 760),
(013, 305, 516, 601), (153, 325, 526, 671), (154, 465, 507, 701), (174, 425, 517, 731), (314, 427, 726, 613), (324, 437, 736, 623),
(354, 475, 576, 643), (527, 746, 653, 375).

Let $V(K_9^{(3)})$ be $\{0,1,2,3,4,5,6,7,8\}$. Then the following 21 4-cycles decompose $K_9^{(3)}$:

(021, 128, 803, 310), (132, 230, 054, 401), (243, 341, 125, 502), (354, 402, 216, 613), (425, 583, 317, 704), (506, 604, 408, 805),
(657, 715, 570, 086), (728, 816, 651, 187), (870, 017, 742, 208), (061, 158, 813, 370), (142, 260, 043, 461), (253, 351, 145, 526),
(384, 462, 276, 623), (465, 573, 327, 714), (536, 684, 418, 825), (637, 785, 510, 036), (738, 836, 671, 127), (810, 067, 752, 238),
(586, 643, 347, 745), (687, 764, 428, 826), (748, 845, 530, 027).

The Representatives in a Choice Design on H_6 with $|bc*| = 5$ for all $b \in \{0,2,4,6,8,10\}$ and $c \in \{1,3,5,7,9,11\}$:

$0\,1* = \{2,6,10,5,9\}$	$0\,3* = \{4,8,5,9,11\}$	$0\,5* = \{4,8,1,7,11\}$
$0\,7* = \{2,8,1,3,9\}$	$0\,9* = \{4,8,10,5,11\}$	$0\,11* = \{6,8,10,1,7\}$
$2\,1* = \{6,10,3,7,11\}$	$2\,3* = \{0,4,8,5,9\}$	$2\,5* = \{0,4,8,1,9\}$
$2\,7* = \{6,10,3,5,9\}$	$2\,9* = \{0,6,8,1,11\}$	$2\,11* = \{0,4,3,5,7\}$
$4\,1* = \{0,2,8,5,9\}$	$4\,3* = \{6,10,1,7,11\}$	$4\,5* = \{6,10,3,7,11\}$
$4\,7* = \{0,2,1,9,11\}$	$4\,9* = \{2,8,10,3,5\}$	$4\,11* = \{0,8,10,1,9\}$
$6\,1* = \{4,8,3,7,11\}$	$6\,3* = \{0,2,8,5,9\}$	$6\,5* = \{0,2,10,1,9\}$
$6\,7* = \{0,4,8,3,5\}$	$6\,9* = \{0,4,1,7,11\}$	$6\,11* = \{2,4,3,5,7\}$
$8\,1* = \{0,2,3,7,11\}$	$8\,3* = \{4,10,5,7,11\}$	$8\,5* = \{4,6,10,1,9\}$
$8\,7* = \{2,4,10,5,9\}$	$8\,9* = \{6,10,1,3,11\}$	$8\,11* = \{2,6,10,5,7\}$
$10\,1* = \{4,6,8,5,9\}$	$10\,3* = \{0,2,6,1,5\}$	$10\,5* = \{0,2,7,9,11\}$
$10\,7* = \{0,4,6,1,3\}$	$10\,9* = \{2,6,3,7,11\}$	$10\,11* = \{2,6,1,3,7\}$

The Representatives in a Choice Design on H_8 with $|bc*| = 7$ for all $b \in \{0, 2, \ldots, 14\}$ and $c \in \{1, 3, \ldots, 15\}$:

$0\,1* = \{2, 6, 10, 14, 5, 9, 15\}$ $0\,3* = \{4, 8, 12, 5, 9, 11, 15\}$ $0\,5* = \{4, 8, 12, 1, 7, 11, 15\}$
$0\,7* = \{2, 8, 14, 1, 3, 9, 13\}$ $0\,9* = \{4, 8, 10, 14, 5, 11, 15\}$ $0\,11* = \{6, 8, 10, 12, 1, 7, 13\}$
$0\,13* = \{4, 8, 12, 1, 3, 5, 9\}$ $0\,15* = \{2, 6, 10, 14, 7, 11, 13\}$

$2\,1* = \{6, 10, 14, 3, 7, 11, 15\}$ $2\,3* = \{0, 4, 8, 12, 5, 9, 13\}$ $2\,5* = \{0, 4, 8, 12, 1, 9, 15\}$
$2\,7* = \{6, 10, 14, 3, 5, 9, 13\}$ $2\,9* = \{0, 6, 8, 14, 1, 11, 13\}$ $2\,11* = \{0, 4, 12, 3, 5, 7, 15\}$
$2\,13* = \{0, 4, 12, 14, 1, 5, 11\}$ $2\,15* = \{6, 12, 14, 3, 7, 9, 13\}$

$4\,1* = \{0, 2, 8, 12, 5, 9, 13\}$ $4\,3* = \{6, 10, 14, 1, 7, 11, 15\}$ $4\,5* = \{6, 10, 12, 3, 7, 11, 15\}$
$4\,7* = \{0, 2, 12, 1, 9, 11, 13\}$ $4\,9* = \{2, 8, 10, 14, 3, 5, 13\}$ $4\,11* = \{0, 8, 10, 12, 1, 9, 15\}$
$4\,13* = \{6, 8, 10, 3, 5, 11, 15\}$ $4\,15* = \{0, 2, 8, 10, 1, 7, 9\}$

$6\,1* = \{4, 8, 12, 3, 7, 11, 15\}$ $6\,3* = \{0, 2, 8, 14, 5, 9, 13\}$ $6\,5* = \{0, 2, 10, 14, 1, 9, 13\}$
$6\,7* = \{0, 4, 8, 12, 3, 5, 15\}$ $6\,9* = \{0, 4, 12, 1, 7, 11, 13\}$ $6\,11* = \{2, 4, 14, 3, 5, 7, 15\}$
$6\,13* = \{0, 2, 8, 1, 7, 11, 15\}$ $6\,15* = \{4, 8, 10, 14, 3, 5, 9\}$

$8\,1* = \{0, 2, 12, 3, 7, 11, 15\}$ $8\,3* = \{4, 10, 14, 5, 7, 11, 15\}$ $8\,5* = \{4, 6, 10, 12, 1, 9, 13\}$
$8\,7* = \{2, 4, 10, 14, 5, 9, 15\}$ $8\,9* = \{6, 10, 12, 1, 3, 11, 13\}$ $8\,11* = \{2, 6, 10, 14, 5, 7, 13\}$
$8\,13* = \{2, 10, 12, 14, 1, 3, 7\}$ $8\,15* = \{0, 2, 12, 5, 9, 11, 13\}$

$10\,1* = \{4, 6, 8, 14, 5, 9, 13\}$ $10\,3* = \{0, 2, 6, 12, 1, 5, 13\}$ $10\,5* = \{0, 2, 14, 7, 9, 11, 15\}$
$10\,7* = \{0, 4, 6, 12, 1, 3, 13\}$ $10\,9* = \{2, 6, 14, 3, 7, 11, 15\}$ $10\,11* = \{2, 6, 12, 1, 3, 7, 15\}$
$10\,13* = \{0, 2, 6, 5, 9, 11, 15\}$ $10\,15* = \{2, 8, 12, 14, 1, 3, 7\}$

$12\,1* = \{0, 2, 10, 14, 3, 7, 11\}$ $12\,3* = \{4, 6, 8, 12, 5, 9, 11, 13\}$ $12\,5* = \{6, 10, 14, 1, 7, 11, 15\}$
$12\,7* = \{0, 2, 8, 14, 3, 9, 15\}$ $12\,9* = \{0, 2, 4, 10, 1, 5, 13\}$ $12\,11* = \{6, 8, 14, 7, 9, 13, 15\}$
$12\,13* = \{4, 6, 10, 14, 1, 5, 7\}$ $12\,15* = \{0, 4, 6, 1, 3, 9, 13\}$

$14\,1* = \{4, 6, 8, 3, 7, 11, 13\}$ $14\,3* = \{0, 2, 10, 12, 5, 9, 15\}$ $14\,5* = \{0, 2, 4, 8, 1, 7, 13\}$
$14\,7* = \{4, 6, 10, 3, 9, 11, 13\}$ $14\,9* = \{6, 8, 12, 1, 5, 13, 15\}$ $14\,11* = \{0, 2, 4, 10, 3, 5, 9\}$
$14\,13* = \{0, 4, 6, 10, 3, 11, 15\}$ $14\,15* = \{4, 8, 12, 1, 5, 7, 11\}$

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