AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **71(2)** (2018), Pages 184–195

On decomposability of cyclic triple systems

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Abstract

We enumerate, up to isomorphism, the indecomposable cyclic triple systems $\mathrm{TS}(v, \lambda)$ in each of the following cases: $v \leq 18$ and $\lambda \leq 7$; $v = 19$ and $\lambda \leq 4$; $v = 21$ and $\lambda \leq 3$; $v = 25$ and $\lambda \leq 2$.

In particular, our census enables us to answer in the affirmative the questions left open in the book "Triple systems" by C.J. Colbourn and A. Rosa, about the existence of an indecomposable $\mathrm{TS}(v, \lambda)$ for $(v, \lambda) = (13, 5)$, $(15, 5)$, $(11, 6)$, $(12, 6)$, $(13, 6)$, $(15, 6)$, and $(16, 6)$. In the first five cases there are "many" cyclic solutions (all of which are non-simple) but there is a unique indecomposable cyclic $\mathrm{TS}(16, 6)$. It is simple and it is shown that it lies in an infinite class of $\mathrm{TS}(4k, 6)$ that we obtain, theoretically, via k-extended Skolem sequences.

1 Introduction

A λ-fold triple system $(V, \mathcal{B})$ of order v, denoted $\mathrm{TS}(v, \lambda)$, is a collection $\mathcal{B}$ of 3-subsets (*triples* or *blocks*) from a v-set V of *points*, such that every given pair of elements of V occurs in exactly λ triples. A one-fold triple system is called a Steiner triple system $\mathrm{STS}(v)$, and a λ-fold triple system of order v is also known as 2-$(v, 3, \lambda)$ design. A triple system is *simple* if every triple occurs at most once in $\mathcal{B}$.

A $\mathrm{TS}(v, \lambda)$ is *cyclic* if it admits an automorphism cyclically permuting all its points. In this case its points are naturally identified with the elements of $\mathbb{Z}_v$ and the mentioned automorphism with addition by 1 modulo v. Two cyclic $\mathrm{TS}(v, \lambda)$ are *(multiplier) equivalent* if there is an isomorphism between the two systems of the form $i \mapsto \mu \cdot i \bmod v$; see further in [12].

Bays and Lambossy proved in 1931 that in the case v prime, two cyclic triple systems are isomorphic if and only if they are multiplier equivalent. Later, Brand and Phelps showed that the condition v prime is essential for this. See [13, pp. 109, 110] for more details and see [19] for a generalization of the Bays-Lambossy theorem.

Necessary and sufficient conditions for the existence of cyclic triple systems are well-known [13, Thm. 7.11]:

Theorem 1.1 *There is a cyclic* $TS(v, \lambda)$ *if and only if:*

1. $\lambda \equiv 1, 5 \pmod 6$ *and* $v \equiv 1, 3 \pmod 6$, $(v, \lambda) \neq (9, 1)$;
2. $\lambda \equiv 2, 10 \pmod{12}$ *and* $v \equiv 0, 1, 3, 4, 7, 9 \pmod{12}$, $(v, \lambda) \neq (9, 2)$;
3. $\lambda \equiv 3 \pmod 6$ *and* $v \equiv 1 \pmod 2$;
4. $\lambda \equiv 4, 8 \pmod{12}$ *and* $v \equiv 0, 1 \pmod 3$;
5. $\lambda \equiv 6 \pmod{12}$ *and* $v \equiv 0, 1, 3 \pmod 4$; *or*
6. $\lambda \equiv 0 \pmod{12}$ *and* $v \geq 3$.

Indecomposable triple systems. A $\mathrm{TS}(v, \lambda)$ is *decomposable* if its collection of triples is the (multiset) union of the triples of a $\mathrm{TS}(v, \lambda_1)$ and a $\mathrm{TS}(v, \lambda_2)$, where λ_1 and λ_2 are non-negative integers which sum to λ. It is *indecomposable* otherwise; see further in [12, 16].

In [13, Ch. 23] the known results on the decomposability of triple systems are summarized.

Theorem 1.2 ([13, Thm. 23.1]) *(Indecomposable TSs with small index)*

1. *A simple indecomposable* $\mathrm{TS}(v, 2)$ *exists if and only if*
$$v \equiv 0, 1 \pmod 3, v \geq 4, v \neq 7.$$
2. *A simple indecomposable* $\mathrm{TS}(v, 3)$ *exists if and only if*
$$v \equiv 1 \pmod 2, v \geq 5.$$
3. *A simple indecomposable* $\mathrm{TS}(v, 4)$ *exists if and only if*
$$v \equiv 0, 1 \pmod 3, v \geq 10.$$
4. *A simple indecomposable* $\mathrm{TS}(v, 5)$ *exists if and only if*
$$v \equiv 1, 3 \pmod 6, v \geq 13,$$
except possibly for $v \in \{13, 15\}$.

5. *A simple indecomposable* TS$(v, 6)$ *exists if and only if*

$$v \geq 8, v \neq 9, 10,$$

except possibly for $v \in \{11, 12, 13, 15, 16\}$.

In [13, p. 419] the open parameter sets are posed as open questions, even for non-simple triple systems:

- Do indecomposable TS$(13, 5)$ and TS$(15, 5)$ exist?
- Do there exist indecomposable TS$(v, 6)$ for $v = 11, 12, 13, 15, 16$?

Indecomposable cyclic triple systems. Since the publication of [13], Rees and Shalaby [21] studied indecomposable cyclic triple systems for $\lambda = 2$. Very recently, Shalaby, Sheppard, Silvesan [23] presented infinite series of indecomposable cyclic triple systems for $\lambda = 3$.

In this paper we can answer some of these open questions. In Table 1 we will see that the number of non-isomorphic indecomposable cyclic TS$(13, 5)$ and TS$(15, 5)$ as well as TS$(v, 6)$ for $v = 11, 12, 13, 15, 16$ is greater than zero, i.e., indecomposable triple systems with these parameters do exist. Further, we give the precise number of non-isomorphic indecomposable cyclic triple systems for some small values of v and λ.

We do this by complete enumeration of all cyclic triple systems (simple and non-simple), up to isomorphism, which are indecomposable.

Our approach to enumerate these triple systems is divided into several steps. In the first step all cyclic triple systems for some values of v and λ are generated. This is done by the methods described in [1, 5, 25]. After sieving out the (multiplier) equivalent copies, the remaining triple systems are tested for indecomposability by the methods described in [7, 25]; see Section 2.

The results are listed in Table 1. Looking at this table, one parameter set clearly stands out: there is a unique indecomposable simple cyclic TS$(16, 6)$. We present and analyse this triple system in Sections 4 and 5, and show that it falls into an infinite class of TS$(v, 6)$ obtainable with the help of extended Skolem sequences.

The results of this paper were first presented at the Combinatorics conference in Gaeta, Italy in June 2014.

2 The enumeration algorithm

Our method of finding all indecomposable cyclic triple systems is rather straightforward. First, we generate for fixed parameters all triple systems having the cyclic group $\mathbb{Z}_v$ as automorphism group. Then, we sieve out all (multiplier) equivalent copies. The remaining cyclic triple systems are tested one by one to see whether there exists a decomposition into cyclic triple systems. That means we test the triple system for decomposability. This step already sieves out many decomposable triple systems. Finally, if a triple system cannot be decomposed into two cyclic triple systems, it is tested for (general) indecomposability.

First step: Construction of triple systems with prescribed automorphism group. For v and λ we want to construct all $\mathrm{TS}(v,\lambda)$ with $G = \mathbb{Z}_v$ as a group of automorphism. This is done by the well-known method of Kramer and Mesner [17]. Let

$$B_1^G \cup B_2^G \cup \ldots \cup B_n^G \subseteq \binom{V}{3}$$

denote the orbits of G acting on the triples of V and let

$$T_1^G \cup T_2^G \cup \ldots \cup T_\ell^G = \binom{V}{2}$$

denote the orbits of G acting on the pairs of V. Then the matrix $M = (m_{i,j})$ is defined by

$$m_{i,j} := \left|\{B \in B_j^G \mid T_i \subset B\}\right|.$$

Theorem 2.1 (Kramer and Mesner [17]) *If x is a vector of non-negative integers solving the equation*

$$M \cdot x = \begin{pmatrix} \lambda \\ \lambda \\ \vdots \\ \lambda \end{pmatrix},$$

then the multiset union of x_j copies of B_j^G for $j = 1, \ldots, n$ forms a TS(v,λ) having G as an automorphism group. Every TS(v,λ) having G as an automorphism group is in one-to-one correspondence with a non-negative integer vector solving the above equation.

In case x is a $\{0,1\}$ vector, then the TS(v,λ) is a simple TS(v,λ) having G as an automorphism group.

In [1], there is a description of an approach—called the *method of partial differences*—showing how to compute the matrix M for $G = \mathbb{Z}_v$ efficiently. The method is feasible even for hand calculation in quite large instances.

Once the matrix M has been computed, one has to solve the Diophantine linear system $M \cdot x = (\lambda, \lambda, \ldots \lambda)^\top$. We used the algorithm from [25] to determine all solutions of this system of equations.

Second step: Sieve out (multiplier) equivalent copies. For $v \le 20$ and $\lambda \le 7$ this can be done by brute force, since it is not a time critical step. The authors implemented a Python program for this.

Third step: Test for cyclic decomposability. For the test for decomposability into cyclic triple systems, the matrix M can be reused. A cyclic $\mathrm{TS}(v,\lambda_1)$, $\lambda_1 < \lambda$, decomposes a given cyclic $\mathrm{TS}(v,\lambda)$ with triples $\mathcal{B}$ if and only if there exists a $\{0,1\}$

solution of the equation

$$M' \cdot x = \begin{pmatrix} \lambda_1 \\ \lambda_1 \\ \vdots \\ \lambda_1 \end{pmatrix},$$

where the matrix M' consists of the columns of the matrix M which correspond to the orbits $B_1^G, B_2^G, \ldots, B_n^G \in \mathcal{B}$ of the cyclic $\mathrm{TS}(v, \lambda)$.

It is sufficient to search for solutions of the above equation for values $\lambda_1 = 1, \ldots, \lfloor \lambda/2 \rfloor$. For $\lambda_1 > 1$ we again use the method from [25], and for $\lambda_1 = 1$ we use the exact cover algorithm by Knuth [16].

Fourth step: Test for decomposability. If a cyclic $\mathrm{TS}(v, \lambda)$ with triples $\mathcal{B}$ is "cyclically indecomposable", we use a brute force method to test it for (general) decomposability.

We compute the incidence matrix M'' between pairs (row labels) and triples (column labels) of a fixed $\mathrm{TS}(v, \lambda)$. A $\mathrm{TS}(v, \lambda_1)$, $\lambda_1 < \lambda$, decomposes a given $\mathrm{TS}(v, \lambda)$ if and only if there exists a $\{0, 1\}$ solution of the equation

$$M'' \cdot x = \begin{pmatrix} \lambda_1 \\ \lambda_1 \\ \vdots \\ \lambda_1 \end{pmatrix}.$$

Again, it is sufficient to search for solutions of the above equation for values $\lambda_1 = 1, \ldots, \lfloor \lambda/2 \rfloor$ and we use the methods from [16, 25].

Fifth step: Isomorphism test. At this point we have the number of indecomposable cyclic $\mathrm{TS}(v, \lambda)$ up to equivalence. By an old result by Pálfy [20] this number coincides with the number of non-isomorphic ones when $\gcd(v, \phi(v)) = 1$, where ϕ is Euler's totient function. In all the other cases the GAP design package by Leonard Soicher [24]—which is based on *nauty* [18]—allows one to do an exact classification of all multiplier-equivalent candidates. We found that the number of inequivalent and non-isomorphic indecomposable cyclic $\mathrm{TS}(v, \lambda)$ coincide also in these cases except when $(v, \lambda) = (16, 4)$. In this case the number of inequivalent ones is 2770 and they fall into two types: 2254 systems have a full automorphism group of order 16; the remaining 16 (all of which are non-simple) have a full automorphism group of order 32 and they can be matched into pairs of isomorphic systems. So the number of non-isomorphic indecomposable cyclic $\mathrm{TS}(16, 4)$ is 2262. We will give the "special" sixteen $\mathrm{TS}(16, 4)$ explicitly in the last section.

3 Results

By applying the algorithm described in the previous section we are able to compute the number of non-isomorphic indecomposable cyclic $\mathrm{TS}(v, \lambda)$ for small values of

v and λ. The results are listed in the following table. We have an empty cell in correspondence of pairs (v, λ) for which the existence conditions of Theorem 1.1 are not fulfilled. In the other cells we have either one number when all systems are simple, or two numbers giving the count for simple systems and for all systems (simple and non-simple), respectively. Boldface entries indicate that the numbers are new to the best of our knowledge. Most computations could be done in a matter of minutes. The greatest CPU time was spent in the case TS$(18, 4)$, which took nearly two days on a standard PC. The correctness of the results is supported by the fact that all previously known results were confirmed by our computations.

Looking at Table 1 we can answer the questions posed in Chapter 23 of [12]:

Theorem 3.1 *There exists a unique simple indecomposable cyclic TS*$(16, 6)$ *and there exist "many" non-simple indecomposable cyclic TS*$(v, 5)$ *for* $v = 13, 15$ *and TS*$(v, 6)$ *for* $v = 11, 12, 13, 15$.

4 The simple indecomposable cyclic TS$(16, 6)$

The 15 base blocks of the unique simple indecomposable cyclic triple system TS$(16, 6)$ are

$$\begin{array}{llll}
(0,\ 1,\ 3), & (0,\ 2,\ 6), & (0,\ 3,\ 8), & (0,\ 4,\ 8),\\
(0,\ 1,\ 5), & (0,\ 2,\ 7), & (0,\ 3,\ 9), & \\
(0,\ 1,\ 7), & (0,\ 2,\ 11), & (0,\ 3,\ 10), & \\
(0,\ 1,\ 10), & (0,\ 2,\ 12), & (0,\ 3,\ 11), & \\
(0,\ 1,\ 12), & & & \\
(0,\ 1,\ 14). & & &
\end{array}$$

With suitable cyclic transformations the base blocks can be displayed as in Figure 1. The full automorphism group of the triple system has been determined by GAP [15] to be $\mathbb{D}_{32}$ the dihedral group of order 32; see Remark 5.5.

5 A construction by Skolem sequences

In this section, given positive integers a and b, we denote by $[a, b]$ the set of all integers x such that $a \le x \le b$. We recall the following definition.

Definition 5.1 *Given k, with $1 \le k \le 2n + 1$, a k*-extended Skolem sequence *of order n is a sequence $(s_1, \ldots, s_n)$ of n integers such that*

$$\bigcup_{i=1}^{n} \{s_i, s_i + i\} = [1, 2n + 1] \setminus \{k\}.$$

Example 5.2 *The sequence* $(s_1, \ldots, s_7) := (11, 1, 5, 10, 2, 9, 6)$ *is a* 4*-extended Skolem sequence of order* 7*. Indeed* $\bigcup_{i=1}^{7} \{s_i, s_i + i\}$ *is given by*

$$\{11, 12\} \cup \{1, 3\} \cup \{5, 8\} \cup \{10, 14\} \cup \{2, 7\} \cup \{9, 15\} \cup \{6, 13\}$$

which is exactly $[1, 15] \setminus \{4\}$.

$v \setminus \lambda$	1	2	3	4	5	6	7
4		1		0		0	
5			1			0	
6				**0**			
7	1	0	1	**0**	**0**	**0**	**0**
8						1 12	
9	0		0 1	0 **1**	0 **2**	**0**	**0**
10				**0** **5**			
11			6 8			**0** **49**	
12		0 4		**0** **3**	0	**0** **369**	
13	1	**6**	**14** **24**	**4** **56**	**0** **92**	**0** **77**	**0** **58**
14							
15	2	0 5	**209** **355**	**91** **936**	0 **378**	**0** **7032**	**0** **2579**
16		67 71		**35** **2262**		**1** ≥ 1	
17			990 1550			**255** $\geq$ **255**	
18				**1118** **60041**			
19	4	**151** **161**	**5501** **8839**	**47681** $\geq$ **47681**	? ?	? ?	? ?
21	77	**0** **109**	**124300** **202578**	? ?	? ?	? ?	? ?
25	1212	**16745** **18201**	? ?	? ?	? ?	? ?	? ?

Table 1: Number of pairwise non-isomorphic indecomposable cyclic TS(v, λ).

The existence problem for k-extended Skolem sequences of order n with $k = 2n+1$ (*ordinary* Skolem sequences) and with $k = 2n$ (*hooked* Skolem sequences) was solved a long time ago (see [2]). The solution for arbitrary k has been given by Baker [3].

Theorem 5.3 *There exists a k-extended Skolem sequence of order n with k odd [even] if and only if $n \equiv 0$ or 1 [$n \equiv 2$ or 3] (mod 4).*

Skolem sequences and their variants [22] can be successfully used in many combinatorial problems. For this, we refer to the survey [14] and the references therein.

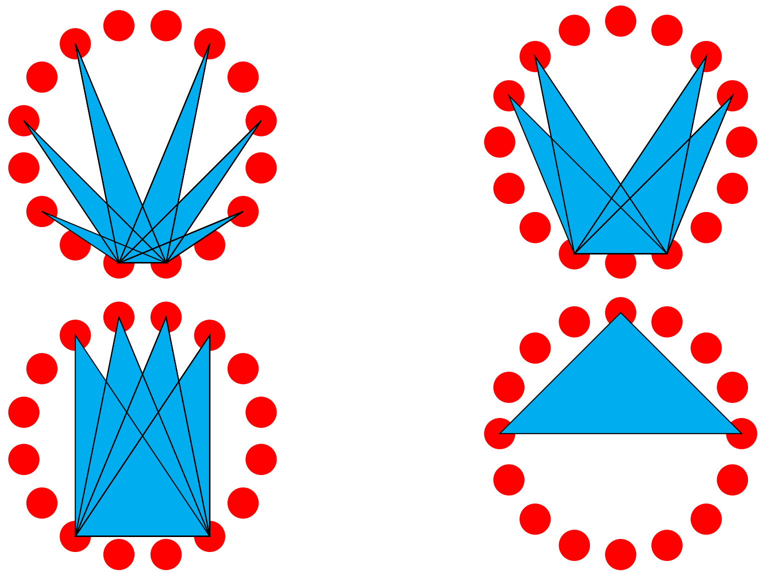

Figure 1: The base blocks of the simple indecomposable cyclic TS$(16, 6)$

It should be pointed out, however, that a series of papers of the first author, such as [10], remained unnoticed in the survey. Here we also mention the more recent [6] and [11], where extended Skolem sequences have been used for determining the set of values of v for which there exists a 1-*rotational* TS(v) (with some open cases for $v \equiv 1$ modulo 96) and a 3-*pyramidal* TS(v), respectively.

Note that for k odd or even we have $2k - 1 \equiv 1$ or 3 (mod 4), respectively. Thus, Baker's theorem assures the existence both of a k-extended and of a $3k$-extended Skolem sequence of order $2k - 1$ for every positive k.

Now we need to recall the notion of a *difference family* (see [1] or [4, Ch. VII]). If B is a subset of $\mathbb{Z}_v$, the list of differences of B is the multiset ΔB of all possible differences $x - y \pmod v$ with (x, y) an ordered pair of distinct elements of B. A (v, k, λ) difference family is a collection $\mathcal{F} = \{B_1, \ldots, B_n\}$ of k-subsets of $\mathbb{Z}_v$, called *base blocks*, such that the multiset union $\Delta B_1 \cup \ldots \cup \Delta B_n$ covers $\mathbb{Z}_v \setminus \{0\}$ exactly λ times. Such a difference family $\mathcal{F}$ generates a cyclic 2-(v, k, λ) design, called a *development* of $\mathcal{F}$ and denoted by $dev(\mathcal{F})$, whose points are the elements of $\mathbb{Z}_v$ and whose blocks are all possible translates $B_i + z$ of all the base blocks. Thus, in particular, a $(v, 3, \lambda)$ difference family generates a cyclic TS(v, λ).

If $\mathcal{F}$ is a difference family as above and μ is any unit of $\mathbb{Z}_v$, then it is clear that the collection $\mu\mathcal{F} := \{\mu B_1, \ldots, \mu B_n\}$ is also a (v, k, λ) difference family. One says that μ is a *multiplier* of $\mathcal{F}$ in the case that $dev(\mu\mathcal{F}) = dev(\mathcal{F})$. This happens if we have $\mu B_i = B_{\pi(i)} + z_i$ for a suitable permutation π of $\{1, \ldots, n\}$ and a suitable n-tuple $(z_1, \ldots, z_n) \in \mathbb{Z}_v^n$. Of course any multiplier of $\mathcal{F}$ is an automorphism of $dev(\mathcal{F})$.

If M is a group of units of $\mathbb{Z}_v$ each of which is a multiplier of $\mathcal{F}$ and $\hat{\mathbb{Z}}_v$ is the group of translations of $\mathbb{Z}_v$, then the semidirect product $M \ltimes \mathbb{Z}_v$ is a group of automorphisms of $dev(\mathcal{F})$. Thus, if -1 is a multiplier of $\mathcal{F}$, then $\mathbb{D}_{2v}$, the dihedral group of order $2v$, is an automorphism of $dev(\mathcal{F})$ since we have $\{1, -1\} \ltimes \hat{\mathbb{Z}}_v \cong \mathbb{D}_{2v}$.

For any positive integer k, we can construct many cyclic $\mathrm{TS}(4k,6)$ using the following construction.

Theorem 5.4 *Let Σ be the set of all $(2k-1)$-tuples each of which is either a k-extended or a $3k$-extended Skolem sequence of order $2k-1$. Given two sequences $S=(s_1,\dots,s_{2k-1})$, $T=(t_1,\dots,t_{2k-1})$ of Σ and given two units u, v of $\mathbb{Z}_{4k}$, define $B_i = u\cdot\{0,i,-s_i\}$ and $C_i = v\cdot\{0,i,-t_i\}$ for $1\le i\le 2k-1$. Also define $A=\{k,2k,3k\}$. Then*

$$\mathcal{F}(S,T,u,v) := \{A\} \ \cup\ \{B_i \mid 1\le i\le 2k-1\} \ \cup\ \{C_i \mid 1\le i\le 2k-1\}$$

is a $(4k,3,6)$ difference family.

Proof. First, let us suppose $u=1$. We have $\Delta(B_i)=\pm\{i,s_i,s_i+i\}$ and, considering that S is an x-extended Skolem sequence of order $2k-1$ with $x\in\{k,3k\}$, we can write:

$$\bigcup_{i=1}^{2k-1}\Delta(B_i) = \pm[1,2k-1] \ \cup\ \pm([1,4k-1]\setminus\{x\}).$$

Taking into account that $2x\equiv 2k \pmod{4k}$ and $3x\equiv 4k-x \pmod{4k}$ for $x\in\{k,3k\}$, we note:

- $\pm[1,2k-1]=\mathbb{Z}_{4k}\setminus\{0,2k\}$;
- $[1,4k-1]\setminus\{x\}=\mathbb{Z}_{4k}\setminus\{0,x\}$;
- $-([1,4k-1]\setminus\{x\})=\mathbb{Z}_{4k}\setminus\{0,3x\}$.

Then it is clear that $\bigcup_{i=1}^{2k-1}\Delta(B_i)$ covers every element of $\mathbb{Z}_{4k}\setminus\{k,2k,3k\}$ exactly three times and every element of $\{k,2k,3k\}$ exactly twice.

Now, in the general case the B_i's are multiplied by a unit u of $\mathbb{Z}_{4k}$, which acts as a permutation of the elements of $\mathbb{Z}_{4k}$ but leaves $\{k,2k,3k\}$ invariant. Therefore, also for an arbitrary unit u of $\mathbb{Z}_{4k}$, the multiset $\bigcup_{i=1}^{2k-1}\Delta(B_i)$ covers every element of $\mathbb{Z}_{4k}\setminus\{k,2k,3k\}$ exactly three times and every element of $\{k,2k,3k\}$ exactly twice.

In the same way we see that this is true also for $\bigcup_{i=1}^{2k-1}\Delta(C_i)$. Finally note that $\Delta(A)$ is two times $\{k,2k,3k\}$. We conclude that the list of differences of $\mathcal{F}(S,T,u,v)$ covers every non-zero element of $\mathbb{Z}_{4k}$ exactly 6 times, namely $\mathcal{F}(S,T,u,v)$ is a $(4k,3,6)$ difference family. $\square$

We note, in particular, that if $S=T$ and $v=-u$, then -1 is evidently a multiplier of $\mathcal{F}(S,T,u,v)$. Therefore, $\mathcal{F}(S,S,1,-1)$ generates a $\mathrm{TS}(4k,6)$ admitting $\mathbb{D}_{8k}$ as an automorphism group for any $S\in\Sigma$.

Remark 5.5 *The cyclic indecomposable $TS(16,6)$ constructed in the previous section is the development of $\mathcal{F}(S,S,1,-1)$ where S is the extended Skolem sequence of Example 5.2. Hence it admits $\mathbb{D}_{32}$ as an automorphism group.*

6 The indecomposable cyclic TS$(16,4)$ with full automorphism group of order 32

As already commented, up to multiplier equivalence, there are 2770 indecomposable cyclic TS$(16,4)$. They all have full automorphism group of order 16 except sixteen of them whose full automorphism group has order 32. These special TS$(16,4)$ can be quickly presented using *similar difference families*, a concept introduced in [8] which will be more deeply exploited in [9].

Given a (v,k,λ) difference family $\mathcal{F}=\{B_1,\dots,B_n\}$ and any subset I of $\{1,\dots,n\}$, set $\mathcal{F}(I)=\{\sigma_1B_1,\dots,\sigma_nB_n\}$ where $\sigma_i=-1$ or $+1$ according to whether i belongs or does not belong to I, respectively. It is clear that $\mathcal{F}(I)$ is also a (v,k,λ) difference family which is said to be *similar* to $\mathcal{F}$. Thus, two difference families are similar if one family can be obtained from the other by flipping the signs of some base blocks.

It is straightforward to check that each of the following four collections of triples is a $(16,3,4)$ difference family.

$$\mathcal{F}_1=\{\{0,1,3\},\{0,1,8\},\{0,1,11\},\{0,1,13\},\{0,2,7\},\\ \{0,2,12\},\{0,2,12\},\{0,3,9\},\{0,3,11\},\{0,4,11\}\}$$

$$\mathcal{F}_2=\{\{0,1,3\},\{0,1,3\},\{0,1,4\},\{0,1,9\},\{0,2,7\},\\ \{0,2,7\},\{0,3,8\},\{0,4,9\},\{0,4,10\},\{0,4,10\}\}$$

$$\mathcal{F}_3=\{\{0,1,3\},\{0,1,3\},\{0,1,9\},\{0,1,12\},\{0,2,7\},\\ \{0,2,7\},\{0,3,7\},\{0,3,8\},\{0,4,10\},\{0,4,10\}\}$$

$$\mathcal{F}_4=\{\{1,2,4]\},\{1,2,9\},\{1,2,12\},\{1,2,13\},\{1,3,8\},\\ \{1,3,13\},\{1,3,13\},\{1,4,8\},\{1,4,9\},\{1,4,10\}\}$$

With the sign flipping method from above, $\mathcal{F}_j(I)$ is also a $(16,3,4)$ difference family for $1\le j\le 4$ and any subset I of $\{1,\dots,10\}$. Up to multiplier equivalence, the sixteen indecomposable cyclic TS$(16,4)$ with full automorphism group of order 32 are those generated by the following difference families:

$\mathcal{F}_1(I)$ for $I=\emptyset,\{6,7\},\{4,9,10\},\{2,4,6,7,9,10\},\{2,4,9,10\},\{4,6,7,9,10\}$;

$\mathcal{F}_2(I)$ for $I=\emptyset,\{4\},\{3,4,8\},\{3,4,7,8\}$;

$\mathcal{F}_3(I)$ for $I=\emptyset,\{3\},\{3,4,7\},\{3,4,7,8\}$;

$\mathcal{F}_4(I)$ for $I=\emptyset,\{5,6\}$.

The number of pairwise non-isomorphic designs among the previous ones is eight; indeed, setting $D_j(I)=dev(\mathcal{F}_j(I))$, one may check that the involutory permutation (2 10)(3 11)(6 14)(7 15) switches the four pairs

$$\mathcal{D}_1(\emptyset)-\mathcal{D}_1(\{6,7\}),\qquad \mathcal{D}_1(\{4,9,10\})-\mathcal{D}_1(\{4,6,7,9,10\}),$$
$$\mathcal{D}_1(\{2,4,9,10\})-\mathcal{D}_1(\{2,4,6,7,9,10\}),\qquad \mathcal{D}_4(\emptyset)-\mathcal{D}_4(\{5,6\}),$$

and that the involutory permutation (1 11)(2 14)(3 9)(4 12)(5 7)(6 10)(13 15) switches the four pairs

$$\mathcal{D}_2(\emptyset)-\mathcal{D}_3(\{3,4,7,8\}),\quad \mathcal{D}_2(\{4\})-\mathcal{D}_3(\{3,4,7\}),$$
$$\mathcal{D}_2(\{3,4,8\})-\mathcal{D}_3(\{3\}),\qquad \mathcal{D}_3(\emptyset)-\mathcal{D}_2(\{3,4,7,8\}).$$

Acknowledgements

This work has been performed under the auspices of the G.N.S.A.G.A. of the C.N.R. (National Research Council) of Italy. The authors thank the anonymous referees for helpful suggestions and for insisting on the classification up to isomorphism.

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(Received 22 Apr 2016; revised 20 Feb 2018, 13 Mar 2018)