AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **71(1)** (2018), Pages 113–120

Proof of Northshield's conjecture concerning an analogue of Stern's sequence for $\mathbb{Z}[\sqrt{2}]$

Michael Coons

School of Mathematical and Physical Sciences
University of Newcastle
Callaghan, NSW 2308
Australia
Michael.Coons@newcastle.edu.au

Abstract

We prove a conjecture of Northshield by determining the maximal order of his analogue of Stern's sequence for $\mathbb{Z}[\sqrt{2}]$. In particular, if b is Northshield's analogue, we prove that

$$\limsup_{n\to\infty} \frac{2b(n)}{(2n)^{\log_3(\sqrt{2}+1)}} = 1.$$

1 Introduction

Stern's diatomic sequence (commonly called *Stern's sequence*) is given by $a(0) = 0$, $a(1) = 1$, and when $n \geqslant 1$ by

$$a(2n) = a(n) \quad \text{and} \quad a(2n+1) = a(n) + a(n+1).$$

As an analogue of Stern's sequence for the ring $\mathbb{Z}[\sqrt{2}]$, Northshield [10] introduced the sequence defined by $b(0) = 0$, $b(1) = 1$, and in general by

$$\begin{aligned} b(3n) &= b(n) \\ b(3n+1) &= \sqrt{2} \cdot b(n) + b(n+1) \\ b(3n+2) &= b(n) + \sqrt{2} \cdot b(n+1). \end{aligned}$$

In joint work with Tyler [7], answering a question of Berlekamp, Conway, and Guy [3, page 115] and improving on a result of Calkin and Wilf [4], we determined the maximal order of Stern's sequence; in particular, we proved that

$$\limsup_{n\to\infty} \frac{a(n)}{n^{\log_2 \varphi}} = \frac{3^{\log_2 \varphi}}{\sqrt{5}},$$

where $\varphi = (1+\sqrt{5})/2$ is the golden mean. Here and throughout this paper, we write $\log_k c$ for the base-k logarithm of the real number c. Concerning his analogue, Northshield [10, Cor. 5] showed that

$$\limsup_{n\to\infty} \frac{2b(n)}{(2n)^{\log_3(\sqrt{2}+1)}} \geqslant 1, \tag{1}$$

and he conjectured that equality holds.

In this paper, using the method developed by Coons and Tyler [7] (see also Coons and Spiegelhofer [6]), we prove Northshield's conjecture.

Theorem 1. *Let $\{b(n)\}_{n\geqslant 0}$ denote Northshield's analogue of Stern sequence as defined above. Then*

$$\limsup_{n\to\infty} \frac{2b(n)}{(2n)^{\log_3(\sqrt{2}+1)}} = 1.$$

This paper is organised as follows. In Section 2, we define a piecewise linear function and provide several lemmas comparing it to Northshield's sequence. In Section 3, we record a few additional lemmas and also prove Theorem 1. Finally, in Section 4, we give some further comparisons with Stern's sequence and related values and functions.

2 Preliminaries

We proceed along the same lines as the arguments of Coons and Tyler [7] and Coons and Spiegelhofer [6]. In particular, we will define a piecewise linear function h, which will serve as an upper bound for the sequence b. The benefit in this situation is that h is continuous and (except at a few points) differentiable. As well, the function h will be close to the sequence b for the maximal values of b. This closeness will allow us to use the asymptotic properties of h to determine the desired asymptotics concerning b.

We start by formally defining the function h and a special sequence of points.

Definition 2. For $n \geqslant 1$, let $x_n := 3^n/2$, $y_n := (\sqrt{2}+1)^n/2$ and let $h : \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ be the piecewise linear function connecting the set of points $\{(0,0)\} \cup \{(x_n, y_n) : n \geqslant 1\}$.

Northshield proved that[1]

$$\max\{b(m) : m \in (3^{n-1}, 3^n]\} = \frac{(\sqrt{2}+1)^n + (\sqrt{2}-1)^n}{2},$$

and, moreover, the first such maximum in this interval occurs at $m = (3^n+1)/2$. The points $\{(x_n, y_n) : n \geqslant 1\}$ were chosen to be very close to the points where b achieves its maximal values.

Lemma 1. *For $m \geqslant 2$, we have $b(m) \leqslant h(m) + (\sqrt{2}+1)\lfloor \log_3(m) \rfloor$.*

[1] Our version corrects a small typo in [10].

Proof. Throughout this proof, we use freely the fact that for $m > 1$,

$$(\sqrt{2}+1)\lfloor \log_3(m) \rfloor > \lfloor \log_3(m) \rfloor.$$

Also, note that in the interval $[x_n, x_{n+1}]$, we have that

$$\begin{aligned} h(x) &= \frac{h(x_{n+1}) - h(x_n)}{x_{n+1} - x_n}(x - x_n) + h(x_n) \\ &= \frac{\sqrt{2}}{2}\left(\frac{\sqrt{2}+1}{3}\right)^n x + (\sqrt{2}+1)^n\left(\frac{2-\sqrt{2}}{4}\right). \end{aligned} \tag{2}$$

We will proceed by induction. Using (2), we now can check, as a base case, that the result of the lemma holds in the interval $(3^0, 3^2] = (1, 9]$; see Table 1 for these values.

Table 1: Values (showing only three decimal places) demonstrating that $b(m) \leqslant h(m) + \lfloor \log_3(m) \rfloor$ for $m = 2, \ldots, 9$; that is, all m in the interval $(3^0, 3^2] = (1, 9]$.

m	2	3	4	5	6	7	8	9
$b(m)$	1.414	1	2.828	3	1.414	3	2.828	1
$h(m) + \lfloor \log_3(m) \rfloor$	1.491	3.060	3.629	4.198	4.767	5.336	5.905	7.474

Suppose that the result holds in $(3^{n-1}, 3^n]$ and consider $(3^n, 3^{n+1}]$. As mentioned above, the first occurring maximum value of b in $(3^n, 3^{n+1}]$ is

$$b\left(\frac{3^{n+1}+1}{2}\right) = \frac{(\sqrt{2}+1)^{n+1} + (\sqrt{2}-1)^{n+1}}{2}.$$

As $(3^{n+1}+1)/2 \in (x_{n+1}, x_{n+2}]$, by (2), at this value we have

$$\begin{aligned} h\left(\frac{3^{n+1}+1}{2}\right) + \left\lfloor \log_3\left(\frac{3^{n+1}+1}{2}\right)\right\rfloor &= \frac{\sqrt{2}}{2}\left(\frac{\sqrt{2}+1}{3}\right)^{n+1}\left(\frac{3^{n+1}+1}{2}\right) \\ &\quad + (\sqrt{2}+1)^{n+1}\left(\frac{2-\sqrt{2}}{4}\right) + n \\ &= \left(\frac{\sqrt{2}}{4\cdot 3^{n+1}} + \frac{1}{2}\right)(\sqrt{2}+1)^{n+1} + n \qquad (3) \\ &> \frac{(\sqrt{2}+1)^{n+1} + (\sqrt{2}-1)^{n+1}}{2} \\ &= b\left(\frac{3^{n+1}+1}{2}\right), \end{aligned}$$

so the lemma holds for the value $(3^{n+1}+1)/2$.

Now if $m \in [(3^{n+1}+1)/2, 3^{n+1}]$, since the lemma holds for the value $(3^{n+1}+1)/2$ and b takes its maximal value in $(3^n, 3^{n+1}]$ at $(3^{n+1}+1)/2$, we have

$$b(m) \leqslant b\left(\frac{3^{n+1}+1}{2}\right) \leqslant h\left(\frac{3^{n+1}+1}{2}\right) + \left\lfloor \log_3\left(\frac{3^{n+1}+1}{2}\right)\right\rfloor \leqslant h(m) + \lfloor \log_3(m)\rfloor,$$

where the last inequality follows from the fact that h is monotonically increasing. Thus the lemma holds in the interval $[(3^{n+1}+1)/2, 3^{n+1}]$. It remains to show that the result holds for $m \in (3^n, (3^{n+1}-1)/2]$.

If $m = 3k \in (3^n, (3^{n+1}-1)/2]$, then $k \in (3^{n-1}, 3^n]$. By Northshield's definition and the induction hypothesis, we have

$$b(m) = b(3k) = b(k) \leqslant h(k) + \lfloor \log_3(k)\rfloor \leqslant h(m) + \lfloor \log_3(m)\rfloor,$$

where as above, the last inequality follows from the monotonicity of h.

If $m = 3k+1 \in (3^n, (3^{n+1}-1)/2]$, then $k+1 \in (3^{n-1}, (3^n+1)/2]$. Note that in this case, using (2), we have

$$\begin{aligned} h(3k+1) &- (\sqrt{2}+1)h(k+1) \\ &= \frac{\sqrt{2}}{2}\left(\frac{\sqrt{2}+1}{3}\right)^{n+1}(3k+1) - \frac{3\sqrt{2}}{2}\left(\frac{\sqrt{2}+1}{3}\right)^{n+1}(k+1) \\ &= -\sqrt{2}\left(\frac{\sqrt{2}+1}{3}\right)^{n+1} \in (-1,0). \quad (4) \end{aligned}$$

Now

$$\begin{aligned} b(m) = b(3k+1) &= \sqrt{2}\cdot b(k) + b(k+1) \\ &\leqslant (\sqrt{2}+1)\cdot \max\{b(k), b(k+1)\} \\ &\leqslant (\sqrt{2}+1)\left(h(k+1) + \lfloor \log_3(k+1)\rfloor\right), \end{aligned}$$

again appealing to the monotonicity of h. Combining this with (4) and using the induction hypothesis, we have

$$\begin{aligned} b(m) = b(3k+1) &\leqslant h(3k+1) + (\sqrt{2}+1)\lfloor \log_3(k+1)\rfloor + 1 \\ &\leqslant h(3k+1) + (\sqrt{2}+1)\lfloor \log_3(3k+1)\rfloor, \end{aligned}$$

since here $\lfloor \log_3(k+1)\rfloor = n$ and $\lfloor \log_3(3k+1)\rfloor = n+1$. Thus the result holds for $m = 3k+1 \in (3^n, (3^{n+1}-1)/2]$.

The remaining case is $m = 3k+2 \in (3^n, (3^{n+1}-1)/2]$. But this follows easily from the monotonicity of h, as again we have

$$b(m) = b(3k+2) = b(k) + \sqrt{2}\cdot b(k+1) \leqslant (\sqrt{2}+1)\cdot \max\{b(k), b(k+1)\}.$$

Thus the previous case along with the monotonicity of h gives

$$\begin{aligned} b(m) = b(3k+2) &\leqslant h(3k+1) + (\sqrt{2}+1)\lfloor \log_3(3k+1)\rfloor \\ &\leqslant h(3k+2) + (\sqrt{2}+1)\lfloor \log_3(3k+2)\rfloor. \end{aligned}$$

This finishes the proof of the lemma. □

3 Proof of Northshield's conjecture

In this section, we provide two essential lemmas, and give the proof of Northshield's conjecture.

Lemma 2. *We have*

$$\limsup_{m\to\infty} \frac{b(m)}{h(m)} = 1.$$

Proof. Set $m_n := (3^{n+1}+1)/2$. Note that $b(m_n) \sim (\sqrt{2}+1)^{n+1}/2$ and also, recalling (3), that

$$h\left(\frac{3^{n+1}+1}{2}\right) = \left(\frac{\sqrt{2}}{4\cdot 3^{n+1}} + \frac{1}{2}\right)(\sqrt{2}+1)^{n+1} \sim \frac{(\sqrt{2}+1)^{n+1}}{2}.$$

Thus

$$1 = \lim_{n\to\infty} \frac{b(m_n)}{h(m_n)} \leqslant \limsup_{m\to\infty} \frac{b(m)}{h(m)} \leqslant \limsup_{m\to\infty} \frac{h(m) + (\sqrt{2}+1)\lfloor \log_3 m\rfloor}{h(m)} = 1,$$

where the last inequality is given by Lemma 1 and the final equality follows since for $m \in [x_n, x_{n+1}]$, we have

$$\frac{(\sqrt{2}+1)\lfloor \log_3 m\rfloor}{h(m)} \leqslant \frac{3\lfloor \log_3 x_{n+1}\rfloor}{h(x_n)} \leqslant \frac{6(n+1)}{(\sqrt{2}+1)^n}. \qquad \square$$

Lemma 3. *For $x > 3/2$, we have $2\cdot h(x) \leqslant (2x)^{\log_3(\sqrt{2}+1)}$.*

Proof. Firstly, note that for the sequence x_n as given in Definition 2 and $n \geqslant 1$, we have $\log_3 x_n = n - \log_3 2$, so that

$$2\cdot h(x_n) = 2\cdot y_n = (\sqrt{2}+1)^n = (\sqrt{2}+1)^{\log_3 x_n + \log_3 2} = (2x_n)^{\log_3(\sqrt{2}+1)},$$

which shows the lemma holds for the values x_n.

Write

$$H(x) := 2\cdot h(x) - (2x)^{\log_3(\sqrt{2}+1)}.$$

If $H(x) > 0$ for some $x \in [x_n, x_{n+1}]$, then since H is differentiable in (x_n, x_{n+1}) there is some $w \in (x_n, x_{n+1})$ where H attains a maximum value. But

$$\begin{aligned}\frac{d^2}{dx^2}H(x) &= \frac{d^2}{dx^2}\left\{-(2x)^{\log_3(\sqrt{2}+1)}\right\}\\ &= -2^{\log_3(\sqrt{2}+1)}\log_3(\sqrt{2}+1)(\log_3(\sqrt{2}+1)-1)x^{\log_3(\sqrt{2}+1)-2},\end{aligned}$$

which is positive for all $x \in [x_n, x_{n+1}]$. Thus $H(x) \leqslant 0$ for all $x > x_1 = 3/2$ proving the lemma. $\square$

Proof of Theorem 1. By Lemmas 2 and 3 we have

$$1 \leqslant \limsup_{m\to\infty} \frac{2b(m)}{(2m)^{\log_3(\sqrt{2}+1)}} \leqslant \limsup_{m\to\infty} \frac{b(m)}{h(m)} = 1,$$

where the first inequality, recorded in (1), is due to Northshield. $\square$

4 Further remarks

Both Stern's sequence and Northshield's analogue are examples of k-regular sequences as defined by Allouche and Shallit in their seminal paper [1]; see also their monograph, *Automatic Sequences* [2]. For an integer $k \geqslant 2$, an integer-valued sequence f is called *k-regular* provided there exist a positive integer d, a finite set of matrices $\mathcal{M} = \{\mathbf{M}_0, \dots, \mathbf{M}_{k-1}\} \subseteq \mathbb{Z}^{d\times d}$, and vectors $\mathbf{v}, \mathbf{w} \in \mathbb{Z}^d$ such that

$$f(n) = \mathbf{w}^T \mathbf{M}_w \mathbf{v},$$

where $\mathbf{M}_w = \mathbf{M}_{i_0} \cdots \mathbf{M}_{i_s}$ and $w = i_0 \cdots i_s$ is the reversal of the base-k expansion $(n)_k = i_s \cdots i_0$; see [1, Lemma 4.1]. We call the tuple $(\mathbf{w}, \mathcal{M}, \mathbf{v})$ the *linear representation* of the k-regular sequence f.

Stern's sequence a is 2-regular and has linear representation

$$\left([1\ 0], \{\mathbf{A}_0, \mathbf{A}_1\} = \left\{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\right\}, [1\ 0]\right),$$

whereas Northshield's sequence b is 3-regular (though not integer-valued) and has linear representation

$$\left([1\ 0], \{\mathbf{B}_0, \mathbf{B}_1, \mathbf{B}_2\} = \left\{\begin{bmatrix} 1 & 0 \\ \sqrt{2} & 1 \end{bmatrix}, \begin{bmatrix} \sqrt{2} & 1 \\ 1 & \sqrt{2} \end{bmatrix}, \begin{bmatrix} 1 & \sqrt{2} \\ 0 & 1 \end{bmatrix}\right\}, [0\ 1]\right).$$

This representation of k-regular sequences looks a lot like the matrix version of a linear recurrence (coefficients of rational power series), and indeed, k-regular sequences are sometimes known as 'radix-rational' sequences.

The method used here can give analogous results for other k-regular sequences. Essentially this can be done using the following recipe for a k-regular sequence f:

1. Determine the maximal values of f between consecutive powers of k and where they first occur.

2. Find a piecewise linear function h that is both monotonically increasing and close enough to the above determined maximal values of f so that one has $\limsup_{n\to\infty} f(n)/h(n) = 1$.

3. Show that the desired maximal order holds for h and deduce from Step 2 that it also holds for f.

Compared to Step 1, in general, Steps 2 and 3 should be relatively easy. The difficulty in Step 1 is related to questions surrounding the joint spectral radius of finite sets of (in this case) integer matrices.

The *joint spectral radius* of a finite set of matrices $\mathcal{M} = \{\mathbf{M}_0, \mathbf{M}_1, \dots, \mathbf{M}_{k-1}\}$, denoted $\rho(\mathcal{M})$, is defined as the real number

$$\rho(\mathcal{M}) = \limsup_{n\to\infty} \max_{0\leqslant i_0, i_1, \dots, i_{n-1} \leqslant k-1} \left\| \mathbf{M}_{i_0} \mathbf{M}_{i_1} \cdots \mathbf{M}_{i_{n-1}} \right\|^{1/n},$$

where $\|\cdot\|$ is any (submultiplicative) matrix norm. It is quite clear that when all of the $\mathbf{M}_i$ are equal, say to a matrix $\mathbf{M}$, the joint spectral radius of $\mathcal{M}$ is equal to the spectral radius of $\mathbf{M}$. The joint spectral radius was introduced by Rota and Strang [11] and has a wide range of applications. For an extensive treatment, see Jungers's monograph [8].

For the examples of Stern's and Northshield's sequences, the joint spectral radii are the golden and silver means, respectively. That is,

$$\rho(\{\mathbf{A}_0, \mathbf{A}_1\}) = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad \rho(\{\mathbf{B}_0, \mathbf{B}_1, \mathbf{B}_1\}) = \rho(\mathbf{B}_1) = \sqrt{2}+1.$$

The result for the Stern sequence has been known for some decades already, and for Northshield's sequence, Theorem 1 provides proof; see Coons [5] for additional details.

If one can find the joint spectral radius of the set $\mathcal{M}$ associated to f, then one can probably find the maximal values of f, though in practice, this has been done in the other direction within the research of this area.

Where these maximal values occur is related to an interesting and still-open question due to Lagarias and Wang [9]. The finite set of integer matrices $\mathcal{M}$ is said to satisfy the *finiteness property* provided there is a specific finite product $\mathbf{M}_{i_0}\cdots\mathbf{M}_{i_{m-1}}$ of matrices from $\mathcal{M}$ such that $\rho(\mathbf{M}_{i_0}\cdots\mathbf{M}_{i_{m-1}})^{1/m} = \rho(\mathcal{M})$. Currently, there is no general way to determine if such a set $\mathcal{M}$ satisfies the finiteness property.

In the cases of Stern's and Northshield's sequences, both sets of matrices satisify the finiteness property. For Stern's sequence the finite product is $\mathbf{A}_0\mathbf{A}_1$, and for Northshield's sequence it is the single matrix $\mathbf{B}_1$.

Acknowledgements

This research was done while visiting the Alfréd Rényi Institute of the Hungarian Academy of Sciences; we thank the Institute and its members for their kindness and support. We also thank the referees for their careful readings and valuable comments.

References

[1] Jean-Paul Allouche and Jeffrey Shallit, The ring of k-regular sequences, *Theoret. Comput. Sci.* 98(2) (1992), 163–197.

[2] Jean-Paul Allouche and Jeffrey Shallit, *Automatic sequences*, Cambridge University Press, Cambridge, 2003.

[3] Elwyn R. Berlekamp, John H. Conway, and Richard K. Guy, *Winning ways for your mathematical plays, Vol. 1*, Academic Press Inc. [Harcourt Brace Jovanovich Publishers], London, 1982. Games in general.

[4] Neil J. Calkin and Herbert S. Wilf, Binary partitions of integers and Stern-Brocot-like trees, unpublished, updated version 5 Aug. 2009, 19 pp., 1998.

[5] Michael Coons, Regular sequences and the joint spectral radius, *Internat. J. Found. Comput. Sci.* 28(2) (2017), 135–140.

[6] Michael Coons and Lukas Spiegelhofer, The maximal order of hyper-(b-ary)-expansions, *Electron. J. Combin.* 24(1) (2017), Paper 1.15, 8 pp.

[7] Michael Coons and Jason Tyler, The maximal order of Stern's diatomic sequence, *Mosc. J. Comb. Number Theory* 4(3) (2014), 3–14.

[8] Raphaël Jungers, *The joint spectral radius*, *Lec. Notes in Control and Information Sciences* Vol. 385, Springer-Verlag, Berlin, 2009. Theory and applications.

[9] Jeffrey C. Lagarias and Yang Wang, The finiteness conjecture for the generalized spectral radius of a set of matrices, *Linear Algebra Appl.* 214 (1995), 17–42.

[10] Sam Northshield, An analogue of Stern's sequence for $\mathbb{Z}[\sqrt{2}]$, *J. Integer Seq.* 18(11) (2015), Art. 15.11.6, 22pp.

[11] Gian-Carlo Rota and Gilbert Strang, A note on the joint spectral radius, *Nederl. Akad. Wetensch. Proc. Ser. A 63, also: Indag. Math.* 22 (1960), 379–381.

(Received 7 Sep 2017)