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A matrix approach to the Yang multiplication theorem

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Dedicated to the memory of Professor Noboru Ito

Abstract

In this paper, we use two-variable Laurent polynomials attached to matrices to encode properties of compositions of sequences. The Lagrange identity in the ring of Laurent polynomials is then used to give a short and transparent proof of a theorem about the Yang multiplication.

1 Introduction

Many classes of complementary sequences have been investigated in the literature (see [1]). A quadruple of (± 1)-sequences $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d})$ of length m, m, n, n, respectively, is called *base sequences* if

$$N_{\boldsymbol{a}}(j) + N_{\boldsymbol{b}}(j) + N_{\boldsymbol{c}}(j) + N_{\boldsymbol{d}}(j) = 0$$

for all positive integers j, where

$$N_{\boldsymbol{s}}(j) = \begin{cases} \sum_{i=0}^{l-j-1} s_i s_{i+j} & \text{if } 0 \le j < l, \\ 0 & \text{otherwise,} \end{cases}$$

for $\boldsymbol{s} = (s_0, \ldots, s_{l-1}) \in \{\pm 1\}^l$. We denote by $BS(m, n)$ the set of base sequences of length m, m, n, n. If $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}) \in BS(m, n)$, then it is complementary with weight $2(m + n)$. In [9], Yang proved the following theorem, which is known as one version of the *Yang multiplication theorem*:

Theorem 1.1 ([9, Theorem 4]). *If $BS(m + 1, m) \neq \emptyset$ and $BS(n + 1, n) \neq \emptyset$, then $BS(m', m') \neq \emptyset$ with $m' = (2m + 1)(2n + 1)$.*

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The well-known Hadamard conjecture states that Hadamard matrices of order $4n$ exist for every positive integer n. A consequence of Theorem 1.1 is the existence of a Hadamard matrix of order $8m'$ for a positive integer m' satisfying the hypotheses. Indeed, a class of sequences called T-sequences with length $2m'$ can be obtained from $BS(m', m')$ [8], and Hadamard matrices of order $8m'$ can be produced from T-sequences with length $2m'$ by using Goethals–Seidel arrays [10]. For more information on T-sequences, we refer the reader to [1, 2, 3, 4].

In order to prove Theorem 1.1, Yang used the Lagrange identity for polynomial rings. Let $\mathbb{Z}[x^{\pm 1}]$ be the ring of Laurent polynomials over $\mathbb{Z}$ and $* : \mathbb{Z}[x^{\pm 1}] \to \mathbb{Z}[x^{\pm 1}]$ be the involutive automorphism defined by $x \mapsto x^{-1}$. Let $\boldsymbol{a} = (a_0, \dots, a_{l-1}) \in \mathbb{Z}^l$. We define the *Hall polynomial* $\phi_{\boldsymbol{a}}(x) \in \mathbb{Z}[x^{\pm 1}]$ of $\boldsymbol{a}$ by

$$\phi_{\boldsymbol{a}}(x) = \sum_{i=0}^{l-1} a_i x^i.$$

It is easy to see that a quadraple (± 1)-sequences $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d})$ of length m, m, n, n, respectively, is a base sequences if and only if

$$(\phi_{\boldsymbol{a}}\phi_{\boldsymbol{a}}^* + \phi_{\boldsymbol{b}}\phi_{\boldsymbol{b}}^* + \phi_{\boldsymbol{c}}\phi_{\boldsymbol{c}}^* + \phi_{\boldsymbol{d}}\phi_{\boldsymbol{d}}^*)(x) = 2(m+n).$$

Suppose $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}) \in BS(n+1, n)$ and $(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}, \boldsymbol{e}) \in BS(m+1, m)$. The proof of Theorem 1.1 in [9] is by establishing the identity

$$\begin{aligned}&(\phi_{\boldsymbol{q}}\phi_{\boldsymbol{q}}^* + \phi_{\boldsymbol{r}}\phi_{\boldsymbol{r}}^* + \phi_{\boldsymbol{s}}\phi_{\boldsymbol{s}}^* + \phi_{\boldsymbol{t}}\phi_{\boldsymbol{t}}^*)(x) \\ &= (\phi_{\boldsymbol{a}}\phi_{\boldsymbol{a}}^* + \phi_{\boldsymbol{b}}\phi_{\boldsymbol{b}}^* + \phi_{\boldsymbol{c}}\phi_{\boldsymbol{c}}^* + \phi_{\boldsymbol{d}}\phi_{\boldsymbol{d}}^*)(x^2)(\phi_{\boldsymbol{e}}\phi_{\boldsymbol{e}}^* + \phi_{\boldsymbol{f}}\phi_{\boldsymbol{f}}^* + \phi_{\boldsymbol{g}}\phi_{\boldsymbol{g}}^* + \phi_{\boldsymbol{h}}\phi_{\boldsymbol{h}}^*)(x^{2(2m+1)}), \end{aligned} \tag{1}$$

after defining the sequences $\boldsymbol{q}$, $\boldsymbol{r}$, $\boldsymbol{s}$, $\boldsymbol{t}$ appropriately such that, in particular,

$$\begin{aligned}\phi_{\boldsymbol{q}}(x) = {}& \phi_{\boldsymbol{a}}(x^2)\phi_{\boldsymbol{f}^*}(x^{2(2m+1)}) + x\phi_{\boldsymbol{c}}(x^2)\phi_{\boldsymbol{g}}(x^{2(2m+1)}) \\ & - x^{2(2m+1)}\phi_{\boldsymbol{b}^*}(x^2)\phi_{\boldsymbol{e}}(x^{2(2m+1)}) + x^{2(2m+1)+1}\phi_{\boldsymbol{d}}(x^2)\phi_{\boldsymbol{h}}(x^{2(2m+1)}).\end{aligned}$$

A key to the proof is the Lagrange identity (see [9, Theorem L]): given a, b, c, d, e, f, g, h in a commutative ring with an involutive automorphism $*$, set

$$\begin{aligned} q &= af^* + cg - b^*e + dh, \\ r &= bf^* + dg^* + a^*e - ch^*, \\ s &= ag^* - cf - bh - d^*e, \\ t &= bg - df + ah^* + c^*e. \end{aligned} \tag{2}$$

Then

$$qq^* + rr^* + ss^* + tt^* = (aa^* + bb^* + cc^* + dd^*)(ee^* + ff^* + gg^* + hh^*). \tag{3}$$

However, the derivation of (1) from (3) is not so immediate since one has to define a, b, c, d, e, f, g, h, as

$$\begin{aligned}&\phi_{\boldsymbol{a}}(x^2), \phi_{\boldsymbol{b}}(x^2), x\phi_{\boldsymbol{c}}(x^2), x\phi_{\boldsymbol{d}}(x^2), \\ &x^{2m+(1-n)(2m+1)}\phi_{\boldsymbol{e}}(x^{2(2m+1)}), x^{-n(2m+1)}\phi_{\boldsymbol{f}}(x^{2(2m+1)}), \\ &x^{-n(2m+1)}\phi_{\boldsymbol{g}}(x^{2(2m+1)}), x^{(1-n)(2m+1)}\phi_{\boldsymbol{h}}(x^{2(2m+1)}),\end{aligned}$$

rather than

$$\phi_{\boldsymbol{a}}(x^2), \phi_{\boldsymbol{b}}(x^2), \phi_{\boldsymbol{c}}(x^2), \phi_{\boldsymbol{d}}(x^2), \phi_{\boldsymbol{e}}(x^{2(2m+1)}), \phi_{\boldsymbol{f}}(x^{2(2m+1)}), \phi_{\boldsymbol{g}}(x^{2(2m+1)}), \phi_{\boldsymbol{h}}(x^{2(2m+1)}),$$

respectively. We note that Đoković and Zhao [7] observed some connection between the Yang multiplication theorem and the octonion algebra. More information on the Yang multiplication theorem and constructions of complementary sequences can be found in [5].

In this paper, we give a more straightforward proof of Theorem 1.1. Our approach is by constructing a matrix Q from the eight sequences $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}, \boldsymbol{e}, \boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}$ and produce Laurent polynomials $\psi_{\boldsymbol{s}}(x)$ for $\boldsymbol{s} \in \{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}, \boldsymbol{e}, \boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}\}$ of single variable and a Laurent polynomial $\psi_Q(x, y)$ of two variables for a matrix Q, such that

$$\psi_Q(x, y) = \psi_{\boldsymbol{a}}(x)\psi_{\boldsymbol{f}}(y) + \psi_{\boldsymbol{c}}(x)\psi_{\boldsymbol{g}}(y) + \psi_{\boldsymbol{b}}(x)\psi_{\boldsymbol{e}}(y) + \psi_{\boldsymbol{d}}(x)\psi_{\boldsymbol{h}}(y).$$

This gives an interpretation of the Lagrange identity in term of sequences and matrices, i.e. there exist matrices Q, R, S, T such that

$$\begin{aligned}
&(\psi_Q\psi_Q^* + \psi_R\psi_R^* + \psi_S\psi_S^* + \psi_T\psi_T^*)(x, y) \\
&= (\psi_{\boldsymbol{a}}\psi_{\boldsymbol{a}}^* + \psi_{\boldsymbol{b}}\psi_{\boldsymbol{b}}^* + \psi_{\boldsymbol{c}}\psi_{\boldsymbol{c}}^* + \psi_{\boldsymbol{d}}\psi_{\boldsymbol{d}}^*)(x)(\psi_{\boldsymbol{e}}\psi_{\boldsymbol{e}}^* + \psi_{\boldsymbol{f}}\psi_{\boldsymbol{f}}^* + \psi_{\boldsymbol{g}}\psi_{\boldsymbol{g}}^* + \psi_{\boldsymbol{h}}\psi_{\boldsymbol{h}}^*)(y).
\end{aligned}$$

Then (1) follows immediately by noticing $\psi_Q(x, x^{(2n+1)}) = \psi_{\boldsymbol{q}}(x)$ and $(\psi_{\boldsymbol{a}}\psi_{\boldsymbol{a}}^*)(x) = (\phi_{\boldsymbol{a}}\phi_{\boldsymbol{a}}^*)(x^2)$.

The paper is organized as follows. In Section 2, we will define a Laurent polynomial $\psi_{\boldsymbol{a}}(x)$ for a sequence $\boldsymbol{a}$ and introduce basic properties of $\psi_{\boldsymbol{a}}(x)$. We will also show how to combine sequences and matrices to produce new sequences and matrices, eventually leading to a construction of a matrix from a given set of eight sequences. Finally, in Section 3, we will prove Theorem 1.1 as a consequence of the Lagrange identity in the ring of Laurent polynomials of two variables. We note here that Theorem 1.1 [9, Theorem 4] is known as one of the Yang multiplication theorem. Other versions of the Yang multiplication theorem will be investigated in subsequent papers.

2 Preliminary Results

Let $\mathcal{R}$ be a commutative ring with identity and let $*$ be an involutive automorphism of $\mathcal{R}$. Moreover, let $\mathcal{R}[x^{\pm 1}]$ be the ring of Laurent polynomials over $\mathcal{R}$ and $* : \mathcal{R}[x^{\pm 1}] \to \mathcal{R}[x^{\pm 1}]$ be the extension of the involutive automorphism $*$ of $\mathcal{R}$ defined by $x \mapsto x^{-1}$.

Definition 2.1. Let $\boldsymbol{a} = (a_0, \dots a_{l-1}) \in \mathcal{R}^l$. We define the *Hall polynomial* $\phi_{\boldsymbol{a}}(x) \in \mathcal{R}[x^{\pm 1}]$ of $\boldsymbol{a}$ by

$$\phi_{\boldsymbol{a}}(x) = \sum_{i=0}^{l-1} a_i x^i.$$

We define a Laurent polynomial $\psi_{\boldsymbol{a}}(x) \in \mathcal{R}[x^{\pm 1}]$ by

$$\psi_{\boldsymbol{a}}(x) = x^{1-l}\phi_{\boldsymbol{a}}(x^2).$$

Hall polynomials have been used not only by Yang, but also others. See [6] and references therein. For a sequence $\boldsymbol{a} = (a_0, \dots, a_{l-1}) \in \mathcal{R}^l$ of length l we define $\boldsymbol{a}^* \in \mathcal{R}^l$ by $(a_{l-1}^*, \dots, a_0^*)$. It follows immediately that $\boldsymbol{a}^{**} = \boldsymbol{a}$ for every $\boldsymbol{a} \in \mathcal{R}^l$.

Definition 2.2. For a sequence $\boldsymbol{a} = (a_0, \dots, a_{l-1})$ of length l with entries in $\mathcal{R}$, we define *the non-periodic autocorrelation* $N_{\boldsymbol{a}}$ *of* $\boldsymbol{a}$ by

$$N_{\boldsymbol{a}}(j) = \begin{cases} \sum_{i=0}^{l-j-1} a_i a_{i+j}^* & \text{if } 0 \le j < l, \\ 0 & \text{otherwise.} \end{cases}$$

We say that a set of sequences $\{\boldsymbol{a}_1, \dots, \boldsymbol{a}_n\}$ not necessarily all of the same length, is *complementary with weight* w if

$$\sum_{i=1}^{n} N_{\boldsymbol{a}_i}(j) = \begin{cases} w & \text{if } j = 0, \\ 0 & \text{otherwise.} \end{cases}$$

By Definition 2.2 with $\mathcal{R} = \mathbb{Z}$, we see that $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}) \in BS(m,n)$ if and only if $\{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}\}$ is complementary with weight $2(m+n)$.

Lemma 2.3. *Let l be a positive integer and $\boldsymbol{a} \in \mathcal{R}^l$. Then*

$$\psi_{\boldsymbol{a}^*}(x) = \psi_{\boldsymbol{a}}^*(x).$$

Proof. Straightforward. □

Lemma 2.4. *For sequences $\boldsymbol{a}_1, \dots, \boldsymbol{a}_n$ with entries in $\mathcal{R}$, the following are equivalent.*

(i) *$\boldsymbol{a}_1, \dots, \boldsymbol{a}_n$ are complementary with weight w,*

(ii) $\sum_{i=1}^{n} (\phi_{\boldsymbol{a}_i} \phi_{\boldsymbol{a}_i}^*)(x) = w$,

(iii) $\sum_{i=1}^{n} (\psi_{\boldsymbol{a}_i} \psi_{\boldsymbol{a}_i}^*)(x) = w$.

Proof. It is straightforward to check that (i) is equivalent to (ii). Equivalence of (ii) and (iii) is clear since for any sequence $\boldsymbol{a}$, $\phi_{\boldsymbol{a}}(x^2)\phi_{\boldsymbol{a}}^*(x^2) = \psi_{\boldsymbol{a}}(x)\psi_{\boldsymbol{a}}^*(x)$ from Definition 2.1. □

Definition 2.5. Let $\boldsymbol{a} = (a_0, \dots, a_{l-1}) \in \mathcal{R}^l$. Define

$$\boldsymbol{a}/0 = (a_0, 0, a_1, \dots, 0, a_{l-1}) \in \mathcal{R}^{2l-1}, \quad 0/\boldsymbol{a} = (0, a_0, 0, \dots, a_{l-1}, 0) \in \mathcal{R}^{2l+1}.$$

Lemma 2.6. *For every $\boldsymbol{a} \in \mathcal{R}^l$,*

$$\psi_{\boldsymbol{a}/0}(x) = \psi_{0/\boldsymbol{a}}(x) = \psi_{\boldsymbol{a}}(x^2).$$

Proof. By Definition 2.1 and Definition 2.5 , we have

$$\begin{aligned} \psi_{\boldsymbol{a}/0}(x) &= x^{1-(2l-1)}\phi_{\boldsymbol{a}/0}(x^2) = x^{2-2l}\phi_{\boldsymbol{a}}(x^4) = \psi_{\boldsymbol{a}}(x^2), \\ \psi_{0/\boldsymbol{a}}(x) &= x^{1-(2l+1)}\phi_{0/\boldsymbol{a}}(x^2) = x^{-2l}x^2\phi_{\boldsymbol{a}}(x^4) = \psi_{\boldsymbol{a}}(x^2). \end{aligned}$$

□

Now, we will define a Laurent polynomial of two variables for arbitrary matrices. Let $\mathcal{R}[x^{\pm 1}, y^{\pm 1}]$ be the ring of Laurent polynomials in two variables x, y. We define an involutive ring automorphism $* : \mathcal{R}[x^{\pm 1}, y^{\pm 1}] \to \mathcal{R}[x^{\pm 1}, y^{\pm 1}]$ by $x \mapsto x^{-1}$, $y \mapsto y^{-1}$ and $a \mapsto a^*$ for $a \in \mathcal{R}$.

Definition 2.7. For $A \in \mathcal{R}^{m\times n}$, we denote the row vectors of a matrix A by $\boldsymbol{a}_0, \dots, \boldsymbol{a}_{m-1}$. Define

$$\mathrm{seq}(A) = (\boldsymbol{a}_0 \mid \boldsymbol{a}_1 \mid \cdots \mid \boldsymbol{a}_{m-1}) \in \mathcal{R}^{mn},$$

where $\mid$ denotes concatenation, and

$$\psi_A(x, y) = \sum_{i=0}^{m-1} \psi_{\boldsymbol{a}_i}(x) y^{2i+1-m}.$$

Clearly, we have $\psi_{A\pm B}(x, y) = \psi_A(x, y) \pm \psi_B(x, y)$ for every $A, B \in \mathcal{R}^{m\times n}$. Note that we may regard $\mathcal{R}^n$ as $\mathcal{R}^{1\times n}$. So, for every $\boldsymbol{a} \in \mathcal{R}^n$, we have $\boldsymbol{a}^t \in \mathcal{R}^{n\times 1}$ where t denotes the transpose of a matrix.

Lemma 2.8. *Let $\boldsymbol{f} \in \mathcal{R}^m$ and $\boldsymbol{a} \in \mathcal{R}^n$. Then*

$$\psi_{\boldsymbol{f}^t\boldsymbol{a}}(x, y) = \psi_{\boldsymbol{a}}(x)\psi_{\boldsymbol{f}}(y).$$

Proof. Let $\boldsymbol{f} = (f_0, \dots, f_{m-1})$. Then

$$\begin{aligned}\psi_{\boldsymbol{f}^t\boldsymbol{a}}(x, y) &= \sum_{i=0}^{m-1} \psi_{(\boldsymbol{f}^t\boldsymbol{a})_i}(x) y^{2i+1-m} \\ &= \sum_{i=0}^{m-1} f_i \psi_{\boldsymbol{a}}(x) y^{2i+1-m} \\ &= \psi_{\boldsymbol{a}}(x) \sum_{i=0}^{m-1} f_i y^{2i+1-m} \\ &= \psi_{\boldsymbol{a}}(x)\psi_{\boldsymbol{f}}(y).\end{aligned}$$

□

Lemma 2.9. *If $A \in \mathcal{R}^{m\times n}$, then*

$$\psi_{\mathrm{seq}(A)}(x) = \psi_A(x, x^n).$$

Proof. Let $\boldsymbol{a}_0, \dots, \boldsymbol{a}_{m-1}$ be the row vectors of A. Since $\phi_{\mathrm{seq}(A)}(x) = \sum_{i=0}^{m-1} x^{ni}\phi_{\boldsymbol{a}_i}(x)$, we have

$$\begin{aligned}
\psi_{\mathrm{seq}(A)}(x) &= x^{1-mn}\phi_{\mathrm{seq}(A)}(x^2) \\
&= x^{1-mn}\sum_{i=0}^{m-1} x^{2ni}\phi_{\boldsymbol{a}_i}(x^2) \\
&= x^{1-mn}\sum_{i=0}^{m-1} x^{2ni+n-1}\psi_{\boldsymbol{a}_i}(x) \\
&= \sum_{i=0}^{m-1} x^{n(2i+1-m)}\psi_{\boldsymbol{a}_i}(x) \\
&= \psi_A(x, x^n).
\end{aligned}$$

□

3 Main Result

We will present our result by three steps. The following lemma is essential to describe the Yang multiplication theorem by using matrix approach.

Lemma 3.1. *Let*

$$\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d} \in \mathcal{R}^n, \quad \boldsymbol{e}, \boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h} \in \mathcal{R}^m.$$

Set

$$\begin{aligned}
Q &= \boldsymbol{f}^{*t}\boldsymbol{a} + \boldsymbol{g}^t\boldsymbol{c} - \boldsymbol{e}^t\boldsymbol{b}^* + \boldsymbol{h}^t\boldsymbol{d}, \\
R &= \boldsymbol{f}^{*t}\boldsymbol{b} + \boldsymbol{g}^{*t}\boldsymbol{d} + \boldsymbol{e}^t\boldsymbol{a}^* - \boldsymbol{h}^{*t}\boldsymbol{c}, \\
S &= \boldsymbol{g}^{*t}\boldsymbol{a} - \boldsymbol{f}^t\boldsymbol{c} - \boldsymbol{h}^t\boldsymbol{b} - \boldsymbol{e}^t\boldsymbol{d}^*, \\
T &= \boldsymbol{g}^t\boldsymbol{b} - \boldsymbol{f}^t\boldsymbol{d} + \boldsymbol{h}^{*t}\boldsymbol{a} + \boldsymbol{e}^t\boldsymbol{c}^*.
\end{aligned}$$

Then

$$\begin{aligned}
&(\psi_Q\psi_Q^* + \psi_R\psi_R^* + \psi_S\psi_S^* + \psi_T\psi_T^*)(x, y) \\
&= (\psi_{\boldsymbol{a}}\psi_{\boldsymbol{a}}^* + \psi_{\boldsymbol{b}}\psi_{\boldsymbol{b}}^* + \psi_{\boldsymbol{c}}\psi_{\boldsymbol{c}}^* + \psi_{\boldsymbol{d}}\psi_{\boldsymbol{d}}^*)(x)(\psi_{\boldsymbol{e}}\psi_{\boldsymbol{e}}^* + \psi_{\boldsymbol{f}}\psi_{\boldsymbol{f}}^* + \psi_{\boldsymbol{g}}\psi_{\boldsymbol{g}}^* + \psi_{\boldsymbol{h}}\psi_{\boldsymbol{h}}^*)(y).
\end{aligned}$$

Proof. By Lemma 2.3 and Lemma 2.8, we have

$$\begin{aligned}
\psi_Q(x, y) &= \psi_{\boldsymbol{a}}(x)\psi_{\boldsymbol{f}}^*(y) + \psi_{\boldsymbol{c}}(x)\psi_{\boldsymbol{g}}(y) - \psi_{\boldsymbol{b}}^*(x)\psi_{\boldsymbol{e}}(y) + \psi_{\boldsymbol{d}}(x)\psi_{\boldsymbol{h}}(y), \\
\psi_R(x, y) &= \psi_{\boldsymbol{b}}(x)\psi_{\boldsymbol{f}}^*(y) + \psi_{\boldsymbol{d}}(x)\psi_{\boldsymbol{g}}^*(y) + \psi_{\boldsymbol{a}}^*(x)\psi_{\boldsymbol{e}}(y) - \psi_{\boldsymbol{c}}(x)\psi_{\boldsymbol{h}}^*(y), \\
\psi_S(x, y) &= \psi_{\boldsymbol{a}}(x)\psi_{\boldsymbol{g}}^*(y) - \psi_{\boldsymbol{c}}(x)\psi_{\boldsymbol{f}}(y) - \psi_{\boldsymbol{b}}(x)\psi_{\boldsymbol{h}}(y) - \psi_{\boldsymbol{d}}^*(x)\psi_{\boldsymbol{e}}(y), \\
\psi_T(x, y) &= \psi_{\boldsymbol{b}}(x)\psi_{\boldsymbol{g}}(y) - \psi_{\boldsymbol{d}}(x)\psi_{\boldsymbol{f}}(y) + \psi_{\boldsymbol{a}}(x)\psi_{\boldsymbol{h}}^*(y) + \psi_{\boldsymbol{c}}^*(x)\psi_{\boldsymbol{e}}(y).
\end{aligned}$$

Thus, by applying the Lagrange identity, the result follows. □

For the remainder of this section, we fix a multiplicatively closed subset $\mathcal{T}$ of $\mathcal{R}\setminus\{0\}$ satisfying $-1\in\mathcal{T}=\mathcal{T}^*$. Also, we denote $\mathcal{T}_0=\mathcal{T}\cup\{0\}$. Denote by $\mathrm{supp}(\boldsymbol{a})$ and $\mathrm{supp}(A)$ the set of indices of nonzero entries of a sequence $\boldsymbol{a}=(a_0,\dots,a_{l-1})\in\mathcal{R}^l$ and a matrix $A=[a_{ij}]_{0\le i\le m-1,0\le j\le n-1}\in\mathcal{R}^{m\times n}$, respectively. We say that sequences $\boldsymbol{a},\boldsymbol{b}$ are *disjoint* if $\mathrm{supp}(\boldsymbol{a})\cap\mathrm{supp}(\boldsymbol{b})=\emptyset$. Matrices A,B are also said to be *disjoint* if $\mathrm{supp}(A)\cap\mathrm{supp}(B)=\emptyset$.

Lemma 3.2. *Let m and n be positive integers,*

$$\begin{aligned}&\boldsymbol{a},\boldsymbol{b}\in\mathcal{T}^{n+1},\\&\boldsymbol{c},\boldsymbol{d}\in\mathcal{T}^{n},\\&\boldsymbol{f},\boldsymbol{g}\in\mathcal{T}^{m+1},\\&\boldsymbol{h},\boldsymbol{e}\in\mathcal{T}^{m}.\end{aligned}$$

Set

$$\begin{aligned}\boldsymbol{a}'&=\boldsymbol{a}/0,\quad &\boldsymbol{b}'&=\boldsymbol{b}/0,\quad &\boldsymbol{c}'&=0/\boldsymbol{c},\quad &\boldsymbol{d}'&=0/\boldsymbol{d},\\ \boldsymbol{f}'&=\boldsymbol{f}/0,\quad &\boldsymbol{g}'&=\boldsymbol{g}/0,\quad &\boldsymbol{h}'&=0/\boldsymbol{h},\quad &\boldsymbol{e}'&=0/\boldsymbol{e}.\end{aligned}$$

Write

$$Q=\boldsymbol{f}'^{*t}\boldsymbol{a}'+\boldsymbol{g}'^{t}\boldsymbol{c}'-\boldsymbol{e}'^{t}\boldsymbol{b}'^{*}+\boldsymbol{h}'^{t}\boldsymbol{d}',\tag{4}$$
$$R=\boldsymbol{f}'^{*t}\boldsymbol{b}'+\boldsymbol{g}'^{*t}\boldsymbol{d}'+\boldsymbol{e}'^{t}\boldsymbol{a}'^{*}-\boldsymbol{h}'^{*t}\boldsymbol{c}',\tag{5}$$
$$S=\boldsymbol{g}'^{*t}\boldsymbol{a}'-\boldsymbol{f}'^{t}\boldsymbol{c}'-\boldsymbol{h}'^{t}\boldsymbol{b}'-\boldsymbol{e}'^{t}\boldsymbol{d}'^{*},\tag{6}$$
$$T=\boldsymbol{g}'^{t}\boldsymbol{b}'-\boldsymbol{f}'^{t}\boldsymbol{d}'+\boldsymbol{h}'^{*t}\boldsymbol{a}'+\boldsymbol{e}'^{t}\boldsymbol{c}'^{*}.\tag{7}$$

Then $Q,R,S,T\in\mathcal{T}^{(2m+1)\times(2n+1)}$ satisfy

$$\begin{aligned}&(\psi_Q\psi_Q^*+\psi_R\psi_R^*+\psi_S\psi_S^*+\psi_T\psi_T^*)(x,y)\\&=(\psi_{\boldsymbol{a}}\psi_{\boldsymbol{a}}^*+\psi_{\boldsymbol{b}}\psi_{\boldsymbol{b}}^*+\psi_{\boldsymbol{c}}\psi_{\boldsymbol{c}}^*+\psi_{\boldsymbol{d}}\psi_{\boldsymbol{d}}^*)(x^2)(\psi_{\boldsymbol{e}}\psi_{\boldsymbol{e}}^*+\psi_{\boldsymbol{f}}\psi_{\boldsymbol{f}}^*+\psi_{\boldsymbol{g}}\psi_{\boldsymbol{g}}^*+\psi_{\boldsymbol{h}}\psi_{\boldsymbol{h}}^*)(y^2).\end{aligned}$$

Proof. Notice that $\boldsymbol{a}',\boldsymbol{b}',\boldsymbol{c}',\boldsymbol{d}'\in\mathcal{T}_0^{2n+1}$ and $\boldsymbol{e}',\boldsymbol{f}',\boldsymbol{g}',\boldsymbol{h}'\in\mathcal{T}_0^{2m+1}$.

Since $\mathrm{supp}(\boldsymbol{s}'^*)=\mathrm{supp}(\boldsymbol{s}')$ for every $\boldsymbol{s}\in\{\boldsymbol{a},\boldsymbol{b},\boldsymbol{c},\boldsymbol{d},\boldsymbol{e},\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}\}$ and $(\boldsymbol{s}',\boldsymbol{t}')$ is disjoint whenever

$$\boldsymbol{s}\in\{\boldsymbol{a},\boldsymbol{b}\},\boldsymbol{t}\in\{\boldsymbol{c},\boldsymbol{d}\}\quad\text{or}\quad\boldsymbol{s}\in\{\boldsymbol{f},\boldsymbol{g}\},\boldsymbol{t}\in\{\boldsymbol{h},\boldsymbol{e}\},$$

matrices A and B are disjoint whenever $A\neq B$ and

$$A,B\in\{\boldsymbol{f}'^{*t}\boldsymbol{a}',\boldsymbol{g}'^{t}\boldsymbol{c}',\boldsymbol{e}'^{t}\boldsymbol{b}'^{*},\boldsymbol{h}'^{t}\boldsymbol{d}'\}.$$

Also,

$$\begin{aligned}\mathrm{supp}(\boldsymbol{a}')\cup\mathrm{supp}(\boldsymbol{c}')=\mathrm{supp}(\boldsymbol{b}'^{*})\cup\mathrm{supp}(\boldsymbol{d}')=\{0,\dots,2n\},\\ \mathrm{supp}(\boldsymbol{f}'^{*})=\mathrm{supp}(\boldsymbol{g}'),\quad\mathrm{supp}(\boldsymbol{e}')=\mathrm{supp}(\boldsymbol{h}').\end{aligned}$$

Hence

$$\begin{aligned}\operatorname{supp}(Q) &= \operatorname{supp}(\boldsymbol{f}'^{*t}\boldsymbol{a}') \cup \operatorname{supp}(\boldsymbol{g}'^{t}\boldsymbol{c}') \cup \operatorname{supp}(\boldsymbol{e}'^{t}\boldsymbol{b}'^{*}) \cup \operatorname{supp}(\boldsymbol{h}'^{t}\boldsymbol{d}') \\ &= \{(i,j) : i \in \operatorname{supp}(\boldsymbol{g}'), j \in \operatorname{supp}(\boldsymbol{a}') \cup \operatorname{supp}(\boldsymbol{c}')\} \\ &\qquad \cup \{(i,j) : i \in \operatorname{supp}(\boldsymbol{e}'), j \in \operatorname{supp}(\boldsymbol{b}'^{*}) \cup \operatorname{supp}(\boldsymbol{d}')\} \\ &= \{(i,j) : i \in \operatorname{supp}(\boldsymbol{g}') \cup \operatorname{supp}(\boldsymbol{e}'), j \in \{0,\dots,2n\}\} \\ &= \{0,\dots,2m\} \times \{0,\dots,2n\}.\end{aligned}$$

By a similar argument, we obtain

$$\operatorname{supp}(R) = \operatorname{supp}(S) = \operatorname{supp}(T) = \{0,\dots,2m\} \times \{0,\dots,2n\}.$$

Therefore, $Q, R, S, T \in \mathcal{T}^{(2m+1)\times(2n+1)}$. The claimed identity follows from Lemma 2.6 and Lemma 3.1. □

Theorem 3.3. *Let m, n be positive integers, and suppose*

$$\begin{aligned}&\boldsymbol{a}, \boldsymbol{b} \in \mathcal{T}^{n+1}, \\ &\boldsymbol{c}, \boldsymbol{d} \in \mathcal{T}^{n}, \\ &\boldsymbol{f}, \boldsymbol{g} \in \mathcal{T}^{m+1}, \\ &\boldsymbol{h}, \boldsymbol{e} \in \mathcal{T}^{m}\end{aligned}$$

satisfy

$$\begin{aligned}(\psi_{\boldsymbol{a}}\psi_{\boldsymbol{a}}^{*} + \psi_{\boldsymbol{b}}\psi_{\boldsymbol{b}}^{*} + \psi_{\boldsymbol{c}}\psi_{\boldsymbol{c}}^{*} + \psi_{\boldsymbol{d}}\psi_{\boldsymbol{d}}^{*})(x) = 2(2n+1), \\ (\psi_{\boldsymbol{e}}\psi_{\boldsymbol{e}}^{*} + \psi_{\boldsymbol{f}}\psi_{\boldsymbol{f}}^{*} + \psi_{\boldsymbol{g}}\psi_{\boldsymbol{g}}^{*} + \psi_{\boldsymbol{h}}\psi_{\boldsymbol{h}}^{*})(x) = 2(2m+1).\end{aligned}$$

Then there exist $\boldsymbol{q}, \boldsymbol{r}, \boldsymbol{s}, \boldsymbol{t} \in \mathcal{T}^{(2m+1)(2n+1)}$ such that

$$(\psi_{\boldsymbol{q}}\psi_{\boldsymbol{q}}^{*} + \psi_{\boldsymbol{r}}\psi_{\boldsymbol{r}}^{*} + \psi_{\boldsymbol{s}}\psi_{\boldsymbol{s}}^{*} + \psi_{\boldsymbol{t}}\psi_{\boldsymbol{t}}^{*})(x) = 4(2m+1)(2n+1).$$

Proof. Define Q, R, S, T as in (4), (5), (6), (7), respectively. Write

$$\boldsymbol{q} = \operatorname{seq}(Q), \quad \boldsymbol{r} = \operatorname{seq}(R), \quad \boldsymbol{s} = \operatorname{seq}(S), \quad \boldsymbol{t} = \operatorname{seq}(T).$$

By Lemma 3.2, $\boldsymbol{q}, \boldsymbol{r}, \boldsymbol{s}, \boldsymbol{t} \in \mathcal{T}^{(2m+1)(2n+1)}$. Applying Lemma 2.9 and Lemma 3.2, we have

$$\begin{aligned}&(\psi_{\boldsymbol{q}}\psi_{\boldsymbol{q}}^{*} + \psi_{\boldsymbol{r}}\psi_{\boldsymbol{r}}^{*} + \psi_{\boldsymbol{s}}\psi_{\boldsymbol{s}}^{*} + \psi_{\boldsymbol{t}}\psi_{\boldsymbol{t}}^{*})(x) \\ &= (\psi_{Q}\psi_{Q}^{*} + \psi_{R}\psi_{R}^{*} + \psi_{S}\psi_{S}^{*} + \psi_{T}\psi_{T}^{*})(x, x^{2n+1}) \\ &= (\psi_{\boldsymbol{a}}\psi_{\boldsymbol{a}}^{*} + \psi_{\boldsymbol{b}}\psi_{\boldsymbol{b}}^{*} + \psi_{\boldsymbol{c}}\psi_{\boldsymbol{c}}^{*} + \psi_{\boldsymbol{d}}\psi_{\boldsymbol{d}}^{*})(x^{2})(\psi_{\boldsymbol{e}}\psi_{\boldsymbol{e}}^{*} + \psi_{\boldsymbol{f}}\psi_{\boldsymbol{f}}^{*} + \psi_{\boldsymbol{g}}\psi_{\boldsymbol{g}}^{*} + \psi_{\boldsymbol{h}}\psi_{\boldsymbol{h}}^{*})(x^{2(2n+1)}) \\ &= 4(2m+1)(2n+1).\end{aligned}$$

Hence the proof is complete. □

Finally, we see that Theorem 1.1 follows from Theorem 3.3 by setting $\mathcal{T} = \{\pm 1\} \subseteq \mathbb{Z}$. Hence, our method gives a more transparent proof of Theorem 1.1. Indeed, by taking $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}) \in BS(n+1, n)$ and $(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}, \boldsymbol{e}) \in BS(m+1, m)$, the hypotheses in Theorem 3.3 are satisfied by Lemma 2.4. Then the resulting sequences $(\boldsymbol{q}, \boldsymbol{r}, \boldsymbol{s}, \boldsymbol{t})$ belong to $BS(m', m')$ by Lemma 2.4 where $m' = (2m+1)(2n+1)$.

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