AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **71(2)** (2018), Pages 272–292

On the directed Oberwolfach Problem with equal cycle lengths: the odd case

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Abstract

We show that the complete symmetric digraph K^*_{2m} admits a resolvable decomposition into directed cycles of length m for all odd m, $5 \leq m \leq 49$. Consequently, K^*_n admits a resolvable decomposition into directed cycles of length m for all $n \equiv 0 \pmod{2m}$ and odd m, $5 \leq m \leq 49$.

1 Introduction

The *complete symmetric digraph* of order n, denoted K^*_n, is the digraph with n vertices, and with arcs (u, v) and (v, u) for each pair of distinct vertices u and v. In this paper, we are concerned with the problem of decomposing the complete symmetric digraph K^*_n into spanning subdigraphs, each a vertex-disjoint union of directed cycles of length m. Thus, we are interested in the following problem.

Problem 1.1 *Determine the necessary and sufficient conditions on m and n for the complete symmetric digraph K^*_n to admit a resolvable decomposition into directed m-cycles.*

In the design-theoretic literature, such decompositions have also been called Mendelsohn designs [8]. Problem 1.1 can also be viewed as the directed version

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of the well-known Oberwolfach Problem with uniform cycle lengths, which was completely solved in [2, 3, 9].

It is easily seen that K_n^* admits a resolvable decomposition into directed m-cycles only if m divides n, and this condition is obviously sufficient if $m = 2$. Problem 1.1 has also been solved previously for $m = 3$ [6] and for $m = 4$ [1, 4]: the necessary conditions are sufficient except for $(m, n) = (3, 6)$ and $(4, 4)$. More recently, two of the present authors showed the following.

Theorem 1.2 *[7] Let m and n be integers with $5 \leq m \leq n$. Then the following hold.*

1. *Let m be even, or m and n be both odd. Then there exists a resolvable decomposition of K_n^* into directed m-cycles if and only if m divides n and $(m, n) \neq (6, 6)$.*
2. *If there exists a resolvable decomposition of K_{2m}^* into directed m-cycles, then there exists a resolvable decomposition of K_n^* into directed m-cycles whenever $n \equiv 0 \pmod{2m}$.*

In the same paper, we also posed the following conjecture.

Conjecture 1.3 *[7] Let m be a positive odd integer. Then K_{2m}^* admits a resolvable directed m-cycle decomposition if and only if $m \geq 5$.*

Observe that proving Conjecture 1.3 (which appears to be difficult) would complete the solution to Problem 1.1. In this paper, we confirm the above conjecture for all $m \leq 49$. Thus, we prove the following result.

Theorem 1.4 *Let m be an odd integer, $5 \leq m \leq 49$. Then K_{2m}^* admits a resolvable decomposition into directed m-cycles.*

Except for the smallest case $m = 5$, the above theorem is proved using a general construction that is complemented with a computational result. We expect that with more computing power, this approach can be used to extend our result to even larger values of m.

Theorems 1.2 and 1.4 immediately yield the following.

Corollary 1.5 *Let m be an odd integer, $5 \leq m \leq 49$. Then K_n^* admits a resolvable decomposition into directed m-cycles whenever $n \equiv 0 \pmod{2m}$.*

2 Preliminaries

In this paper, the term *digraph* will mean a directed graph with no loops or multiple arcs. For a digraph $D = (V, A)$, a subset $V' \subseteq V$ of its vertex set, and subset

$A' \subseteq A$ of its arc set, the symbols $D[V']$ and $D - A'$ will denote the subdigraph of D induced by V', and the subdigraph obtained from D by deleting all arcs in A', respectively. If D is a spanning subdigraph of the complete symmetric digraph K_n^* and $A' \subseteq A(K_n^*) - A(D)$, then $D + A'$ will denote the digraph $(V(D), A(D) \cup A')$. That is, $D + A'$ is obtained from the digraph D by adjoining the (new) arcs from the set A'.

A *decomposition* of a digraph D is a collection $\{H_1, H_2, \dots, H_k\}$ of subdigraphs of D whose arc sets partition the arc set of D. If each of the digraphs H_i is isomorphic to a digraph H, then $\{H_1, H_2, \dots, H_k\}$ is called an *H-decomposition* of the digraph G.

A *resolution class* (or *parallel class*) of a decomposition $\mathcal{D} = \{H_1, H_2, \dots, H_k\}$ of D is a subset $\{H_{i_1}, H_{i_2}, \dots, H_{i_t}\}$ of $\mathcal{D}$ with the property that the vertex sets of the digraphs $H_{i_1}, H_{i_2}, \dots, H_{i_t}$ partition the vertex set of D. A decomposition is called *resolvable* if it can be partitioned into resolution classes.

By $\vec{C}_m$ we shall denote the directed cycle of length m. The terms $\vec{C}_m$-decomposition and resolvable $\vec{C}_m$-decomposition will be abbreviated as $\vec{C}_m$-D and R$\vec{C}_m$-D, respectively.

For a positive integer m and $S \subseteq \mathbb{Z}_m^*$, the digraph with vertex set $\mathbb{Z}_m$ and arc set $\{(i, i+d) : i \in \mathbb{Z}_m, d \in S\}$, denoted $\mathrm{Circ}(m; S)$, is called the *directed circulant of order m with connection set S*. (Note that the symbol $\mathrm{Circ}(m; S)$ will be used only for directed circulants.)

A well-known result by Bermond et al. [5] shows that every 4-regular connected Cayley graph on a finite abelian group can be decomposed into two Hamilton cycles. The following corollary will be an important ingredient in our constructions.

Lemma 2.1 *Let m be a positive integer and $S \subseteq \mathbb{Z}_m^*$. Assume S can be partitioned into sets of the form*

- $\{d\}$ *such that* $\gcd(d, m) = 1$, *and*
- $\{\pm d, \pm d'\}$ *of cardinality four such that* $\gcd(d, d', m) = 1$.

Then the directed circulant $\mathrm{Circ}(m; S)$ *can be decomposed into directed m-cycles.*

Proof. By the assumption, $\mathrm{Circ}(m; S)$ can be decomposed into directed circulants of the form $\mathrm{Circ}(m; S')$, where either $S' = \{d\}$ for some $d \in \mathbb{Z}_m^*$ such that $\gcd(d, m) = 1$, or $S' = \{\pm d, \pm d'\}$ for some $d, d' \in \mathbb{Z}_m^*$ such that $\gcd(d, d', m) = 1$ and $|\{\pm d, \pm d'\}| = 4$. In the former case, $\mathrm{Circ}(m; S')$ itself is a directed m-cycle. In the latter case, let G' be the undirected graph obtained from $\mathrm{Circ}(m; S')$ by replacing each pair of opposite arcs with an undirected edge. Then G' is a 4-regular (undirected) circulant, which is connected because $\gcd(d, d', m) = 1$. Thus, G' is a connected Cayley graph on a cyclic group, and hence by [5] admits a decomposition into two Hamilton cycles, say C_1 and C_2. Taking two copies of each of C_1 and C_2, and directing the two copies in opposite ways results in a decomposition of $\mathrm{Circ}(m; S')$ into four directed m-cycles. Hence $\mathrm{Circ}(m; S)$ can be decomposed into directed m-cycles. □

3 Results

Lemma 3.1 *There exists a $R\vec{C}_5$-D of K_{10}^*.*

PROOF. Label the vertices of K_{10}^* by $x_0, x_1, \dots, x_9$. It can be verified that the following resolution classes (obtained by a computer search) form a $R\vec{C}_5$-D of K_{10}^*.

$$\begin{aligned}
R_0 &= \{x_0x_1x_2x_3x_4x_0, x_5x_6x_7x_8x_9x_5\}\\
R_1 &= \{x_0x_2x_1x_3x_5x_0, x_4x_6x_8x_7x_9x_4\}\\
R_2 &= \{x_0x_3x_1x_4x_2x_0, x_5x_7x_6x_9x_8x_5\}\\
R_3 &= \{x_0x_4x_1x_5x_8x_0, x_2x_6x_3x_9x_7x_2\}\\
R_4 &= \{x_0x_5x_2x_8x_3x_0, x_1x_7x_4x_9x_6x_1\}\\
R_5 &= \{x_0x_6x_2x_5x_9x_0, x_1x_8x_4x_3x_7x_1\}\\
R_6 &= \{x_0x_7x_3x_8x_6x_0, x_1x_9x_2x_4x_5x_1\}\\
R_7 &= \{x_0x_8x_2x_9x_1x_0, x_3x_6x_4x_7x_5x_3\}\\
R_8 &= \{x_0x_9x_3x_2x_7x_0, x_1x_6x_5x_4x_8x_1\}
\end{aligned}$$

□

The rest of the proof of Theorem 1.4 is divided into two main cases, $m \not\equiv 0 \pmod 3$, which is dealt with in Proposition 3.4, and $m \equiv 0 \pmod 3$, which is considered in Proposition 3.6, as well as two small cases, $m = 11$ and $m = 9$, which require a modification of the general approach. All of these cases, however, have the following construction in common.

Construction 3.2 Let $m \geq 5$ be an odd integer, and write $m = 2k+1$. Let the vertex set of $D = K_{2m}^*$ be $X \cup Y$, where $X = \{x_0, x_1, \dots, x_{2k}\}$ and $Y = \{y_0, y_1, \dots, y_{2k}\}$. We shall call arcs of the form (x_i, x_{i+d}) and (y_i, y_{i+d}) arcs of *pure left* and *pure right difference d*, respectively, and arcs of the form (x_i, y_{i+d}) and (y_i, x_{i+d}) arcs of *mixed difference d*. All subscripts will be evaluated modulo $m = 2k + 1$.

Start by defining directed m-cycles

$$C_0 = x_0y_0x_1y_1x_2y_2 \dots x_kx_0 \qquad \text{and} \qquad C_0' = y_kx_{k+1}y_{k+1} \dots y_{2k}y_k.$$

Observe that cycles C_0 and C_0' jointly use up all arcs of the form (x_j, y_j) except (x_k, y_k), all arcs of the form (y_j, x_{j+1}) except (y_{2k}, x_0), and they also use the arcs (x_k, x_0) and (y_{2k}, y_k).

For $i \in \mathbb{Z}_m$, obtain C_i and C_i' from C_0 and C_0', respectively, by adding i to the subscripts of the vertices in X, and $2i$ to the subscripts of the vertices in Y. Observe that cycles C_i and C_i' jointly use up all arcs of the form (x_j, y_{j+i}) except (x_{k+i}, y_{k+2i}), all arcs of the form (y_j, x_{j+i+1}) except (y_{2k+2i}, x_i), and they also use the arcs (x_{k+i}, x_i) and (y_{2k+2i}, y_{k+2i}).

Next, form resolution classes

$$R_i = \{C_i, C_i'\}, \qquad \text{for } i \in \mathbb{Z}_m.$$

Observe that $R_0, \dots, R_{m-1}$ use up all arcs of pure left difference $k+1$, all arcs of pure right difference $k+1$, and all arcs of mixed differences except for the arcs

$$(x_{k+i}, y_{k+2i}) \quad \text{and} \quad (y_{2k+2i}, x_i) \quad \text{for all } i \in \mathbb{Z}_m. \tag{1}$$

Let L denote the subdigraph of D induced by the set of these leftover arcs. Then L contains all vertices of D, and decomposes into directed 2-paths of the form

$$y_{2k+2i}\ x_i\ y_{(2k+2i)+(k+2)}, \qquad \text{for all } i \in \mathbb{Z}_m, \tag{2}$$

that is, into directed $(y_j, y_{j+(k+2)})$-paths of length 2, for all $j \in \mathbb{Z}_m$. The union of these directed 2-paths is a directed $2m$-cycle if and only if $\gcd(k+2, 2k+1) = 1$, that is, if and only if $\gcd(3, 2k+1) = 1$. This case will be considered in Lemma 3.3 and Proposition 3.4. If, however, $\gcd(k+2, 2k+1) \neq 1$, then $3|m$ and the leftover digraph is composed of three disjoint directed cycles of length $\frac{2m}{3}$. This case will be covered in Lemma 3.5 and Proposition 3.6.

On the other hand, the digraph L can also be decomposed into directed 2-paths of the form

$$\begin{cases} x_{k+i}\ y_{k+2i}\ x_{(k+i)+\frac{3k+3}{2}} & \text{if } k \text{ is odd} \\ x_{k+i}\ y_{k+2i}\ x_{(k+i)+\frac{k+2}{2}} & \text{if } k \text{ is even} \end{cases}, \tag{3}$$

for all $i \in \mathbb{Z}_m$. In other words, L decomposes into directed (x_j, x_{j+p})-paths of length 2, for all $j \in \mathbb{Z}_m$, where

$$p = \begin{cases} \frac{3k+3}{2} & \text{if } k \text{ is odd} \\ \frac{k+2}{2} & \text{if } k \text{ is even} \end{cases}.$$

These observations will help us complete the constructions in Propositions 3.4 and 3.6. □

Next, we examine the case $m = 11$, which requires a modified construction, but serves as a good introduction to the general approach in the case $m \not\equiv 0 \pmod 3$ that will be described in Proposition 3.4.

Lemma 3.3 *There exists a $R\vec{C}_{11}$-D of K_{22}^*.*

Proof. With $m = 11$, adopt the notation and define resolution classes $R_0, \dots, R_{10}$ as in Construction 3.2. Since $11 \not\equiv 0 \pmod 3$, as shown above, the 22 leftover arcs of mixed differences in (1) form a directed 22-cycle

$$C = x_5 y_5 \dots x_5.$$

Using Observations (2) and (3), we decompose C into the following directed paths:

$$\begin{aligned} P_1 &= x_5 y_5 \dots x_6, & P_2 &= x_6 y_7, \\ P_3 &= y_7 x_4, & P_4 &= x_4 y_3, \\ P_5 &= y_3 \dots y_2, & P_6 &= y_2 x_7, \\ P_7 &= x_7 y_9, & P_8 &= y_9 x_5, \end{aligned}$$

where P_1 and P_5 are of length 10 and 6, respectively. Use the P_i for i odd to form the resolution class

$$R_{11} = \{P_1x_6x_5, P_3x_4x_7P_7y_9y_3P_5y_2y_7\}.$$

We shall use the P_i for i even in the next resolution class. Notice that in $D[Y]$ we have used all arcs of right pure difference 6 and two arcs — namely, (y_9, y_3) and (y_2, y_7) — of right pure difference 5. The remaining arcs of right pure difference 5 form a directed (y_3, y_2)-path Q'_1 of length 2, and a directed (y_7, y_9)-path Q'_2 of length 7. If we can find vertex-disjoint directed (x_7, x_4)-path of length 7 (call it Q_1) and (x_5, x_6)-path of length 2 (call it Q_2) in $D[X]$, then the next resolution class will be

$$R_{12} = \{P_2Q'_2P_8Q_2, P_4Q'_1P_6Q_1\}.$$

What will then remain of $D[Y]$ is a $\mathrm{Circ}(11; \{\pm 1, \pm 2, \pm 3, \pm 4\})$, which admits a $\vec{C}_{11}$-D by Lemma 2.1. It thus suffices to appropriately decompose the remaining subdigraph of $D[X]$. In particular, it suffices to find a set of differences $S \subseteq \mathbb{Z}^*_{11}$ such that

(X_1) $6 \notin S$, as left pure difference 6 has already been used;

(X_2) $3, 10 \in S$, as only arcs (x_6, x_5) and (x_4, x_7) of these left pure differences have already been used;

(X_3) $\mathrm{Circ}(11; \mathbb{Z}^*_{11} - S - \{6\})$ admits a decomposition into directed 11-cycles; and

(X_4) $\mathrm{Circ}(11; S) - \{(6, 5), (4, 7)\}$ admits a decomposition into directed 11-cycles, and vertex-disjoint directed paths: a $(5, 6)$-path of length 2 and a $(7, 4)$-path of length 7.

Such a set S was found using a computer search. The set S, as well as a suitable decomposition, is shown in the appendix. □

Proposition 3.4 *Let m be an odd integer such that $m \not\equiv 0 \pmod 3$, $m \geq 7$, and $m \neq 11$. Let $k = \frac{m-1}{2}$, and define parameters d, s'_i, t'_i, s_i, t_i (for $i = 1, 2$) as indicated below.*

Parameter \ Case	$k \equiv 0 \pmod 4$	$k \equiv 1 \pmod 4$	$k \equiv 2 \pmod 4$	$k \equiv 3 \pmod 4$
d	$(7k+8)/4$	$(5k+7)/4$	$(3k+6)/4$	$(k+5)/4$
s'_1	$k/4$	$(3k+1)/4$	$(5k+2)/4$	$(7k+3)/4$
s'_2	$(3k+4)/4$	$(k+3)/4$	$(7k+6)/4$	$(5k+5)/4$
t'_2	$(k-2)/2$	$(3k-1)/2$	$(k-2)/2$	$(3k-1)/2$
t_1	$(3k+2)/2$	$(k+1)/2$	$(3k+2)/2$	$(k+1)/2$

In addition, let $t'_1 = s_2 = k$, $s_1 = 2k-1$, and $t_2 = t'_2$.

Then $\gcd(d, m) = 1$, and hence for each $i = 1, 2$, there exists a unique $r_i \in \mathbb{Z}_m$ such that $s'_i + r_i d = t'_i$ (in $\mathbb{Z}_m$). Furthermore, define $a_i = (t_i, s_i)$ and $d^Y_i = s_i - t_i$ (in $\mathbb{Z}_m$).

Now assume there exists a set $S \subseteq \mathbb{Z}^*_m$ such that:

(Y_1) $k+1 \notin S$;

(Y_2) $d_1^Y, d_2^Y \in S$;

(Y_3) $\mathrm{Circ}(m; \mathbb{Z}_m^* - (S \cup \{k+1\}))$ admits a $\vec{C}_m$-D; and

(Y_4) $\mathrm{Circ}(m; S) - \{a_1, a_2\}$ admits a decomposition into directed m-cycles and two vertex-disjoint directed paths: an (s_1, t_1)-path of length r_1 and an (s_2, t_2)-path of length r_2.

Then K_{2m}^* admits a $R\vec{C}_m$-D.

Proof. Adopt the notation and define resolution classes $R_0, \dots, R_{m-1}$ as in Construction 3.2. As shown earlier, since $m \not\equiv 0 \pmod 3$, the $2m$ leftover arcs of mixed differences in (1) form a directed $2m$-cycle

$$C = y_k \dots x_k y_k.$$

Write $C = P_1 P_2 \dots P_8$ as a concatenation of directed paths such that P_1 is of length $m-1$, P_5 is of length $m-5$, and the rest are of length 1. Using Observations (2) and (3), it can be shown for each congruency class of k modulo 4 that the paths are

$$\begin{aligned} P_1 &= y_{s_2} \dots y_{t_2}, & P_2 &= y_{t_2} x_{s_1'}, \\ P_3 &= x_{s_1'} y_{t_1}, & P_4 &= y_{t_1} x_{s_2'}, \\ P_5 &= x_{s_2'} \dots x_{t_2'}, & P_6 &= x_{t_2'} y_{s_1}, \\ P_7 &= y_{s_1} x_{t_1'}, & P_8 &= x_{t_1'} y_{s_2}, \end{aligned}$$

where the parameters s_i, t_i, s_i', t_i' (for $i = 1, 2$) are as defined in the statement of the proposition. We use the P_i for i odd, together with 4 linking arcs (two of pure left, and two of pure right difference) to form the resolution class

$$R_m = \{P_1 y_{t_2} y_{s_2}, P_5 x_{t_2'} x_{s_1'} P_3 y_{t_1} y_{s_1} P_7 x_{t_1'} x_{s_2'}\}.$$

The linking arcs are:

$$(x_{t_2'}, x_{s_1'}) \text{ and } (x_{t_1'}, x_{s_2'}) \text{ of pure left difference } d = s_1' - t_2' = s_2' - t_1',$$

$$a_1 = (y_{t_1}, y_{s_1}) \text{ of pure right difference } d_1^Y = s_1 - t_1, \text{ and}$$

$$a_2 = (y_{t_2}, y_{s_2}) \text{ of pure right difference } d_2^Y = s_2 - t_2,$$

with d, d_1^Y, d_2^Y as defined in the statement of the proposition. Since $m \neq 11$, observe that none of these pure differences are equal to $k+1$ (which has already been used in $R_0, \dots, R_{m-1}$).

The P_i for i even will be used in the next resolution class as shown below. But first we verify that $\gcd(2k+1, d) = 1$. If $k \equiv 0 \pmod 4$, then $d = \frac{7k+8}{4}$. Using the Euclidean algorithm, we have $2k+1 = \frac{7k+8}{4} + \frac{k-4}{4}$ and $\frac{7k+8}{4} = 7\frac{k-4}{4} + 9$. Hence $\gcd(2k+1, \frac{7k+8}{4})$ divides 9, but since 3 does not divide $2k+1$, we must have $\gcd(2k+$

$1, \frac{7k+8}{4}) = 1$. Similarly it can be verified that $\gcd(2k+1, d) = 1$ for the remaining congruency classes of k modulo 4.

It follows that the arcs of pure left difference d form a directed m-cycle, and in particular, those that have not been used in R_m form a directed $(x_{s'_1}, x_{t'_1})$-path Q'_1 of length r_1 and a directed $(x_{s'_2}, x_{t'_2})$-path Q'_2 of length r_2, where r_1 and r_2 are as defined in the statement of the proposition.

Now let S be a subset of $\mathbb{Z}_m^*$ satisfying Conditions (Y_1)–(Y_4) of the proposition, and let Q_1 and Q_2 be the corresponding vertex-disjoint directed (y_{s_1}, y_{t_1})-path of length r_1 and (y_{s_2}, y_{t_2})-path of length r_2, respectively. We then let the next resolution class be

$$R_{m+1} = \{P_2Q'_1P_8Q_2, P_4Q'_2P_6Q_1\}.$$

All arcs of mixed differences have now been used in resolution classes $R_0, \ldots, R_{m+1}$. In $D[X]$, we have also used up all arcs of differences $k+1$ and d. Since $\gcd(2k+1, k+1) = \gcd(2k+1, d) = 1$, Lemma 2.1 now guarantees that the remaining subdigraph of $D[X]$ admits a $\vec{C}_m$-D.

In $D[Y]$, however, we have used up:

- all arcs of difference $k+1$;
- arcs a_1 and a_2 of differences d_1^Y and d_2^Y, respectively; and
- arcs used in the directed paths Q_1 and Q_2.

Assumptions (Y_1)–(Y_4) now guarantee that the remaining subdigraph of $D[Y]$ admits a $\vec{C}_m$-D. Finally, the directed m-cycles from the remaining subdigraphs of $D[X]$ and $D[Y]$ can be arranged into resolution classes that complete our R$\vec{C}_m$-D of K_{2m}^*. □

We now turn our attention to the case $m \equiv 0 \pmod 3$. As before, a small case ($m = 9$) requires a modified construction and will also serve as an introduction to the general approach.

Lemma 3.5 *There exists a R$\vec{C}_9$-D of K_{18}^*.*

Proof. Adopt the notation and construction of resolution classes $R_0, \ldots, R_8$ from Construction 3.2. The 18 leftover arcs of mixed differences from (1) now form three directed 6-cycles, which we write as a concatenation of directed paths of length 2 and linking arcs as follows:

$$\begin{aligned} C_{(1)} &= x_0y_5x_3y_2x_6y_8x_0 = P_1^X x_3y_2P_1^Y y_8x_0, \\ C_{(2)} &= x_1y_7x_4y_4x_7y_1x_1 = P_2^X x_4y_4P_2^Y y_1x_1, \\ C_{(3)} &= x_2y_0x_5y_6x_8y_3x_2 = P_3^X x_5y_6P_3^Y y_3x_2. \end{aligned}$$

We use the directed paths P_i^X, P_i^Y (for $i = 1, 2, 3$), together with 6 linking arcs of pure differences, to form the resolution class R_9:

$$R_9 = \{P_1^X x_3x_1P_2^X x_4x_2P_3^X x_5x_0, P_1^Y y_8y_6P_3^Y y_3y_4P_2^Y y_1y_2\}.$$

We have thus used the following linking arcs:

$$\begin{array}{lcll} b_1^X & = & (x_3, x_1) & \text{of pure left difference } d_1^X = 7, \\ b_2^X & = & (x_4, x_2) & \text{of pure left difference } d_1^X = 7, \\ b_3^X & = & (x_5, x_0) & \text{of pure left difference } d_2^X = 4, \\ b_1^Y & = & (y_1, y_2) & \text{of pure right difference } d_1^Y = 1, \\ b_2^Y & = & (y_3, y_4) & \text{of pure right difference } d_1^Y = 1, \\ b_3^Y & = & (y_8, y_6) & \text{of pure right difference } d_2^Y = 7. \end{array}$$

Note that none of these differences are equal to 5, which has been used in $R_0, \ldots, R_8$.

We have now used up all arcs of mixed differences except for the arcs (x_3, y_2), $(x_4, y_4), (x_5, y_6)$ and arcs $(y_8, x_0), (y_1, x_1), (y_3, x_2)$.

To form the resolution class R_{10}, we want to find three vertex-disjoint directed paths with sources x_0, x_1, x_2 and terminals x_3, x_4, x_5 using some of the remaining arcs in $D[X]$, and three vertex-disjoint directed paths with sources y_2, y_4, y_6 and terminals y_8, y_1, y_3 using some of the remaining arcs in $D[Y]$; these paths, together with all the remaining arcs of mixed differences, will form two vertex-disjoint directed 9-cycles. In particular, we can define

$$R_{10} = \{Q'_1 x_3 y_2 Q_1 y_3 x_2 Q'_2 x_4 y_4 Q_2 y_1 x_1, Q'_3 x_5 y_6 Q_3 y_8 x_0\}$$

as long as we have suitable directed paths

$$\begin{array}{ll} Q'_1: & (x_1, x_3)\text{-path of length } 1, \\ Q'_2: & (x_2, x_4)\text{-path of length } 1, \\ Q'_3: & (x_0, x_5)\text{-path of length } 4, \\ Q_1: & (y_2, y_3)\text{-path of length } 1, \\ Q_2: & (y_4, y_1)\text{-path of length } 2, \text{ and} \\ Q_3: & (y_6, y_8)\text{-path of length } 3 \end{array}$$

that use only hitherto unused arcs of pure differences. More precisely, it suffices to find sets $S^X, S^Y \subseteq \mathbb{Z}_9^*$ such that the following hold.

(X_1) $5 \notin S^X$, as left pure difference 5 has already been used;

(X_2) $4, 7 \in S^X$, as arcs $(x_3, x_1), (x_4, x_2), (x_5, x_0)$ have already been used;

(X_3) $\mathrm{Circ}(9; \mathbb{Z}_9^* - S^X - \{5\})$ admits a decomposition into directed 9-cycles; and

(X_4) $\mathrm{Circ}(9; S^X) - \{(3, 1), (4, 2), (5, 0)\}$ admits a decomposition into directed 9-cycles and pairwise vertex-disjoint directed $(1, 3)$-path of length 1, $(2, 4)$-path of length 1, and $(0, 5)$-path of length 4;

(Y_1) $5 \notin S^Y$, as right pure difference 5 has already been used;

(Y_2) $1, 7 \in S^Y$, as arcs $(y_1, y_2), (y_3, y_4), (y_8, y_6)$ have already been used;

(Y_3) $\mathrm{Circ}(9; \mathbb{Z}_9^* - S^Y - \{5\})$ admits a decomposition into directed 9-cycles; and

(Y_4) $\mathrm{Circ}(9; S^Y) - \{(1,2), (3,4), (8,6)\}$ admits a decomposition into directed 9-cycles and pairwise vertex-disjoint directed paths: a $(2,3)$-path of length 1, a $(4,1)$-path of length 2, and a $(6,8)$-path of length 3.

Such sets S^X and S^Y were found using a computer search. These sets, as well as suitable decompositions, are shown in the appendix. □

Proposition 3.6 *Let m be an odd integer such that $m \equiv 0 \pmod 3$, $m \geq 15$. Let $k = \frac{m-1}{2}$, and define parameters s_1 and t_1 as indicated in the table below.*

Parameter \ Case	$k \equiv 0 \pmod 4$	$k \equiv 1 \pmod 4$	$k \equiv 2 \pmod 4$	$k \equiv 3 \pmod 4$
s_1	$k/2$	$(3k+1)/2$	$k/2$	$(3k+1)/2$
t_1	$3k/4$	$(k-1)/4$	$(7k+2)/4$	$(5k+1)/4$

In addition, for $i = 1, 2$, let $s_{1+i} = s_1 + 2i$ and $t_{1+i} = t_1 + i$ (all evaluated in $\mathbb{Z}_m$).

Furthermore, define arcs:

$$b_1^X = (t_1, 1), \quad b_1^Y = (1, s_1), \quad c_1 = (t_1, 0),$$
$$b_2^X = (t_2, 2), \quad b_2^Y = (3, s_2), \quad c_2 = (t_2, 1),$$
$$b_3^X = (t_3, 0), \quad b_3^Y = (-1, s_3), \quad c_3 = (t_3, 2).$$

Now assume there exist sets $S^X, S^Y \subseteq \mathbb{Z}_m^*$ such that:

(X_1) $k+1, -t_1 \notin S^X$;

(X_2) $1 - t_1, -2 - t_1 \in S^X$;

(X_3) $\mathrm{Circ}(m; \mathbb{Z}_m^* - (S^X \cup \{k+1, -t_1\}))$ admits a $\vec{C}_m$-D;

(X_4) $\mathrm{Circ}(m; S^X) - \{b_1^X, b_2^X, b_3^X\} + \{c_1, c_2, c_3\}$ admits a $\vec{C}_m$-D;

(Y_1) $k+1 \notin S^Y$;

(Y_2) $s_1 - 1, s_1 + 5 \in S^Y$;

(Y_3) $\mathrm{Circ}(m; \mathbb{Z}_m^* - (S^Y \cup \{k+1\}))$ admits a $\vec{C}_m$-D; and

(Y_4) $\mathrm{Circ}(m; S^Y) - \{b_1^Y, b_2^Y, b_3^Y\}$ admits a decomposition into directed m-cycles and three pairwise vertex-disjoint directed paths: an $(s_1, -1)$-path of length $\frac{2m}{3} - 1$, an $(s_2, 3)$-path of some length $q \in \{1, \ldots, \frac{m}{3} - 3\}$, and an $(s_3, 1)$-path of length $\frac{m}{3} - 2 - q$.

Then K_{2m}^* admits a $R\vec{C}_m$-D.

Proof. Adopt the notation and construction of resolution classes $R_0, \dots, R_{m-1}$ from Construction 3.2. We have seen that, since $m \equiv 0 \pmod 3$, the $2m$ remaining arcs of mixed differences in (1) form three directed $\frac{2m}{3}$-cycles. Using Observations (2) and (3), we write each of these three cycles as a concatenation of directed paths of length $\frac{m}{3} - 1$ and linking arcs as follows:

$$\begin{aligned} C_{(1)} &= x_0 y_{k+1} \dots y_{-1} x_0 = P_1^X x_{t_1} y_{s_1} P_1^Y y_{-1} x_0, \\ C_{(2)} &= x_1 y_{k+3} \dots y_1 x_1 = P_2^X x_{t_2} y_{s_2} P_2^Y y_1 x_1, \\ C_{(3)} &= x_2 y_{k+5} \dots y_3 x_2 = P_3^X x_{t_3} y_{s_3} P_3^Y y_3 x_2. \end{aligned}$$

It can be verified that, for each congruency class of k modulo 4, the parameters s_i, t_i (for $i = 1, 2, 3$) have values as defined in the statement of the proposition.

We use the directed paths P_i^X, P_i^Y (for $i = 1, 2, 3$), together with 6 linking arcs of pure differences, to form the resolution class R_m:

$$R_m = \{P_1^X x_{t_1} x_1 P_2^X x_{t_2} x_2 P_3^X x_{t_3} x_0, P_1^Y y_{-1} y_{s_3} P_3^Y y_3 y_{s_2} P_2^Y y_1 y_{s_1}\}.$$

We have thus used the following linking arcs:

$$\begin{aligned} b_1^X &= (x_{t_1}, x_1) && \text{of pure left difference } d_1^X = 1 - t_1, \\ b_2^X &= (x_{t_2}, x_2) && \text{of pure left difference } d_1^X = 1 - t_1, \\ b_3^X &= (x_{t_3}, x_0) && \text{of pure left difference } d_2^X = -2 - t_1, \\ b_1^Y &= (y_1, y_{s_1}) && \text{of pure right difference } d_1^Y = s_1 - 1, \\ b_2^Y &= (y_3, y_{s_2}) && \text{of pure right difference } d_1^Y = s_1 - 1, \\ b_3^Y &= (y_{-1}, y_{s_3}) && \text{of pure right difference } d_2^Y = s_1 + 5. \end{aligned}$$

Note that, in all cases, none of these differences are equal to $k + 1$.

We have now used up all arcs of mixed differences except for the arcs (x_{t_i}, y_{s_i}) for $i = 1, 2, 3$, and arcs $(y_{-1}, x_0), (y_1, x_1), (y_3, x_2)$.

To form the resolution class R_{m+1}, we want to find three vertex-disjoint directed paths of appropriate lengths with sources x_0, x_1, x_2 and terminals $x_{t_1}, x_{t_2}, x_{t_3}$ using some of the remaining arcs in $D[X]$, and three vertex-disjoint directed paths with sources $y_{s_1}, y_{s_2}, y_{s_3}$ and terminals y_{-1}, y_1, y_3 using some of the remaining arcs in $D[Y]$; these paths, together with all the remaining arcs of mixed differences, will form two vertex-disjoint directed m-cycles.

It can be shown that $\gcd(m, t_1) = 3$. Namely, since $2k + 1 \equiv 0 \pmod 3$, we have $k \equiv 1 \pmod 3$, and hence we can easily verify that $t_1 \equiv 0 \pmod 3$ for each congruency class of k modulo 4. The Euclidean algorithm for $2k + 1$ and t_1 then results in remainder ± 3, confirming that $\gcd(2k+1, t_1) = 3$. Hence the following are indeed directed $(\frac{m}{3} - 1)$-paths in $D[X]$ with the required sources and terminals:

$$\begin{aligned} Q_1' &= x_0 x_{-t_1} x_{-2t_1} \dots x_{t_1}, \\ Q_2' &= x_1 x_{1-t_1} x_{1-2t_1} \dots x_{t_2}, \text{ and} \\ Q_3' &= x_2 x_{2-t_1} x_{2-2t_1} \dots x_{t_3}. \end{aligned}$$

Observe that these paths use all arcs of difference $d^X = -t_1$ except for arcs $c_1 = (x_{t_1}, x_0)$, $c_2 = (x_{t_2}, x_1)$, and $c_3 = (x_{t_3}, x_2)$.

Now let $S^X, S^Y \subseteq \mathbb{Z}_m^*$ be two sets satisfying Assumptions (X_1)–(X_4),(Y_1)–(Y_4) of the proposition. Furthermore, let Q_1, Q_2, Q_3 be the pairwise vertex-disjoint directed paths in $D[Y]$ whose existence is assured by Condition (Y_4), so that

$$Q_1 \text{ is a directed } (y_{s_1}, y_{-1})\text{-path of length } \tfrac{2m}{3} - 1,$$

$$Q_2 \text{ is a directed } (y_{s_2}, y_3)\text{-path of length } q, \text{ for some } q \in \{1, \ldots, \tfrac{m}{3} - 3\}, \text{ and}$$

$$Q_3 \text{ is a directed } (y_{s_3}, y_1)\text{-path of length } \tfrac{m}{3} - 2 - q.$$

We may then define our next resolution class as

$$R_{m+1} = \{Q_1' x_{t_1} y_{s_1} Q_1 y_{-1} x_0, Q_2' x_{t_2} y_{s_2} Q_2 y_3 x_2 Q_3' x_{t_3} y_{s_3} Q_3 y_1 x_1\}.$$

Now, all arcs of mixed differences have been used in resolution classes $R_1, \ldots, R_{m+1}$. In addition, we have also used up in $D[X]$:

- all arcs of difference $k + 1$;
- arcs b_i^X, for $i = 1, 2, 3$ (of differences $1 - t_1$ and $-2 - t_1$); and
- all arcs of difference $-t_1$ except c_i, for $i = 1, 2, 3$.

Assumptions (X_1)–(X_4) now guarantee that the remaining subdigraph of $D[X]$ admits a $\vec{C}_m$-D. In $D[Y]$, however, we have used up:

- all arcs of difference $k + 1$;
- arcs b_i^Y, for $i = 1, 2, 3$ (of differences $s_1 - 1$ and $s_1 + 5$); and
- arcs used in the directed paths Q_i, for $i = 1, 2, 3$.

Assumptions (Y_1)–(Y_4) now guarantee that the remaining subdigraph of $D[Y]$ admits a $\vec{C}_m$-D. The directed m-cycles from the remaining subdigraphs of $D[X]$ and $D[Y]$ can be arranged into resolution classes that complete our R$\vec{C}_m$-D of K_{2m}^*. □

Proof of Theorem 1.4. Let m be an odd integer, $5 \le m \le 49$. Then K_{2m}^* admits a R$\vec{C}_m$-D by Lemma 3.1 if $m = 5$, by Lemma 3.3 if $m = 11$, and by Lemma 3.5 if $m = 9$. It can be verified that the computational results in Appendix A show that the conditions of Proposition 3.4 hold for all odd m, $7 \le m \le 49$, $m \not\equiv 0 \pmod 3$, $m \ne 11$; hence K_{2m}^* admits a R$\vec{C}_m$-D for all such m. Finally, Appendix B shows that the conditions of Proposition 3.6 hold for all odd m, $15 \le m \le 45$, $m \equiv 0 \pmod 3$; hence K_{2m}^* admits a R$\vec{C}_m$-D for all such m as well. Therefore, the statement holds for all odd m, $5 \le m \le 49$. □

4 Conclusion

In Propositions 3.4 and 3.6 we gave sufficient conditions for the complete symmetric digraph K_{2m}^* to admit a resolvable decomposition into directed m-cycles. These sufficient conditions — missing ingredients to complete Construction 3.2 — were verified computationally for $7 \leq m < 50$. We expect that more computing power, as well as more persistence, would yield similar results for larger values of m. A general result would, of course, be preferable. We therefore leave the reader with the following open problem.

Problem 4.1 *Prove that the sufficient conditions in Propositions 3.4 and 3.6 are satisfied for all admissible values of m, or more generally, complete Construction 3.2 to obtain a resolvable directed m-cycle decomposition of K_{2m}^* for all odd $m \geq 7$.*

Note that solving Problem 4.1 would complete the proof of Conjecture 1.3, which in turn would complete the solution to Problem 1.1.

A Computational results — Case $m \not\equiv 0 \pmod 3$

For each value of m we give a set $S \subseteq \mathbb{Z}_m^*$ satisfying Conditions (Y_1) – (Y_4) of Proposition 3.4 (if $m \neq 11$), or Conditions (X_1) – (X_4) from the proof of Lemma 3.3 (if $m = 11$). The required differences appear in bold type. In addition, we give a desired decomposition into directed m-cycles C_i and vertex-disjoint directed paths Q_1 and Q_2. If m is not prime, we also give a partition of $\mathbb{Z}_m^* - (S \cup \{\frac{m+1}{2}\})$ satisfying the assumptions of Lemma 2.1.

- $m = 7$
  $S = \{2, \mathbf{3}, \mathbf{6}\}$
  $Q_1 = (5, 0, 2)$
  $Q_2 = (3, 6, 1, 4)$
  $C_1 = (0, 3, 5, 4, 6, 2, 1, 0)$
  $C_2 = (0, 6, 5, 1, 3, 2, 4, 0)$

- $m = 11$
  $S = \{\mathbf{3}, 4, 9, \mathbf{10}\}$
  $Q_1 = (7, 10, 9, 2, 0, 3, 1, 4)$
  $Q_2 = (5, 8, 6)$
  $C_1 = (0, 10, 2, 6, 9, 8, 1, 5, 4, 3, 7, 0)$
  $C_2 = (0, 4, 8, 7, 6, 10, 3, 2, 5, 9, 1, 0)$
  $C_3 = (0, 9, 7, 5, 3, 6, 4, 2, 1, 10, 8, 0)$

- $m = 13$
  $S = \{\mathbf{1}, 2, 3, \mathbf{4}\}$
  $Q_1 = (11, 1, 5, 7, 10)$
  $Q_2 = (6, 9, 0, 3, 4, 8, 12, 2)$

$C_1 = (0, 1, 2, 4, 5, 6, 8, 9, 10, 12, 3, 7, 11, 0)$
$C_2 = (0, 4, 7, 8, 11, 2, 5, 9, 12, 1, 3, 6, 10, 0)$
$C_3 = (0, 2, 3, 5, 8, 10, 1, 4, 6, 7, 9, 11, 12, 0)$

- $m = 17$
  $S = \{1, \mathbf{2}, 3, \mathbf{5}\}$
  $Q_1 = (15, 16, 1, 4, 7, 9, 14, 2, 5, 6, 11, 13)$
  $Q_2 = (8, 10, 12, 0, 3)$
  $C_1 = (0, 2, 4, 6, 8, 9, 12, 13, 14, 15, 1, 3, 5, 7, 10, 11, 16, 0)$
  $C_2 = (0, 5, 8, 11, 14, 16, 2, 3, 4, 9, 10, 13, 1, 6, 7, 12, 15, 0)$
  $C_3 = (0, 1, 2, 7, 8, 13, 16, 4, 5, 10, 15, 3, 6, 9, 11, 12, 14, 0)$

- $m = 19$
  $S = \{2, \mathbf{12}, \mathbf{15}\}$
  $Q_1 = (17, 0, 15, 11, 7, 3, 5)$
  $Q_2 = (9, 2, 4, 6, 8, 10, 12, 14, 16, 18, 1, 13)$
  $C_1 = (0, 12, 8, 4, 16, 9, 5, 1, 3, 18, 11, 13, 15, 17, 10, 6, 2, 14, 7, 0)$
  $C_2 = (0, 2, 17, 13, 6, 18, 14, 10, 3, 15, 8, 1, 16, 12, 5, 7, 9, 11, 4, 0)$

- $m = 23$
  $S = \{1, 2, \mathbf{15}, \mathbf{18}\}$
  $Q_1 = (21, 22, 17, 9, 1, 19, 20, 12, 7, 8, 10, 5, 0, 2, 4, 6)$
  $Q_2 = (11, 3, 18, 13, 14, 15, 16)$
  $C_1 = (0, 15, 7, 22, 14, 9, 4, 19, 11, 6, 1, 2, 3, 5, 20, 21, 16, 17, 18, 10, 12, 13, 8, 0)$
  $C_2 = (0, 18, 19, 14, 16, 8, 9, 10, 11, 13, 15, 17, 12, 4, 5, 6, 7, 2, 20, 22, 1, 3, 21, 0)$
  $C_3 = (0, 1, 16, 18, 20, 15, 10, 2, 17, 19, 21, 13, 5, 7, 9, 11, 12, 14, 6, 8, 3, 4, 22, 0)$

- $m = 25$
  $S = \{1, 2, \mathbf{4}, \mathbf{7}\}$
  $Q_1 = (23, 2, 6, 10, 14, 15, 16, 17, 19)$
  $Q_2 = (12, 13, 20, 21, 0, 7, 8, 9, 11, 18, 22, 24, 1, 3, 4, 5)$
  $C_1 = (0, 4, 8, 12, 16, 20, 24, 6, 7, 11, 15, 19, 1, 2, 3, 10, 17, 21, 22, 23, 5, 9, 13, 14, 18, 0)$
  $C_2 = (0, 1, 5, 6, 8, 15, 22, 4, 11, 13, 17, 24, 3, 7, 14, 16, 18, 20, 2, 9, 10, 12, 19, 21, 23, 0)$
  $C_3 = (0, 2, 4, 6, 13, 15, 17, 18, 19, 20, 22, 1, 8, 10, 11, 12, 14, 21, 3, 5, 7, 9, 16, 23, 24, 0)$
  Partition contains: $\{\pm 3, \pm 5\}$, $\{\pm 6, \pm 10\}$, and $\{e\}$ for each remaining difference e

- $m = 29$
  $S = \{1, 2, \mathbf{5}, \mathbf{8}\}$
  $Q_1 = (27, 28, 7, 8, 9, 11, 16, 21, 23, 25, 26, 5, 10, 12, 13, 15, 17, 18, 20, 22)$
  $Q_2 = (14, 19, 24, 0, 1, 2, 3, 4, 6)$
  $C_1 = (0, 5, 13, 18, 23, 28, 4, 9, 14, 22, 1, 6, 7, 15, 20, 25, 27, 3, 8, 16, 24, 26, 2, 10, 11, 12, 17, 19, 21, 0)$
  $C_2 = (0, 8, 10, 15, 23, 2, 7, 9, 17, 22, 24, 25, 4, 12, 20, 21, 26, 28, 1, 3, 5, 6, 11, 13, 14, 16, 18, 19, 27, 0)$
  $C_3 = (0, 2, 4, 5, 7, 12, 14, 15, 16, 17, 25, 1, 9, 10, 18, 26, 27, 6, 8, 13, 21, 22, 23, 24, 3, 11, 19, 20, 28, 0)$

- $m = 31$
  $S = \{1, \mathbf{21}, \mathbf{24}\}$
  $Q_1 = (29, 19, 9, 30, 23, 13, 14, 4, 28, 18, 8)$
  $Q_2 = (15, 16, 17, 10, 11, 12, 5, 6, 7, 0, 1, 2, 3, 24, 25, 26, 27, 20, 21, 22)$
  $C_1 = (0, 21, 11, 4, 5, 26, 16, 9, 10, 3, 27, 17, 7, 28, 29, 22, 12, 2, 23, 24, 14, 15, 8, 1, 25, 18, 19, 20, 13, 6, 30, 0)$
  $C_2 = (0, 24, 17, 18, 11, 1, 22, 23, 16, 6, 27, 28, 21, 14, 7, 8, 9, 2, 26, 19, 12, 13, 3, 4, 25, 15, 5, 29, 30, 20, 10, 0)$

- $m = 35$
  $S = \{1, \mathbf{24}, \mathbf{27}\}$
  $Q_1 = (33, 22, 14, 3, 27, 16, 5, 32, 21, 13, 2, 29, 18, 10, 34, 26, 15, 7, 8, 0, 1, 28, 20, 9)$
  $Q_2 = (17, 6, 30, 19, 11, 12, 4, 31, 23, 24, 25)$
  $C_1 = (0, 24, 16, 8, 9, 1, 25, 26, 27, 28, 17, 18, 19, 20, 12, 13, 14, 6, 7, 31, 32, 33, 34, 23, 15, 4, 5, 29, 21, 10, 2, 3, 30, 22, 11, 0)$
  $C_2 = (0, 27, 19, 8, 32, 24, 13, 5, 6, 33, 25, 14, 15, 16, 17, 9, 10, 11, 3, 4, 28, 29, 30, 31, 19, 21, 22, 23, 12, 1, 2, 26, 18, 7, 34, 0)$
  Partition contains: $\{\pm 5, \pm 7\}$, $\{\pm 10, \pm 14\}$, $\{\pm 15, \pm 2\}$, and $\{e\}$ for each remaining difference e

- $m = 37$
  $S = \{1, \mathbf{7}, \mathbf{10}\}$
  $Q_1 = (35, 36, 0, 1, 11, 12, 13, 14, 15, 25, 26, 27, 28)$
  $Q_2 = (18, 19, 29, 2, 9, 10, 20, 21, 22, 23, 30, 3, 4, 5, 6, 16, 17, 24, 31, 32, 33, 34, 7, 8)$
  $C_1 = (0, 7, 14, 21, 28, 1, 8, 15, 22, 29, 36, 9, 16, 26, 33, 6, 13, 23, 24, 34, 35, 5, 12, 19, 20, 30, 31, 4, 11, 18, 25, 32, 2, 3, 10, 17, 27, 0)$
  $C_2 = (0, 10, 11, 21, 31, 1, 2, 12, 22, 32, 5, 15, 16, 23, 33, 3, 13, 20, 27, 34, 4, 14, 24, 25, 35, 8, 9, 19, 26, 36, 6, 7, 17, 18, 28, 29, 30, 0)$

- $m = 41$
  $S = \{1, \mathbf{8}, \mathbf{11}\}$
  $Q_1 = (39, 6, 14, 15, 23, 24, 32, 40, 7, 18, 26, 34, 1, 2, 10, 11, 19, 27, 35, 36, 3, 4, 12, 13, 21, 22, 30, 31)$
  $Q_2 = (20, 28, 29, 37, 38, 5, 16, 17, 25, 33, 0, 8, 9)$
  $C_1 = (0, 11, 12, 23, 31, 1, 9, 17, 28, 36, 6, 7, 8, 19, 20, 21, 32, 2, 3, 14, 22, 33, 34, 35, 5, 13, 24, 25, 26, 37, 4, 15, 16, 27, 38, 39, 40, 10, 18, 29, 30, 0)$
  $C_2 = (0, 1, 12, 20, 31, 32, 33, 3, 11, 22, 23, 34, 4, 5, 6, 17, 18, 19, 30, 38, 8, 16, 24, 35, 2, 13, 14, 25, 36, 37, 7, 15, 26, 27, 28, 39, 9, 10, 21, 29, 40, 0)$

- $m = 43$
  $S = \{1, \mathbf{30}, \mathbf{33}\}$
  $Q_1 = (41, 28, 15, 2, 32, 19, 6, 36, 23, 10, 0, 33, 34, 24, 11)$
  $Q_2 = (21, 22, 12, 13, 3, 4, 5, 35, 25, 26, 16, 17, 18, 8, 9, 42, 29, 30, 20, 7, 37, 38, 39, 40, 27, 14, 1, 31)$
  $C_1 = (0, 30, 31, 32, 33, 20, 10, 11, 1, 34, 21, 8, 38, 28, 18, 19, 9, 39, 29, 16, 3, 36, 26, 27, 17, 4, 37, 24, 25, 12, 2, 35, 22, 23, 13, 14, 15, 5, 6, 7, 40, 41, 42, 0)$

C_2 = (0, 1, 2, 3, 33, 23, 24, 14, 4, 34, 35, 36, 37, 27, 28, 29, 19, 20, 21, 11, 12, 42, 32, 22, 9, 10, 40, 30, 17, 7, 8, 41, 31, 18, 5, 38, 25, 15, 16, 6, 39, 26, 13, 0)

- $m = 47$
  $S = \{1, \mathbf{33}, \mathbf{36}\}$
  Q_1 = (45, 46, 35, 24, 13, 2, 3, 4, 40, 29, 18, 19, 5, 41, 27, 16, 17, 6, 7, 43, 32, 33, 22, 8, 44, 30, 31, 20, 21, 10, 11, 12)
  Q_2 = (23, 9, 42, 28, 14, 0, 36, 37, 38, 39, 25, 26, 15, 1, 34)
  C_1 = (0, 33, 34, 20, 6, 42, 43, 29, 15, 16, 2, 35, 36, 22, 23, 12, 1, 37, 26, 27, 13, 14, 3, 39, 28, 17, 18, 4, 5, 38, 24, 25, 11, 44, 45, 31, 32, 21, 7, 40, 41, 30, 19, 8, 9, 10, 46, 0)
  C_2 = (0, 1, 2, 38, 27, 28, 29, 30, 16, 5, 6, 39, 40, 26, 12, 13, 46, 32, 18, 7, 8, 41, 42, 31, 17, 3, 36, 25, 14, 15, 4, 37, 23, 24, 10, 43, 44, 33, 19, 20, 9, 45, 34, 35, 21, 22, 11, 0)

- $m = 49$
  $S = \{2, \mathbf{10}, \mathbf{13}\}$
  Q_1 = (47, 8, 18, 28, 38, 48, 9, 19, 21, 31, 44, 46, 10, 23, 25, 35, 37)
  Q_2 = (24, 34, 36, 0, 2, 12, 22, 32, 45, 6, 16, 26, 39, 41, 5, 7, 20, 33, 43, 4, 14, 27, 29, 42, 3, 13, 15, 17, 30, 40, 1, 11)
  C_1 = (0, 10, 20, 30, 43, 7, 9, 22, 35, 45, 47, 11, 13, 23, 36, 38, 40, 4, 17, 19, 32, 42, 6, 8, 21, 34, 44, 5, 18, 31, 33, 46, 48, 12, 14, 24, 26, 28, 41, 2, 15, 25, 27, 37, 1, 3, 16, 29, 39, 0)
  C_2 = (0, 13, 26, 36, 46, 7, 17, 27, 40, 42, 44, 8, 10, 12, 25, 38, 2, 4, 6, 19, 29, 31, 41, 43, 45, 9, 11, 21, 23, 33, 35, 48, 1, 14, 16, 18, 20, 22, 24, 37, 39, 3, 5, 15, 28, 30, 32, 34, 47, 0)
  Partition contains: $\{\pm 7, \pm 1\}$, $\{\pm 14, \pm 3\}$, $\{\pm 21, \pm 4\}$, and $\{e\}$ for each remaining difference e

B Computational results — Case $m \equiv 0 \pmod 3$

For each value of m we give sets $S^X, S^Y \subseteq \mathbb{Z}_m^*$ satisfying Conditions (X_1) – (X_4), (Y_1) – (Y_4) of Proposition 3.6 (if $m \geq 15$), or from the proof of Lemma 3.5 (if $m = 9$). The required differences appear in bold type. In addition, we give a desired decomposition of a subgraph of $D[X]$ into directed m-cycles C'_i and (for $m = 9$ only) pairwise vertex-disjoint directed paths Q'_i, and a desired decomposition of a subgraph of $D[Y]$ into directed m-cycles C_i and pairwise vertex-disjoint directed paths Q_i. We also give a partition of $\mathbb{Z}_m^* - (S \cup \{\frac{m+1}{2}\})$ satisfying the assumptions of Lemma 2.1.

- $m = 9$
  $S^X = \{1, 2, 3, \mathbf{4}, 6, \mathbf{7}\}$
  $Q'_1 = (1, 3)$
  $Q'_2 = (2, 4)$
  $Q'_3 = (0, 6, 7, 8, 5)$
  $C'_1 = (0, 4, 8, 3, 7, 5, 6, 1, 2, 0)$
  $C'_2 = (0, 7, 2, 6, 8, 1, 4, 5, 3, 0)$
  $C'_3 = (0, 3, 6, 4, 7, 1, 5, 2, 8, 0)$
  $C'_4 = (0, 1, 8, 2, 3, 5, 7, 4, 6, 0)$

$C'_5 = (0, 2, 5, 8, 6, 3, 4, 1, 7, 0)$
Partition contains: $\{8\}$

$S^Y = \{\mathbf{1}, 3, 4, 6, \mathbf{7}, 8\}$
$Q_1 = (2, 3)$
$Q_2 = (4, 7, 1)$
$Q_3 = (6, 5, 0, 8)$
$C_1 = (0, 1, 8, 2, 5, 6, 7, 4, 3, 0)$
$C_2 = (0, 7, 8, 5, 3, 1, 4, 2, 6, 0)$
$C_3 = (0, 3, 6, 4, 1, 7, 5, 2, 8, 0)$
$C_4 = (0, 6, 1, 5, 4, 8, 3, 7, 2, 0)$
$C_5 = (0, 4, 5, 8, 7, 6, 3, 2, 1, 0)$
Partition contains: $\{2\}$

- $m = 15$
  $S^X = \{\mathbf{4}, \mathbf{7}, 9\}$
  $C'_1 = (0, 4, 13, 7, 11, 5, 9, 3, 12, 1, 10, 14, 8, 2, 6, 0)$
  $C'_2 = (0, 7, 1, 8, 12, 6, 13, 5, 14, 3, 10, 4, 11, 2, 9, 0)$
  $C'_3 = (0, 9, 13, 2, 11, 3, 7, 14, 6, 10, 1, 5, 12, 4, 8, 0)$
  Partition contains: $\{\pm 3, \pm 5\}$, and $\{e\}$ for each remaining difference e

  $S^Y = \{\mathbf{1}, 5, 6, 9, \mathbf{10}\}$
  $Q_1 = (11, 6, 7, 12, 2, 8, 9, 4, 5, 14)$
  $Q_2 = (13, 3)$
  $Q_3 = (0, 10, 1)$
  $C_1 = (0, 1, 2, 12, 7, 13, 8, 3, 4, 14, 9, 10, 11, 5, 6, 0)$
  $C_2 = (0, 5, 11, 2, 7, 1, 6, 12, 3, 9, 14, 8, 13, 4, 10, 0)$
  $C_3 = (0, 6, 11, 1, 7, 8, 2, 3, 12, 13, 14, 5, 10, 4, 9, 0)$
  $C_4 = (0, 9, 3, 8, 14, 4, 13, 7, 2, 11, 12, 6, 1, 10, 5, 0)$
  Partition contains: $\{\pm 3, \pm 2\}$, and $\{e\}$ for each remaining difference e

- $m = 21$
  $S^X = \{\mathbf{1}, \mathbf{4}, 18\}$
  $C'_1 = (0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 19, 16, 20, 17, 18, 0)$
  $C'_2 = (0, 4, 8, 12, 9, 6, 10, 14, 18, 19, 1, 5, 2, 20, 3, 7, 11, 15, 16, 13, 17, 0)$
  $C'_3 = (0, 18, 15, 12, 16, 17, 14, 11, 8, 5, 9, 13, 10, 7, 4, 1, 19, 20, 2, 6, 3, 0)$
  Partition contains: $\{\pm 6, \pm 7\}$, $\{\pm 9, \pm 2\}$,and $\{e\}$ for each remaining difference e

  $S^Y = \{3, \mathbf{4}, \mathbf{10}, 13, 18\}$
  $Q_1 = (5, 15, 19, 2, 12, 16, 13, 17, 0, 4, 8, 18, 10, 20)$
  $Q_2 = (7, 11, 14, 6, 3)$
  $Q_3 = (9, 1)$
  $C_1 = (0, 10, 14, 18, 1, 11, 15, 4, 7, 17, 6, 19, 8, 12, 9, 13, 16, 5, 2, 20, 3, 0)$
  $C_2 = (0, 13, 2, 15, 12, 4, 1, 14, 11, 3, 6, 10, 7, 20, 17, 9, 19, 16, 8, 5, 18, 0)$
  $C_3 = (0, 3, 16, 19, 1, 4, 14, 17, 20, 2, 6, 9, 12, 15, 18, 7, 10, 13, 5, 8, 11, 0)$
  $C_4 = (0, 18, 15, 7, 4, 17, 14, 3, 13, 10, 2, 5, 9, 6, 16, 20, 12, 1, 19, 11, 8, 0)$
  Partition contains: $\{\pm 6, \pm 7\}$, $\{\pm 9, \pm 2\}$, and $\{e\}$ for each remaining difference e

- $m = 27$
  $S^X = \{3, \mathbf{22}, \mathbf{25}\}$
  $C'_1 = (0, 25, 23, 21, 19, 17, 15, 13, 11, 6, 4, 1, 26, 2, 24, 22, 20, 18, 16, 14, 9, 12, 7, 10, 5, 8, 3, 0)$
  $C'_2 = (0, 22, 25, 20, 15, 18, 13, 16, 19, 14, 17, 12, 10, 8, 11, 9, 4, 7, 2, 5, 3, 6, 1, 23, 26, 21, 24, 0)$
  $C'_3 = (0, 3, 25, 1, 4, 26, 24, 19, 22, 17, 20, 23, 18, 21, 16, 11, 14, 12, 15, 10, 13, 8, 6, 9, 7, 5, 2, 0)$
  Partition contains: $\{\pm 6, \pm 1\}$, $\{\pm 9, \pm 4\}$, $\{\pm 12, \pm 7\}$, and $\{e\}$ for each remaining difference e

  $S^Y = \{3, 4, \mathbf{19}, 24, \mathbf{25}\}$
  $Q_1 = (20, 12, 4, 2, 5, 9, 13, 17, 21, 19, 16, 8, 11, 15, 18, 10, 7, 26)$
  $Q_2 = (22, 14, 6, 25, 23, 0, 3)$
  $Q_3 = (24, 1)$
  $C_1 = (0, 19, 11, 3, 1, 26, 18, 16, 14, 12, 15, 7, 4, 23, 20, 24, 22, 25, 17, 9, 6, 10, 2, 21, 13, 5, 8, 0)$
  $C_2 = (0, 25, 2, 6, 4, 7, 11, 8, 12, 9, 1, 5, 24, 16, 13, 10, 14, 17, 20, 23, 21, 18, 15, 19, 22, 26, 3, 0)$
  $C_3 = (0, 4, 1, 25, 22, 19, 17, 14, 11, 9, 12, 10, 8, 6, 3, 7, 5, 2, 26, 23, 15, 13, 16, 20, 18, 21, 24, 0)$
  $C_4 = (0, 24, 21, 25, 1, 4, 8, 5, 3, 6, 9, 7, 10, 13, 11, 14, 18, 22, 20, 17, 15, 12, 16, 19, 23, 26, 2, 0)$
  Partition contains: $\{\pm 6, \pm 1\}$, $\{\pm 9, \pm 5\}$, $\{\pm 12, \pm 7\}$, and $\{e\}$ for each remaining difference e

- $m = 33$
  $S^X = \{11, 12, \mathbf{19}, \mathbf{22}\}$
  $C'_1 = (0, 19, 5, 24, 10, 29, 15, 1, 20, 6, 28, 14, 25, 11, 30, 8, 27, 16, 2, 13, 32, 21, 7, 18, 4, 26, 12, 23, 9, 31, 17, 3, 22, 0)$
  $C'_2 = (0, 22, 1, 12, 31, 20, 9, 28, 6, 17, 29, 18, 7, 19, 8, 30, 16, 5, 27, 13, 24, 3, 25, 14, 26, 15, 4, 23, 2, 21, 10, 32, 11, 0)$
  $C'_3 = (0, 11, 22, 8, 19, 30, 9, 20, 31, 10, 21, 32, 18, 29, 7, 26, 4, 15, 27, 5, 16, 28, 17, 6, 25, 3, 14, 2, 24, 13, 1, 23, 12, 0)$
  $C'_4 = (0, 12, 24, 2, 14, 3, 15, 26, 5, 17, 28, 7, 29, 8, 20, 32, 10, 22, 11, 23, 1, 13, 25, 4, 16, 27, 6, 18, 30, 19, 31, 9, 21, 0)$
  Partition contains: $\{\pm 3, \pm 1\}$, $\{\pm 6, \pm 2\}$, $\{\pm 9, \pm 4\}$, $\{\pm 15, \pm 5\}$, and $\{e\}$ for each remaining difference e

  $S^Y = \{1, \mathbf{7}, \mathbf{13}, 26\}$
  $Q_1 = (8, 21, 14, 27, 28, 2, 15, 16, 29, 9, 22, 23, 24, 17, 30, 4, 11, 18, 25, 5, 31, 32)$
  $Q_2 = (12, 19, 20, 13, 26, 6, 7, 0, 1)$
  $Q_3 = (10, 3)$
  $C_1 = (0, 7, 20, 27, 1, 14, 21, 28, 8, 15, 22, 29, 30, 23, 16, 9, 2, 3, 4, 17, 10, 11, 24, 31, 5, 12, 13, 6, 32, 25, 18, 19, 26, 0)$
  $C_2 = (0, 13, 14, 7, 8, 1, 2, 9, 10, 17, 18, 11, 12, 25, 26, 27, 20, 21, 22, 15, 28, 29, 3, 16,$

$23, 30, 31, 24, 4, 5, 6, 19, 32, 0)$
$C_3 = (0, 26, 19, 12, 5, 18, 31, 11, 4, 30, 10, 23, 3, 29, 22, 2, 28, 21, 1, 27, 7, 14, 15, 8, 9, 16, 17, 24, 25, 32, 6, 13, 20, 0)$
Partition contains: $\{\pm 3, \pm 11\}$, $\{\pm 6, \pm 2\}$, $\{\pm 9, \pm 4\}$, $\{\pm 12, \pm 5\}$, $\{\pm 15, \pm 8\}$, and $\{e\}$ for each remaining difference e

- $m = 39$
  $S^X = \{\mathbf{13}, \mathbf{16}, 24, 26\}$
  $C'_1 = (0, 13, 26, 3, 16, 29, 6, 19, 32, 9, 22, 35, 12, 25, 38, 15, 28, 2, 18, 31, 5, 21, 34, 8, 24, 37, 11, 27, 1, 14, 30, 4, 17, 33, 7, 20, 36, 10, 23, 0)$
  $C'_2 = (0, 16, 32, 6, 22, 38, 12, 28, 15, 2, 26, 13, 29, 3, 19, 35, 9, 25, 1, 17, 30, 7, 23, 10, 36, 21, 37, 14, 27, 4, 20, 33, 18, 5, 31, 8, 34, 11, 24, 0)$
  $C'_3 = (0, 26, 11, 37, 24, 9, 35, 22, 7, 33, 20, 5, 18, 3, 29, 14, 38, 25, 10, 34, 19, 4, 30, 15, 31, 16, 1, 27, 12, 36, 23, 8, 21, 6, 32, 17, 2, 28, 13, 0)$
  $C'_4 = (0, 24, 11, 35, 20, 7, 31, 18, 34, 21, 8, 32, 19, 6, 30, 17, 4, 28, 5, 29, 16, 3, 27, 14, 1, 25, 12, 38, 23, 36, 13, 37, 22, 9, 33, 10, 26, 2, 15, 0)$
  Partition contains: $\{\pm 3, \pm 1\}$, $\{\pm 6, \pm 2\}$, $\{\pm 9, \pm 4\}$, $\{\pm 12, \pm 5\}$, $\{\pm 18, \pm 7\}$, and $\{e\}$ for each remaining difference e

  $S^Y = \{2, 7, \mathbf{28}, \mathbf{34}\}$
  $Q_1 = (29, 18, 7, 14, 16, 23, 25, 27, 22, 17, 12, 19, 21, 10, 5, 0, 28, 35, 24, 13, 2, 30, 32, 34, 36, 38)$
  $Q_2 = (31, 20, 9, 37, 26, 15, 4, 11, 6, 8, 3)$
  $Q_3 = (33, 1)$
  $C_1 = (0, 34, 23, 12, 1, 35, 3, 37, 5, 7, 2, 36, 4, 32, 21, 16, 18, 25, 20, 15, 17, 6, 13, 8, 10, 38, 27, 29, 31, 33, 22, 24, 26, 28, 30, 19, 14, 9, 11, 0)$
  $C_2 = (0, 7, 9, 16, 11, 13, 15, 10, 17, 19, 8, 36, 25, 32, 27, 34, 2, 4, 38, 6, 1, 3, 5, 12, 14, 21, 28, 23, 18, 20, 22, 29, 24, 31, 26, 33, 35, 30, 37, 0)$
  $C_3 = (0, 2, 9, 4, 6, 34, 29, 36, 31, 38, 1, 8, 15, 22, 11, 18, 13, 20, 27, 16, 5, 33, 28, 17, 24, 19, 26, 21, 23, 30, 25, 14, 3, 10, 12, 7, 35, 37, 32, 0)$
  Partition contains: $\{\pm 3, \pm 13\}$, $\{\pm 6, \pm 1\}$, $\{\pm 9, \pm 4\}$, $\{\pm 12, \pm 8\}$, $\{\pm 15, \pm 10\}$, $\{\pm 18, \pm 14\}$, and $\{e\}$ for each remaining difference e

- $m = 45$
  $S^X = \{\mathbf{4}, \mathbf{7}, 39\}$
  $C'_1 = (0, 4, 8, 12, 16, 20, 24, 28, 32, 36, 43, 5, 9, 13, 17, 21, 25, 29, 33, 37, 41, 2, 6, 10, 14, 18, 22, 26, 30, 34, 38, 42, 1, 40, 44, 3, 7, 11, 15, 19, 23, 27, 31, 35, 39, 0)$
  $C'_2 = (0, 7, 1, 5, 12, 19, 26, 33, 27, 21, 28, 35, 42, 4, 11, 18, 25, 32, 39, 43, 2, 9, 16, 23, 30, 37, 44, 6, 13, 20, 14, 8, 15, 22, 29, 36, 40, 34, 41, 3, 10, 17, 24, 31, 38, 0)$
  $C'_3 = (0, 39, 33, 40, 1, 8, 2, 41, 35, 29, 23, 17, 11, 5, 44, 38, 32, 26, 20, 27, 34, 28, 22, 16, 10, 4, 43, 37, 31, 25, 19, 13, 7, 14, 21, 15, 9, 3, 42, 36, 30, 24, 18, 12, 6, 0)$
  Partition contains: $\{\pm 3, \pm 5\}$, $\{\pm 9, \pm 10\}$, $\{\pm 12, \pm 20\}$, $\{\pm 15, \pm 1\}$, $\{\pm 18, \pm 2\}$, $\{\pm 21, \pm 8\}$, and $\{e\}$ for each remaining difference e

$S^Y = \{\mathbf{10}, \mathbf{16}, 31, 35\}$
$Q_1 = (11, 21, 31, 2, 12, 22, 38, 9, 19, 29, 39, 4, 14, 24, 40, 30, 20, 10, 0, 35, 25, 41, 6, 37, 27, 17, 7, 42, 28, 44)$
$Q_2 = (13, 23, 33, 43, 8, 18, 34, 5, 36, 26, 16, 32, 3)$
$Q_3 = (15, 1)$
$C_1 = (0, 10, 20, 30, 40, 5, 21, 37, 23, 13, 3, 38, 28, 18, 4, 39, 29, 15, 31, 41, 12, 2, 33, 19, 35, 6, 16, 26, 36, 1, 32, 22, 8, 24, 34, 44, 9, 25, 11, 42, 7, 17, 27, 43, 14, 0)$
$C_2 = (0, 16, 6, 22, 32, 42, 13, 29, 19, 9, 44, 30, 1, 17, 3, 34, 20, 36, 7, 38, 24, 10, 41, 31, 21, 11, 27, 37, 2, 18, 8, 43, 33, 23, 39, 25, 15, 5, 40, 26, 12, 28, 14, 4, 35, 0)$
$C_3 = (0, 31, 17, 33, 4, 20, 6, 41, 27, 13, 44, 34, 24, 14, 30, 16, 2, 37, 8, 39, 10, 26, 42, 32, 18, 28, 38, 3, 19, 5, 15, 25, 35, 21, 7, 23, 9, 40, 11, 1, 36, 22, 12, 43, 29, 0)$
Partition contains: $\{\pm 3, \pm 5\}$, $\{\pm 6, \pm 20\}$, $\{\pm 9, \pm 1\}$, $\{\pm 12, \pm 2\}$, $\{\pm 15, \pm 4\}$, $\{\pm 18, \pm 7\}$, $\{\pm 21, \pm 8\}$, and $\{e\}$ for each remaining difference e

Acknowledgements

The authors gratefully acknowledge support by the Natural Sciences and Engineering Research Council of Canada. Special thanks to Patrick Niesink and Ryan Murray, who wrote most of the code used to obtain the computational results, and to Aras Erzurumluoğlu for his careful proofreading of the manuscript.

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(Received 21 June 2017; revised 7 Mar 2018)