AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **69(3)** (2017), Pages 417–441

The metric dimension of the circulant graph $C(n, \pm\{1, 2, 3, 4\})$

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Abstract

Let $G = (V, E)$ be a connected graph and let $d(u, v)$ denote the distance between vertices $u, v \in V$. A metric basis for G is a set $B \subseteq V$ of minimum cardinality such that no two vertices of G have the same distances to all points of B. The cardinality of a metric basis of G is called the metric dimension of G, denoted by $\dim(G)$. In this paper we determine the metric dimension of the circulant graphs $C(n, \pm\{1, 2, 3, 4\})$ for all values of n.

1 Introduction

Let $G(V, E)$ be a simple connected and undirected graph. For $u, v \in V$, let $d(u, v)$ denote the distance between u and v. A vertex $x \in V$ is said to resolve or distinguish two vertices u and v if $d(x, u) \neq d(x, v)$. A set $X \subseteq V$ is said to be a *resolving set* for G, if every pair of vertices of G is distinguished by some element of X. A minimum resolving set is called a *metric basis.* The cardinality of a metric basis is called the *metric dimension* of G, denoted by $\dim(G)$. For an ordered set $X = \{x_1, x_2, \ldots, x_k\} \subseteq V$, we refer to the k-vector (ordered k-tuple) $r(v|X) = (d(v, x_1), d(v, x_2), \ldots, d(v, x_k))$ as the *representation* of v with respect to X. Thus we can have another equivalent definition. The set X is called a *metric basis* if distinct vertices of G have distinct representation with respect to X. The metric dimension problem has been studied in several papers [2, 5, 7, 9], where it is also referred to as the *location number*. Note that metric basis, *minimum locating set* and *reference set* are different names used by different authors to describe the same concept. In this paper we use the terms metric basis and metric dimension.

The problem of finding the metric dimension of a graph was studied by Harary and Melter [5]. Slater described the usefulness of this idea in long range aids to navigation [9]. Melter and Tomescu [8] studied the metric dimension problem for grid graphs. The metric dimension problem has been studied also for trees and multi-dimensional grids by Khulller, Raghavachari and Rosenfield [7]. They also described the application of the concept of metric dimension in robot navigation and in [2] Chartrand, Eroh, Johnson and Oellermann presented an application in drug discovery, where it is to be determined whether the features of a compound are responsible for its pharmacological activity.

Cayley graphs on the cyclic group $\mathbb{Z}_n$ are called *circulant graphs*. We use the special notation $C(n, S)$ for a circulant graph on $\mathbb{Z}_n$ with connection set S. In this paper we focus on connection sets of the form $\{1, \ldots, t\}$. Let $C(n, \pm\{1, 2, \ldots, t\})$ for $1 \leqslant t \leqslant \lfloor n/2 \rfloor$ and $n \geqslant 3$, denote the graph with vertex set $V = \{0, 1, \ldots, n-1\}$ and edge set $E = \{(i, j) : |j-i| \equiv s \pmod n, s \in \{1, 2, \ldots, t\}\}$. Note that $C(n, \pm\{1\})$ is the cycle C_n, and $C(n, \pm\{1, 2, \ldots, \lfloor n/2 \rfloor\})$ is the complete graph K_n. The distance between two vertices i and j in a circulant graph $C(n, \pm\{1, 2, \ldots, t\})$ such that $0 \leqslant i < j < n$ is,

$$d(i, j) = \begin{cases} \lceil (j-i)/t \rceil & \text{for } j - i \in \{0, 1, \ldots \lfloor n/2 \rfloor\}, \\ \lceil (n-(j-i))/t \rceil & \text{for } j - i \in \{\lceil n/2 \rceil, \ldots, n-1\}. \end{cases}$$

Extending the results of Imran, Baig, Bokhary and Javaid [6], in [1] Borchert and Gosselin showed that $\dim(C(n, \pm\{1, 2\}) = 4$ if $n \equiv 1 \pmod 4$ and $\dim(C(n, \pm\{1, 2\})$ $= 3$ otherwise. They also solved the case $t = 3$ by proving

$$\dim(C(n, \pm\{1, 2, 3\})) = \begin{cases} 5 & \text{for } n \equiv 1 \pmod 6, \\ 4 & \text{otherwise.} \end{cases}$$

For general t, the following bounds were obtained by Vetrik in [10].

- For $t \geqslant 2$ and $n \geqslant t^2 + 1$, we have $\dim C(n, \pm\{1, 2, \ldots, t\}) \geqslant t$.
- For $t \geqslant 2$ and $n = 2kt + r$ with $k \geqslant 0$ and $t + 2 \leqslant r \leqslant 2t + 1$ we have $\dim C(n, \pm\{1, 2, \ldots, t\}) \geqslant t + 1$.
- For even t and $n = 2kt + t + 2p$, we have $\dim C(n, \pm\{1, 2, \ldots, t\}) \leqslant t + p$.

Actually, the condition $n \geqslant t^2 + 1$ in the first point above can be relaxed:

- For $t \geqslant 2$ and $n \geqslant 2t + 1$, we have $\dim C(n, \pm\{1, 2, \ldots, t\}) \geqslant t$.

We will prove this in Section 3. In [4], we gave an upper bound as follows:

- For $t \geqslant 2$ and $n = 2kt + r$ with $k \geqslant 1$ and $2 \leqslant r \leqslant t + 2$ we have $\dim(C(n, \pm\{1, 2, \ldots, t\})) \leqslant t + 1$.

Combining all these bounds we obtain the following for $t = 4$:

- $\dim C(n, \pm\{1, 2, 3, 4\}) = 4$ for $n \equiv 4 \pmod 8$,

- $\dim C(n, \pm\{1, 2, 3, 4\}) \in \{4, 5\}$ for $n \equiv 2, 3,$ or $5 \pmod 8$,
- $\dim C(n, \pm\{1, 2, 3, 4\}) = 5$ for $n \equiv 6 \pmod 8$,
- $\dim C(n, \pm\{1, 2, 3, 4\}) \in \{5, 6\}$ for $n \equiv 0 \pmod 8$,
- $\dim C(n, \pm\{1, 2, 3, 4\}) \geqslant 5$ for $n \equiv \pm 1 \pmod 8$.

In this paper, we determine $\dim C(n, \pm\{1, 2, 3, 4\})$ for all values of n as follows.

Theorem 1. *Let $G = C(n, \pm\{1, 2, 3, 4\})$, $n \geqslant 6$, $n \notin \{11, 19\}$. Then*

$$\dim(G) = \begin{cases} 4 & \textit{for} \;\; n \equiv 4 \pmod 8, \\ 5 & \textit{for} \;\; n \equiv \pm 2 \;\textit{or}\; \pm 3 \pmod 8, \\ 6 & \textit{for} \;\; n \equiv \pm 1 \;\textit{or}\; 0 \pmod 8. \end{cases}$$

For $n \in \{5, 11, 19\}$, we have $\dim(G) = 4$.

Remark 1. The cases $n \in \{5, 11, 19\}$ have to be treated separately in Theorem 1, because $\{0, 1, 2, 3\}$, $\{0, 2, 3, 10\}$ and $\{0, 2, 7, 19\}$, respectively, are resolving sets witnessing $\dim(G) = 4$ in these cases.

Remark 2. While we were working on the revision of the present paper we became aware of the recent work by Chau and Gosselin [3] who independently obtained some of our results. In particular, they prove the cases $n \equiv 1 \pmod 8$ and $n \equiv 3 \pmod 8$ of Theorem 1.

The paper is structured as follows. In Section 2 we introduce definitions and notation. In Section 3 we prove some lemmas that are needed in the proof of Theorem 1 which is contained in Sections 4 and 5, where Section 4 is devoted to establishing the lower bounds, while Section 5 contains the upper bounds which do not already follow from the results mentioned above, i.e., the cases $n = 8k + r$ with $r \in \{7, 9\}$.

2 Definitions and notations

Throughout this paper we consider the graph $G = C(n, \pm\{1, \ldots, t\})$. We prove a few results for general t, but mostly we are concerned with the case $t = 4$. For $t = 4$ we write the order of the graph as $n = 8k + r$ with $k \geqslant 1$ and $r \in \{2, 3, \ldots, 9\}$ (the cases $n \leqslant 9$ can be done by a brute force computer search). The graph G has diameter $k + 1$, and for every vertex v, the set

$$D_v = \{v + 4k + j \;:\; j = 1, 2, \ldots, r - 1\}$$

of vertices at diameter distance from v has size $|D_v| = r - 1$.

We generally consider the vertex set of a circulant graph of order n as $V = \{0, 1, 2, \ldots, n - 1\}$, and whenever we refer to a vertex such as $i + j$ or $-4m$ this has to be interpreted modulo n in the obvious way.

Definition 1. For any $S \subseteq V$ we define an equivalence relation $\sim_S$ by $u \sim_S v \iff r(u|S) = r(v|S)$.

A set $S \subseteq V$ is a resolving set if and only if all equivalence classes of $\sim_S$ are singletons. So the equivalence classes can serve as a measure for how far away the set S is from being a resolving set. If we want to extend a set S to become a resolving set, we have to add a set X of vertices that resolve all the non-trivial equivalence classes. In our proof it will be convenient to consider subsets of equivalence classes, which we call S-blocks. This is made more precise in the following definition.

Definition 2. Let $A \subset V$ and $S \subset V$. We call the set A an *S-block*, if all vertices of A are at equal distance from every vertex of S, or equivalently, $r(a|S) = r(b|S)$ for all $a, b \in A$. Slightly abusing notation we denote this common representation vector by $r(A|S)$.

Definition 3. Let $A \subset V$ and $X \subset V$ be such that for all $a, b \in A$ we have $r(a|X) \neq r(b|X)$. Then we say X *resolves* A.

For given S, if we want to find a vertex set X such that $S \cup X$ is a resolving set, then X has to resolve all the S-blocks simultaneously. In the following definition we introduce notation for a collection of S-blocks.

Definition 4. Let $S \subseteq V$. An ℓ-tuple $A = (A_1, A_2, \ldots, A_\ell)$ of S-blocks is called an *S-cluster* if the sets A_i are subsets of distinct $\sim_S$-classes. We say that a set $X \subseteq V - S$ resolves the S-cluster A, if $r(a|X) \neq r(b|X)$ for all $a \neq b \in A_j$, $1 \leqslant j \leqslant \ell$.

Note that the set $\{i, i+1\}$ can be resolved by a vertex x with $d(x, i) < d(x, i+1)$ which implies $x \in \{i - 4j \ : \ 0 \leqslant j \leqslant k\}$, or by a vertex x with $d(x, i) > d(x, i+1)$ which implies $x \in \{i + 1 + 4j \ : \ 0 \leqslant j \leqslant k\}$. So

$$R_i = \{i - 4j \ : \ 0 \leqslant j \leqslant k\} \cup \{i + 1 + 4j \ : \ 0 \leqslant j \leqslant k\}$$

is the set of vertices that resolve the set $\{i, i+1\}$. In particular, every metric basis must contain at least one element from each of the sets R_i.

3 Auxiliary results

In order to prove lower bounds in Theorem 1, we need to go into rather tedious case discussions. The basic idea is to show that a resolving set B of size k cannot exist by looking at all possible ways of starting with a set S, $|S| = k_0 < k$, and exhibit an S-block or S-cluster whose resolution requires more than $k - k_0$ vertices. For this purpose we need many statements of the form "If A is an S-cluster of the form ..., then every set resolving A has at least ... elements." The statements used in the proof of Theorem 1 are the lemmas proved below. Sometimes we need supporting claims in the proofs of the lemmas, and we call them *observations*.

We start with a result which is valid for any t, and use it to prove two general lower bounds.

Lemma 1. *Let $G = C(n, \pm\{1, 2, \ldots, t\})$, and let $A \subseteq \{i, i+1, \ldots, i+t\}$ with $|A| = \ell$, for $2 \leqslant \ell \leqslant t + 1$. If X resolves A, then $|X| \geqslant \ell - 1$.*

Proof. We proceed by induction on ℓ. For $\ell = 2$ the statement is true as we need at least one vertex to resolve a set of size at least 2. For $\ell \geqslant 3$, consider the elements of A in the order $i + a_1, i + a_2, \dots, i + a_\ell$ with $0 \leqslant a_1 < a_2 < \cdots < a_\ell \leqslant t$. By assumption there is an element $x \in X$ with $d(x, i + a_1) \neq d(x, i + a_2)$. If $x = i + a_2$ then $d(x, i+a_s) = 1$ for all $s \in \{1, 3, 4, \dots, \ell\}$. Therefore $X \setminus \{x\}$ resolves $A \setminus \{i+a_2\}$. If $x \neq i + a_2$, then $d(x, i + a_s) = d(x, i + a_2)$ for all $s \in \{2, \dots, \ell\}$. Therefore $X \setminus \{x\}$ resolves $A \setminus \{i + a_1\}$. In both cases $X \setminus \{x\}$ resolves a subset of $\{i, \dots, i + t\}$ of size $\ell - 1$, and by induction $|X \setminus \{x\}| \geqslant \ell - 2$, hence $|X| \geqslant \ell - 1$. □

The following theorem strengthens the lower bound given by Vetrik in [10] by extending the range of n for which the lower bound is valid. Note that if we have a circulant graph $C(n, \{1, 2, \dots, t\})$ with $n < 2t + 2$, then that circulant graph is a complete graph on n vertices and its metric dimension is $n - 1$.

Theorem 2. *Let* $G = C(n, \pm\{1, 2, \dots, t\})$, *with* $n \geqslant 2t + 2$. *Then* $\dim(G) \geqslant t$.

Proof. Suppose B is a resolving set of G with $|B| < t$. Without loss of generality, $S = \{0\} \subseteq B$. The set $A = \{1, 2, \dots, t\}$ is an S-block, and by Lemma 1, $|B - S| \geqslant t - 1$, which is the required contradiction. □

Lemma 1 can also be used to give a short alternative proof of the following result which was originally proved by Vetrik in [10].

Theorem 3 ([10])**.** *Let* $n = 2kt+r$ *where* $t \geqslant 2$, $k \geqslant 1$ *and* $r \in \{t+2, t+3, \dots, 2t+1\}$. *Then*

$$\dim(C(n, \{1, 2, \dots, t\})) \geqslant t + 1.$$

Proof. Suppose B is a resolving set of G with $|B| < t+1$. Without loss of generality, $S = \{0\} \subseteq B$. For $D_0 = \{tk + \ell : \ell = 1, 2, \dots, r - 1\}$, we have $|D_0| = r - 1 \geqslant t + 1$, and $d(0, u) = k + 1$ for all $u \in D_0$. Hence, D_0 is an S-block, and by Lemma 1, $|B - S| \geqslant t$, which is the required contradiction. □

For the rest of this section, $t = 4$.

Lemma 2. *Let* $n = 8k+r$ *with* $r \in \{7, 8, 9\}$, *and let* $G = C(n, \pm\{1, 2, 3, 4\})$. *Suppose* B *is a metric basis for* G *with* $|B| = 5$. *Then* $|i - j| \geqslant r - 5$ *for all* $i \neq j \in B$.

Proof. If the statement is wrong, then without loss of generality, $S = \{0, i\} \subseteq B$ for some $i \in \{1, \dots, r - 6\}$. The set $A = \{4k + i + \ell : \ell = 1, \dots, 5\}$ is an S-block with $r(A|S) = (k + 1, k + 1)$ and, since B is a metric basis, $B - S$ resolves A, and by Lemma 1 this implies $|B - S| \geqslant 4$. Hence, $|B| \geqslant 6$, which is the required contradiction. □

Observation 1. Let $G = C(n, \pm\{1, 2, 3, 4\})$ with $n = 8k + r$, $r \in \{2, \dots, 9\} - \{3\}$ and let

$$A = (\{a, a + 1\}, \{a + 2, a + 3\})$$

be an S-cluster for some S. Then for every resolving set X of A, $|X| \geqslant 2$.

Proof. Without loss of generality $a = 0$. Suppose $x \in V$ resolves A, that is,

$$x \in R_0 \cap R_2 = \{-4m, 1 + 4m : 0 \leqslant m \leqslant k\} \cap \{2 - 4m, 3 + 4m : 0 \leqslant m \leqslant k\} = \emptyset,$$

which is the required contradiction. □

Lemma 3. *Let $n = 8k + 9$, $G = C(n, \pm\{1, 2, 3, 4\})$, and let*

$$A = (\{a, a \pm 1\}, \{a \pm j \pm 4\ell : j \in \{2, 3, 4\}\})$$

be an S-cluster for some S, where $0 \leqslant \ell \leqslant 2k + 1$. If $X \subseteq V$ resolves A, then $|X| \geqslant 3$.

Proof. By symmetry, it is sufficient to prove the statement for $A = (\{0, 1\}, \{j + 4\ell : j \in \{2, 3, 4\}\})$. Suppose $X = \{x, y\}$ resolves A, and without loss of generality $x \in R_0 = \{1, 5, \ldots, 4k+1, 4k+9, 4k+13, \ldots, 8k+1, 8k+5, 0\}$. Then x has the form $x = 4m+1$ (where we interpret 0 as $8k+9 = 4(2k+2)+1$), and for all $j \in \{2, 3, 4\}$,

$$d(x, j + 4\ell) = \begin{cases} \ell - m + 1 & \text{if } m \leqslant \ell \leqslant m + k, \\ m - \ell & \text{if } m > \ell \geqslant m - k - 1, \\ m + (2k + 2 - \ell) & \text{if } \ell > m + k, \\ 2k - m + \ell + 3 & \text{if } \ell < m - k - 1. \end{cases}$$

This implies that $\{j + 4\ell : j \in \{2, 3, 4\}\}$ is an $S \cup \{x\}$-block, and since by assumption $X - \{x\}$ resolves it, Lemma 1 implies $|X - \{x\}| \geqslant 2$, which is the required contradiction. □

Lemma 4. *Let $n = 8k + r$, with $r \in \{5, 6, 7, 8\}$ and $G = C(n, \pm\{1, 2, 3, 4\})$. Let*

$$A = (\{a, a + 1\}, \{a + 2, a + 3, a + 4\})$$

be an S-cluster for some S. If X resolves A, then $|X| \geqslant 3$.

Proof. By symmetry, it is sufficient to prove the statement for $a = 0$. Suppose $X = \{x, y\}$ resolves A, and without loss of generality $x \in R_0 = \{1, 5, \ldots, 4k + 1, 0, -4, \ldots, -4k\}$. For $j \in \{2, 3, 4\}$,

$$d(x, j) = \begin{cases} 1 & \text{if } x = 1, \\ m & \text{if } x = 4m + 1 \text{ and } 1 \leqslant m \leqslant k, \\ m + 1 & \text{if } x = -4m \text{ and } 0 \leqslant m \leqslant k. \end{cases}$$

Thus $\{2, 3, 4\}$ is an $S \cup \{x\}$-block, and Lemma 1 implies that $|X - \{x\}| \geqslant 2$, which is the required contradiction. □

Lemma 5. *Let $n = 8k + r$, with $r \in \{7, 8\}$ and $G = C(n, \pm\{1, 2, 3, 4\})$. Let*

$$A = (\{a, a \pm 1\}, \{a \pm 2 \pm 4\ell, a \pm 3 \pm 4\ell, a \pm 4 \pm 4\ell\})$$

be an S-cluster for some S with $1 \leqslant \ell \leqslant k$. If $X \subseteq V - \{a \mp 4k, a \mp 4(k - 1), \ldots, a \mp 4(k - \ell + 1)\}$ resolves A, then $|X| \geqslant 3$.

Proof. By symmetry we can assume $A = (\{0,1\},\{2+4\ell,3+4\ell,4+4\ell\})$. Suppose $X = \{x,y\}$ resolves A and $X \subset V - \{4k+r, 4k+r+4, \dots 4k+r+4(\ell-1)\}$. Then without loss of generality,

$$\begin{aligned} x \in R_0 &- \{4k+r, 4k+r+4, \dots, 4k+r+4(\ell-1)\} \\ &= \{1,5,\dots,4k+1, 4k+r+4\ell, 4k+r+4\ell+4, \dots, 8k+4-4, 8k+r\}. \end{aligned}$$

For $j \in \{2,3,4\}$,

$$d(x, j+4\ell) = \begin{cases} \ell - m + 1 & \text{if } x = 1+4m \text{ and } 0 \leqslant m \leqslant \ell, \\ m - \ell & \text{if } x = 1+4m \text{ and } \ell < m \leqslant k, \\ m + \ell + 1 & \text{if } x = -4m \text{ and } 0 \leqslant m \leqslant k - \ell. \end{cases}$$

Thus $\{j+4\ell : j \in \{2,3,4\}\}$ is an $S\cup\{x\}$-block, and Lemma 1 implies that $|X-\{x\}| \geqslant 2$, which is the required contradiction. □

Lemma 6. *Let $G = C(n, \pm\{1,2,3,4\})$ be a circulant graph of order $n = 8k+8$. Let*

$$A = (A_0,\, A_1, \dots, A_k,\, B_1,\, B_2,\, B_3)$$

be an S-cluster where, for some $m, m' \in \{1,\dots,k\}$ with $m < m'$,

$$\begin{aligned} &A_0 = \{4k+4,\, 4k+5,\, 4k+6\}, \qquad A_i = \{4k+4+4i,\, 4k+5+4i\} \text{ for } 1 \leqslant i \leqslant k, \\ &B_1 = \{8k+7,\, 1\}, \qquad B_2 = \{4m+2,\, 4m+4\}, \qquad B_3 = \{4m'+1,\, 4m'+3\}. \end{aligned}$$

If $X \subset V - \{0,1,\dots,4m'+1\}$ resolves A, then $|X| \geqslant 3$.

Proof. Suppose $X = \{x,y\}$ resolves A. Without loss of generality x resolves $4k+4$ and $4k+5$, i.e.,

$$x \in R_{4k+4} = \{4k+4-4\ell,\, 4k+5+4\ell \,:\, 0 \leqslant \ell \leqslant k\}.$$

We distinguish several cases, and for each possible choice of x and y we show that there remains an unresolved pair or vertices.

Case 1 $x = 4k+5+4\ell$; $1 \leqslant \ell \leqslant k$. For $S' = S \cup \{x\}$, the set

$$B = (\{4k+5, 4k+6\},\, \{8k+7, 1\},\, \{4m+2, 4m+4\})$$

is an S'-cluster. Let $y \in R_{4k+5} = \{4k+5-4\ell_1,\, 4k+6+4\ell_1 \,:\, 0 \leqslant \ell_1 \leqslant k\}$.

- If $y = 4k+5-4\ell_1$, $0 \leqslant \ell_1 \leqslant k-m$, then $d(y, 4m+2) = d(y, 4m+4) = k - \ell_1 - m + 1$.
- If $y = 4k+6+4\ell_1$, $0 \leqslant \ell_1 \leqslant k$, then $d(y, 8k+7) = d(y,1) = k - \ell_1 + 1$.

Case 2 $x = 4k+5$. For $S' = S \cup \{x\}$, the set

$$B = (\{4k+8, 4k+9\}, \{8k+7, 1\}, \{4m'+1, 4m'+3\})$$

is an S'-cluster. Let $y \in R_{4k+8}$.

- If $y = 4k + 8 - 4\ell_1$, $0 \leqslant \ell_1 \leqslant k - m'$, then $d(y, 4m' + 1) = d(y, 4m' + 3) = k - \ell_1 - m' + 1$.
- If $y = 4k + 9 + 4\ell_1$, $0 \leqslant \ell_1 \leqslant k$, then $d(y, 8k + 7) = d(y, 1) = k - \ell_1 + 1$.

Case 3 $x = 4k + 4$. For $S' = S \cup \{x\}$, the set

$$B = (\{4k + 5, 4k + 6\}, \{8k + 7, 1\}, \{4m' + 1, 4m' + 3\})$$

is an S'-cluster. Let $y \in R_{4k+5} = \{4k + 5 - 4\ell_1, 4k + 6 + 4\ell_1 : 0 \leqslant \ell_1 \leqslant k\}$.

- If $y = 4k + 5 - 4\ell_1$, $0 \leqslant \ell_1 \leqslant k - m'$, then $d(y, 4m' + 1) = d(y, 4m' + 3) = k - \ell_1 - m' + 1$.
- If $y = 4k + 6 + 4\ell_1$, $0 \leqslant \ell_1 \leqslant k$, then $d(y, 8k + 7) = d(y, 1) = k - \ell_1 + 1$.

Case 4 $x = 4k + 4 - 4\ell, 1 \leqslant \ell \leqslant k$. For $S' = S \cup \{x\}$ the set

$$B = (\{4k + 5, 4k + 6\}, \{4m' + 1, 4m' + 3\}, \{8k + 8 - 4\ell, 8k + 9 - 4\ell\})$$

is an S'-cluster. Let $y \in R_{4k+5} = \{4k + 5 - 4\ell_1, 4k + 6 + 4\ell_1 : 0 \leqslant \ell_1 \leqslant k\}$.

- If $y = 4k + 5 - 4\ell_1$, $0 \leqslant \ell_1 \leqslant k - m'$, then $d(y, 4m' + 1) = d(y, 4m' + 3) = k - \ell_1 - m' + 1$.
- If $y = 4k+6+4\ell_1$, $0 \leqslant \ell_1 \leqslant k$, then $d(y, 8k+8-4\ell) = d(y, 8k+9-4\ell) = k + 1$. □

Lemma 7. *Let $G = C(n, \pm(1, 2, 3, 4))$ be of order $n = 8k + 7$. Let $A = (A_0, B_0, A_1, B_1, \ldots, A_k, B_k, A_{k+1})$, where $A_i = \{a + 4i, a + 4i + 1\}$ and $B_i = \{a+2+4i, a+3+4i\}$ for $0 \leqslant i \leqslant k + 1$, be an S-cluster for some S. If X resolves A, then $|X| \geqslant 3$.*

Proof. By symmetry, it is enough to prove it for $A_i = \{1 + 4i, 2 + 4i\}$ and $B_i = \{3+4i, 4+4i\}$. Suppose $X = \{x, y\}$ resolves A. Without loss of generality $x \in R_1 = \{8k + 8 - 4\ell, 2 + 4\ell : 0 \leqslant \ell \leqslant k\}$. Let $S' = S \cup \{x\}$.

Case 1 $x = 8k + 8 - 4\ell$, $0 \leqslant \ell \leqslant k$. Then

$$d(x, 3 + 4(k - \ell)) = d(x, 4 + 4(k - \ell)) = k + 1,$$
$$d(x, 1 + 4(k + 1 - \ell)) = d(x, 2 + 4(k + 1 - \ell)) = k + 1.$$

Hence, $B = (B_{k-\ell},\ A_{k+1-\ell})$ is an S'-cluster, an by Observation 1, $|X - \{x\}| \geqslant 2$.

Case 2 $x = 2 + 4\ell$, $0 \leqslant \ell \leqslant k$. Then

$$d(x, 3 + 4\ell) = d(x, 4 + 4\ell) = 1,$$
$$d(x, 5 + 4\ell) = d(x, 6 + 4\ell) = 1.$$

Hence $B = (B_\ell, A_{\ell+1})$ is an S'-cluster, and by Observation 1, $|X - \{x\}| \geqslant 2$. □

Lemma 8. *Let $G = C(n; \pm\{1, 2, 3, 4\})$ be of order $n = 8k+7$. Let $A = (A_0, A_1, \ldots, A_k, B_1, B_2)$ be an S-cluster, where, for some $m' \in \{1, \ldots, k\}$,*

$$A_i = \{3+4i,\, 4+4i\} \text{ for } 0 \leqslant i \leqslant k-1, \quad A_k = \{3+4k,\, 4+4k,\, 5+4k\},$$
$$B_1 = \{3+4(k+m'),\, 4+4(k+m')\}, \qquad B_2 = \{5+4(k+m'),\, 6+4(k+m')\}.$$

If X resolves A, then $|X| \geqslant 3$.

Proof. Suppose $X = \{x, y\}$ resolves A. Without loss of generality

$$x \in R_3 = \{4+4\ell,\, 8k+10-4\ell \,:\, 0 \leqslant \ell \leqslant k\}.$$

We distinguish the following cases.

Case 1 $x = 4+4\ell$, $0 \leqslant \ell \leqslant k$. For all $u \in B_1$ and all $v \in B_2$,

$$d(x,u) = \begin{cases} k+m'-\ell & \text{if } \ell \geqslant m', \\ k+\ell-m'+2 & \text{if } \ell < m', \end{cases} \quad d(x,v) = \begin{cases} k+m'-\ell+1 & \text{if } \ell \geqslant m', \\ k+\ell-m'+2 & \text{if } \ell < m'. \end{cases}$$

Hence, $B = (B_1, B_2)$ is an S'-cluster for $S' = S \cup \{x\}$. By Observation 1, $|X - \{x\}| \geqslant 2$.

Case 2 $x = 8k+10-4\ell$, $0 \leqslant \ell \leqslant k$. For $S' = S \cup \{x\}$,

$$B = (B_1,\, A_k - \{3+4k\}) = (\{3+4k+4m', 4+4k+4m'\},\, \{4+4k, 5+4k\})$$

is an S'-cluster. Without loss of generality $y \in R_{4+4k} = \{4+4k-4\ell_1,\, 5+4k+4\ell_1 \,:\, 0 \leqslant \ell_1 \leqslant k\}$. If $y = 4+4k-4\ell_1$, $0 \leqslant \ell_1 \leqslant k$, then

$$d(y, 3+4k+4m') = d(y, 4+4k+4m') = \begin{cases} m'+\ell_1 & \text{if } m'+\ell_1 \leqslant k+1, \\ 2k+2-(m'+\ell_1) & \text{if } m'+\ell_1 > k+1. \end{cases}$$

If $y = 5+4k+4\ell_1$, $0 \leqslant \ell_1 \leqslant k$, then

$$d(y, 3+4k+4m') = d(y, 4+4k+4m') = \begin{cases} m'-\ell_1 & \text{if } \ell_1 < m', \\ \ell_1-m'+1 & \text{if } \ell_1 \geqslant m'. \end{cases}$$

In both cases, B_1 is an $S \cup \{x,y\}$-block, which is the required contradiction. ☐

Lemma 9. *Let $G = C(n, \pm\{1, 2, 3, 4\})$ with $n = 8k+7$, and let*

$$A = (\{1,2\}, \{4k+2,\, 4k+3\}, \{4k+4,\, 4k+5\}, \{4(k+\ell)+j \,:\, j \in \{7,8,9\}\})$$

be an S-cluster with $\ell \in \{0, 1, \ldots, k\}$. If $X \subset V - \{2, 6, \ldots, 2+4\ell\}$ resolves A, then $|X| \geqslant 3$.

Proof. Suppose $X = \{x, y\} \subset V - \{2, 6, \ldots, 2+4\ell\}$ is a resolving set of A. Without loss of generality,

$$x \in R_1 - \{2, 6, \ldots, 2+4\ell\} = \{8k+8-4m : 0 \leqslant m \leqslant k\} \cup \{2+4m : \ell+1 \leqslant m \leqslant k\}.$$

Case 1 $x = 1$. From $d(1, 4k + 2) = d(1, 4k + 3) = k + 1$ and $d(1, 4k + 4) = d(1, 4k+5) = k+1$ it follows that $A' = (\{4k+2,\ 4k+3\},\ \{4k+4,\ 4k+5\})$ is an $S \cup \{x\}$-cluster, and by Observation 1, $|X - \{x\}| \geqslant 2$.

Case 2 $x = 8k+8-4m$, $1 \leqslant m \leqslant k$. From $d(x, 4k+2) = d(x, 4k+3) = k-m+2$ and $d(x, 4k+4) = d(x, 4k+5) = k-m+1$ it follows that $A' = (\{4k+2,\ 4k+3\},\ \{4k+4,\ 4k+5\})$ is an $S \cup \{x\}$-cluster, and by Observation 1, $|X - \{x\}| \geqslant 2$.

Case 3 $x = 2 + 4m$, $\ell + 1 \leqslant m \leqslant k$. From $d(x, 4k + 4\ell + 9) = d(x, 4k + 4\ell + 8) = d(x, 4k+4\ell+7) = k+l-m+2$ it follows that $A' = \{4k+4\ell+j : j \in \{7, 8, 9\}\}$ is an $S \cup \{x\}$-cluster, and by Lemma 1, $|X - \{x\}| \geqslant 2$. ☐

Lemma 10. *Let $n = 8k + r$, with $r \in \{2, 5, 6\}$, $G = C(n, \pm\{1, 2, 3, 4\})$, and let*

$$A = (\{a, a \pm 1\},\ \{a \pm j \pm 4\ell : j \in \{2, 3, 4\}\})$$

be an S-cluster for some S, where $0 \leqslant \ell \leqslant k$. If $X \subseteq V$ resolves A, then $|X| \geqslant 3$.

Proof. By symmetry, we can assume $A = (\{0, 1\},\ \{j + 4\ell : j \in \{2, 3, 4\}\})$. Suppose $X = \{x, y\}$ resolves A, and without loss of generality $x \in R_0 = \{1+4m,\ -4m : 0 \leqslant m \leqslant k\}$. For all $j \in \{2, 3, 4\}$, $x = 1 + 4m$ and $0 \leqslant m \leqslant k$,

$$d(x, j + 4\ell) = \begin{cases} \ell - m + 1 & \text{if } x = 1 + 4m \text{ and } m \leqslant \ell, \\ m - \ell & \text{if } x = 1 + 4m \text{ and } \ell + 1 \leqslant m \leqslant k, \\ \ell + m + 1 & \text{if } x = -4m \text{ and } \ell + m \leqslant\ k, \\ 2k - m - \ell & \text{if } x = -4m \text{ and } \ell + m > k \text{ and } r = 2, \\ 2k - m - \ell + 1 & \text{if } x = -4m \text{ and } \ell + m > k \text{ and } r \in \{5, 6\}. \end{cases}$$

Consequently, $\{j + 4\ell : j \in \{2, 3, 4\}\}$ is an $S \cup \{x\}$-block, and Lemma 1 implies $|X - \{x\}| \geqslant 2$, which is the required contradiction. ☐

Observation 2. Let $G = C(n, \pm\{1, 2, 3, 4\})$ with $n = 8k + r$ and $r \in \{2, 5\}$. Let $A = \{a, a + 1, a + 5, a + 6\}$. Then $|X| \geqslant 2$ for every set X that resolves A.

Proof. By symmetry we can assume $A = \{0, 1, 5, 6\}$. Suppose $X = \{x\}$ resolves A. Then $x \in R_0 \cap R_5$ and with

$$R_0 = \{1, 5, \ldots, 4k + 1\} \cup \{0, 8k + r - 4, \ldots, 4k + r\},$$
$$R_5 = \{6, 10, \ldots, 4k + 6\} \cup \{5, 1, 8k + r - 3, \ldots, 4k + r + 5\},$$

it follows that $x \in \{1, 5\}$. If $x = 1$, then $d(x, 0) = d(x, 5) = 1$; and if $x = 5$, then $d(x, 1) = d(x, 6) = 1$. In both cases we obtain the required contradiction. ☐

Observation 3. Let $G = C(n, \pm\{1, 2, 3, 4\})$ with $n = 8k + r$ and $r \in \{2, 5\}$. Let

$$A = (\{a, a + 2, a + 5\}, \{a + 6, a + 7\})$$

be an S-cluster for some S. If $X \subset V$ resolves A, then $|X| \geqslant 3$.

Proof. It is enough to prove it for $A = (\{0, 2, 5\}, \{6, 7\})$. Suppose $X = \{x\}$ resolves A. Then $x \in R_6 = \{7 + 4m,\ 6 - 4m\ :\ 0 \leqslant m \leqslant k\}$.

$n = 8k + 5$ In this case we have the following possibilities for x.

- If $x = 7 + 4m$, $0 \leqslant m \leqslant k - 1$, then $d(x, 0) = d(x, 2) = m + 2$.
- If $x = 7 + 4k$, then $d(x, 0) = d(x, 2) = k$.
- If $x = 6$, then $d(x, 2) = d(x, 5) = 1$.
- If $x = 2$, then $d(x, 0) = d(x, 5) = 1$.
- If $x = 6 - 4m$, $2 \leqslant m \leqslant k$, then $d(x, 0) = d(x, 2) = m - 1$.

$n = 8k + 2$ In this case we have the following possibilities for x.

- If $x = 7 + 4m$, $0 \leqslant m \leqslant k - 2$, then $d(x, 0) = d(x, 2) = m + 2$.
- If $x = 3 + 4k$, then $d(x, 0) = d(x, 5) = k$.
- If $x = 7 + 4k$, then $d(x, 2) = d(x, 5) = k$.
- If $x = 6$, then $d(x, 2) = d(x, 5) = 1$.
- If $x = 2$, then $d(x, 0) = d(x, 5) = 1$.
- If $x = 6 - 4m$, $2 \leqslant m \leqslant k$, then $d(x, 0) = d(x, 2) = m - 1$. □

Lemma 11. *Let $G = C(n, \pm\{1, 2, 3, 4\})$ with $n = 8k + r$, $r \in \{2, 5\}$. If $X \subseteq V$ resolves*

$$A = \{a,\ a + 1,\ a + 2,\ a + 5,\ a + 6,\ a + 7\}.$$

then $|X| \geqslant 3$.

Proof. By symmetry, we can assume $A = \{0, 1, 2, 5, 6, 7\}$. Suppose $X = \{x, y\}$ resolves A, and let $x \in R_0 = \{-4m,\ 1 + 4m\ :\ 0 \leqslant m \leqslant k\}$. Let $S' = S \cup \{x\}$. In Table 1 we summarize the possible choices for x, and in each case we exhibit an S'-block or S'-cluster which implies that $|X - \{x\}| \geqslant 2$, which is the required contradiction:

- For the S'-block $\{5, 6, 7\}$, $|X - \{x\}| \geqslant 2$ follows from Lemma 1.
- For the S'-block $\{1, 2, 6, 7\}$, $|X - \{x\}| \geqslant 2$ follows from Observation 2.
- For the S'-block $(\{0, 2, 5\}, \{6, 7\})$, $|X - \{x\}| \geqslant 2$ follows from Observation 3.

□

Lemma 12. *Let $G = C(n, \pm\{1, 2, 3, 4\})$ with $n = 8k + r$, $r \in \{2, 5\}$, and let*

$$A = (\{a,\ a + 1\},\ \{a + 2,\ a + 3,\ a + 5,\ a + 7,\ a + 8\})$$

be an S-cluster for some S. If $X \subseteq V$ resolves A then $|X| \geqslant 3$.

Proof. By symmetry we can assume $A = (\{0, 1\},\ \{2, 3, 5, 7, 8\})$. Suppose $X = \{x, y\}$ resolves A, and without loss of generality $x \in R_0 = \{-4m,\ 1 + 4m\ :\ 0 \leqslant m \leqslant k\}$. Let $S' = S \cup \{x\}$. In Table 2 we summarize the possible choices for x, and in each case we exhibit an S'-block which implies that $|X - \{x\}| \geqslant 2$, which is the required contradiction:

Table 1: Cases in the proof of Lemma 11. The last column contains the S'-block or S'-cluster which implies $|X - \{x\}| \geqslant 2$.

	x	$d(x,0)$	$d(x,1)$	$d(x,2)$	$d(x,5)$	$d(x,6)$	$d(x,7)$	Witness
$r=5$	$-4m,\ 0 \leqslant m \leqslant k-1$	m	$m+1$	$m+1$	$m+2$	$m+2$	$m+2$	$\{5,6,7\}$
	$4k+5$	k	$k+1$	$k+1$	k	k	k	$\{5,6,7\}$
	$1+4m,\ 2 \leqslant m \leqslant k$	$m+1$	m	m	$m-1$	$m-1$	$m-1$	$\{5,6,7\}$
	5	2	1	1	0	1	1	$\{1,2,6,7\}$
	1	1	0	1	1	2	2	$(\{0,2,5\},\{6,7\})$
$r=2$	$8k+2-4m,\ 0 \leqslant m \leqslant k-2$	m	$m+1$	$m+1$	$m+2$	$m+2$	$m+2$	$\{5,6,7\}$
	$4k+6$	$k-1$	k	k	$k+1$	k	k	$\{1,2,6,7\}$
	$4k+2$	k	$k+1$	k	k	$k-1$	$k-1$	$(\{0,2,5\},\{6,7\})$
	$1+4m,\ 2 \leqslant m \leqslant k$	$m+1$	m	m	$m-1$	$m-1$	$m-1$	$\{5,6,7\}$
	5	2	1	1	0	1	1	$\{1,2,6,7\}$
	1	0	1	1	1	2	2	$(\{0,2,5\},\{6,7\})$

Table 2: Cases in the proof of Lemma 12. The last column contains the S'-block which implies $|X - \{x\}| \geqslant 2$.

	x	$d(x,0)$	$d(x,1)$	$d(x,2)$	$d(x,3)$	$d(x,5)$	$d(x,7)$	$d(x,8)$	S'-block
$r=5$	$-4m,\ 0 \leqslant m \leqslant k-1$	m	$m+1$	$m+1$	$m+1$	$m+2$	$m+2$	$m+2$	$\{5,7,8\}$
	$4k+5$	k	$k+1$	$k+1$	$k+1$	k	k	k	$\{5,7,8\}$
	$1+4m,\ 2 \leqslant m \leqslant k$	$m+1$	m	m	m	$m-1$	$m-1$	$m-1$	$\{5,7,8\}$
	5	2	1	1	1	0	1	1	$\{2,3,7,8\}$
	1	1	0	1	1	1	2	2	$\{2,3,5\}$
$r=2$	$-4m,\ 0 \leqslant m \leqslant k-2$	m	$m+1$	$m+1$	$m+1$	$m+2$	$m+2$	$m+2$	$\{5,7,8\}$
	$4k+6$	$k-1$	k	k	k	$k+1$	k	k	$\{2,3,7,8\}$
	$4k+2$	k	$k+1$	k	k	k	$k-1$	$k-1$	$\{2,3,5\}$
	$1+4m,\ 2 \leqslant m \leqslant k$	$m+1$	m	m	m	$m-1$	$m-1$	$m-1$	$\{5,7,8\}$
	5	2	1	1	1	0	1	1	$\{2,3,7,8\}$
	1	1	0	1	1	1	2	2	$\{2,3,5\}$

- For the S'-blocks $\{5,7,8\}$ and $\{2,3,5\}$, $|X - \{x\}| \geqslant 2$ follows from Lemma 1.
- For the S'-block $\{2,3,7,8\}$, $|X - \{x\}| \geqslant 2$ follows from Observation 2. □

Observation 4. Let $n = 8k+r$, with $r \in \{2,5\}$ and let $A = (\{a,\ a+1\},\ \{a+2,\ a+5,\ a+7\})$ be an S-cluster for some S. If $X \subseteq V$ resolves A then $|X| \geqslant 2$.

Proof. By symmetry we can assume $A = (\{0,1\},\ \{2,5,7\})$. Suppose $X = \{x\}$ is a resolving set of A. Then $x \in R_0 = \{8k+r-4m, 1+4m : 0 \leqslant m \leqslant k\}$.

For $n = 8k+5$ we have the following possibilities:

- If $x = 8k+5-4m$, $0 \leqslant m \leqslant k-1$, then $d(x,5) = d(x,7) = m+2$.
- If $x = 4k+5$, then $d(x,5) = d(x,7) = k$.
- If $x = 1$, then $d(x,2) = d(x,5) = 1$.
- If $x = 5$, then $d(x,2) = d(x,7) = 1$.
- If $x = 1+4m$, $2 \leqslant m \leqslant k$, then $d(x,5) = d(x,7) = m-1$.

For $n = 8k + 2$ we have the following possibilities:

- If $x = 8k + 2 - 4m$, $0 \leqslant m \leqslant k - 2$, then $d(x, 5) = d(x, 7) = m + 2$.
- If $x = 4k + 6$, then $d(x, 2) = d(x, 7) = k$.
- If $x = 4k + 2$, then $d(x, 2) = d(x, 5) = k$.
- If $x = 1$, then $d(x, 2) = d(x, 5) = 1$.
- If $x = 5$, then $d(x, 2) = d(x, 7) = 1$.
- If $x = 1 + 4m$, $2 \leqslant m \leqslant k$, then $d(x, 5) = d(x, 7) = m - 1$. □

Observation 5. Let $n = 8k + r$ with $r \in \{2, 5\}$ and let $A = (\{a, a + 2\}, \{a + 3, a + 5\}, \{a + 6, a + 8\})$ be an S-cluster for some S. If $X \subseteq V$ resolves A then $|X| \geqslant 2$.

Proof. By symmetry we can assume $A = (\{0, 2\}, \{3, 5\}, \{6, 8\})$. Suppose x resolves A. Then $x \in R_0 \cup R_1$, where $R_0 = \{8k + r - 4m, 1 + 4m : 0 \leqslant m \leqslant k\}$ and $R_1 = \{8k + r + 1 - 4m, 2 + 4m : 0 \leqslant m \leqslant k\}$. For $n = 8k + 5$ we have the following possibilities:

- If $x = 8k + 5 - 4m,\ 0 \leqslant m \leqslant k$,
  then $d(x, 6) = d(x, 8) = \begin{cases} m + 2 & \text{when } 0 \leqslant m \leqslant k - 1, \\ k & \text{when } m = k. \end{cases}$
- If $x = 1 + 4m,\ 0 \leqslant m \leqslant k$, then $d(x, 6) = d(x, 8) = \begin{cases} m - 1 & \text{when } 2 \leqslant m \leqslant k, \\ 2 - m & \text{when } m \in \{0, 1\}. \end{cases}$
- If $x = 8k + 6 - 4m$, $0 \leqslant m \leqslant k$, then $d(x, 3) = d(x, 5) = m + 1$.
- If $x = 2 + 4m,\ 0 \leqslant m \leqslant k$, then $d(x, 3) = d(x, 5) = \begin{cases} m & \text{when } 1 \leqslant m \leqslant k, \\ 1 & \text{when } m = 0. \end{cases}$

For $n = 8k + 2$ we have the following possibilities:

- If $x = 8k + 2 - 4m,\ 0 \leqslant m \leqslant k$,
  then $d(x, 6) = d(x, 8) = \begin{cases} m + 2 & \text{when } 0 \leqslant m \leqslant k - 2, \\ 2k - m - 1 & \text{when } m \in \{k - 1, k\}. \end{cases}$
- If $x = 1 + 4m,\ 0 \leqslant m \leqslant k$, then $d(x, 6) = d(x, 8) = \begin{cases} m - 1 & \text{when } 2 \leqslant m \leqslant k, \\ 2 - m & \text{when } m \in \{0, 1\}. \end{cases}$
- If $x = 8k + 6 - 4m,\ 0 \leqslant m \leqslant k$,
  then $d(x, 3) = d(x, 5) = \begin{cases} m + 1 & \text{when } 0 \leqslant m \leqslant k - 1, \\ k & \text{when } m = k. \end{cases}$.
- If $x = 2 + 4m,\ 0 \leqslant m \leqslant k$, then $d(x, 3) = d(x, 5) = \begin{cases} m & \text{when } 1 \leqslant m \leqslant k, \\ 1 & \text{when } m = 0. \end{cases}$ □

Lemma 13. *Let $G = C(n, \pm\{1, 2, 3, 4\})$ with $n = 8k + r$ with $r \in \{2, 5\}$, and let*

$$A = (\{a, a + 1, a + 2\}, \{a + 3, a + 5, a + 6, a + 8\})$$

be an S-cluster for some S. If $X \subseteq V$ of A, $|X| \geqslant 3$.

Proof. By symmetry we can assume $A = (\{0, 1, 2\}, \{3, 5, 6, 8\})$. Suppose $X = \{x, y\}$ resolves A. Without loss of generality $x \in R_0 = \{-4m, 1 + 4m : 0 \leqslant m \leqslant k\}$.

Let $S' = S \cup \{x\}$. In Table 3 we summarize the possible choices for x, and in each case we exhibit an S'-block or S'-cluster implying that $|X - \{x\}| \geqslant 2$, which is the required contradiction:

- For the S'-block $\{5, 7, 8\}$, $|X - \{x\}| \geqslant 2$ follows from Lemma 1.
- For the S'-cluster $(\{1, 2\}, \{3, 6, 8\})$, $|X - \{x\}| \geqslant 2$ follows from Observation 4.
- For the S'-cluster $(\{0, 2\}, \{3, 5\}, \{6, 8\})$, $|X - \{x\}| \geqslant 2$ follows from Observation 5. □

Table 3: Cases in the proof of Lemma 13. The last column contains the S'-block or S'-cluster which implies $|X - \{x\}| \geqslant 2$.

	x	$d(x,0)$	$d(x,1)$	$d(x,2)$	$d(x,3)$	$d(x,5)$	$d(x,6)$	$d(x,8)$	Witness
$r=5$	$-4m,\ 0 \leqslant m \leqslant k-1$	m	$m+1$	$m+1$	$m+1$	$m+2$	$m+2$	$m+2$	$\{5,6,8\}$
	$4k+5$	k	$k+1$	$k+1$	$k+1$	k	k	k	$\{5,6,8\}$
	$1+4m,\ 2 \leqslant m \leqslant k$	$m+1$	m	m	m	$m-1$	$m-1$	$m-1$	$\{5,6,8\}$
	5	2	1	1	1	0	1	1	$(\{1,2\},\{3,6,8\})$
	1	1	0	1	1	1	2	2	$(\{0,2\},\{3,5\},\{6,8\})$
$r=2$	$-4m,\ 0 \leqslant m \leqslant k-2$	m	$m+1$	$m+1$	$m+1$	$m+2$	$m+2$	$m+2$	$\{5,6,8\}$
	$4k+6$	$k-1$	k	k	k	$k+1$	k	k	$(\{1,2\},\{3,6,8\})$
	$4k+2$	k	$k+1$	k	k	k	$k-1$	$k-1$	$(\{0,2\},\{3,5\},\{6,8\})$
	$1+4m,\ 2 \leqslant m \leqslant k$	$m+1$	m	m	m	$m-1$	$m-1$	$m-1$	$\{5,6,8\}$
	5	2	1	1	1	0	1	1	$(\{1,2\},\{3,6,8\})$
	1	1	0	1	1	1	2	2	$(\{0,2\},\{3,5\},\{6,8\})$

In the next six lemmas we show that for $n \equiv 3 \pmod 8$ certain S-blocks or S-clusters cannot be resolved by a single vertex x. In the proofs we always assume $a = 0$, and we verify that for every possible x there remains a pair of unresolved vertices.

Lemma 14. *For $n = 8k + 3$, if $X \subseteq V$ resolves $A = \{a, a + 1, a + 5, a + 6\}$ then $|X| \geqslant 2$.*

Proof. If $x \in V$ resolves A then $x \in R_0 = \{8k + 3 - 4m, 1 + 4m : 0 \leqslant m \leqslant k\}$.

- If $x = 8k + 3 - 4m,\ 0 \leqslant m \leqslant k - 1$, then $d(x, 5) = d(x, 6) = m + 2$.
- If $x = 4k + 3$, then $d(x, 5) = d(x, 6) = k$.
- If $x = 1 + 4m,\ 2 \leqslant m \leqslant k$, then $d(x, 5) = d(x, 6) = m - 1$.
- If $x = 5$, then $d(x, 1) = d(x, 6) = 1$.
- If $x = 1$, then $d(x, 0) = d(x, 5) = 1$. □

Lemma 15. *For $n = 8k+3$, if X resolves $A = (\{a, a+1\}, \{a+2, a+5, a+7\}, \{a+9, a+10\})$ then $|X| \geqslant 2$.*

Proof. If $x \in V$ resolves A then $x \in R_0 = \{8k + 3 - 4m,\ 1 + 4m : 0 \leqslant m \leqslant k\}$.

- If $x = 8k + 3 - 4m,\ 0 \leqslant m \leqslant k - 2$, then $d(x, 5) = d(x, 7) = m + 2$.
- If $x = 4k + 7$, then $d(x, 2) = d(x, 7) = k$.
- If $x = 4k + 3$, then $d(x, 9) = d(x, 10) = k - 1$.

- If $x = 1 + 4m,\ 3 \leqslant m \leqslant k$, then $d(x, 5) = d(x, 7) = m - 1$.
- If $x = 9$, then $d(x, 5) = d(x, 7) = 1$.
- If $x = 5$, then $d(x, 2) = d(x, 7) = 1$.
- If $x = 1$, then $d(x, 2) = d(x, 5) = 1$. □

Lemma 16. *For $n = 8k + 3$, in X resolves $A = (\{a, a + 1\}, \{a + 7, a + 8\})$ then $|X| \geqslant 2$.*

Proof. If $x \in V$ resolves A then $x \in R_0 = \{8k + 3 - 4m,\ 1 + 4m\ :\ 0 \leqslant m \leqslant k\}$.

- If $x = 8k + 3 - 4m$, $0 \leqslant m \leqslant k - 2$, then $d(x, 7) = d(x, 8) = m + 2$.
- If $x = 4k + 7$, then $d(x, 7) = d(x, 8) = k$.
- If $x = 4k + 3$, then $d(x, 7) = d(x, 8) = k - 1$.
- If $x = 1 + 4m,\ 2 \leqslant m \leqslant k$, then $d(x, 7) = d(x, 8) = m - 1$.
- If $x = 5$, then $d(x, 7) = d(x, 8) = 1$.
- If $x = 1$, then $d(x, 7) = d(x, 8) = 2$. □

Lemma 17. *For $n = 8k+3$, if X resolves $A = (\{a, a+1\}, \{a+2, a+3\}, \{a+4, a+5\})$ then $|X| \geqslant 2$.*

Proof. If $x \in V$ resolves A then $x \in R_0 = \{8k + 3 - 4m, 1 + 4m : 0 \leqslant m \leqslant k\}$.

- If $x = 8k + 3 - 4m,\ 0 \leqslant m \leqslant k - 1$, then $d(x, 2) = d(x, 3) = m + 1$.
- If $x = 4k + 3$, then $d(x, 4) = d(x, 5) = k$.
- If $x = 1 + 4m,\ 1 \leqslant m \leqslant k$, then $d(x, 2) = d(x, 3) = m$.
- If $x = 1$, then $d(x, 2) = d(x, 3) = 1$. □

Lemma 18. *For $n = 8k + 3$, if X resolves $A = (\{a, a + 1\}, \{a + 3, a + 5, a + 8\})$ then $|X| \geqslant 2$.*

Proof. If $x \in V$ resolves A then $x \in R_0 = \{8k + 3 - 4m,\ 1 + 4m\ :\ 0 \leqslant m \leqslant k\}$.

- If $x = 8k + 3 - 4m,\ 0 \leqslant m \leqslant k - 2$, then $d(x, 5) = d(x, 8) = m + 2$.
- If $x = 4k + 7$, then $d(x, 3) = d(x, 8) = k$.
- If $x = 4k + 3$, then $d(x, 3) = d(x, 5) = k$.
- If $x = 1 + 4m,\ 2 \leqslant m \leqslant k$, then $d(x, 5) = d(x, 8) = m - 1$.
- If $x = 5$, then $d(x, 3) = d(x, 8) = 1$.
- If $x = 1$, then $d(x, 3) = d(x, 5) = 1$. □

Lemma 19. *For $n = 8k + 3$, if X resolves $A = (\{a, a + 1\}, \{a + 3, a + 4\})$ then $|X| \geqslant 2$.*

Proof. If $x \in V$ resolves A then $x \in R_0 = \{8k + 3 - 4m,\ 1 + 4m : 0 \leqslant m \leqslant k\}$.

- If $x = 8k + 3 - 4m,\ 0 \leqslant m \leqslant k - 1$, then $d(x, 3) = d(x, 4) = m + 1$.
- If $x = 4k + 3$, then $d(x, 3) = d(x, 4) = k$.
- If $x = 1 + 4m,\ 1 \leqslant m \leqslant k$, then $d(x, 3) = d(x, 4) = m$.
- If $x = 1$, then $d(x, 3) = d(x, 4) = 1$. □

4 Lower bounds

In this section we prove the lower bounds for Theorem 1, using the lemmas proved in Section 3.

4.1 $n \equiv 1 \pmod 8$

We write $n = 8k + 9$, $G = C(n, \pm\{1, 2, 3, 4\})$, and prove $\dim(G) \geqslant 6$. Suppose $B = \{w_1, w_2, w_3, w_4, w_5\}$ is a metric basis. Without loss of generality $w_1 = 0$ and $w_2, w_3 \in \{1, 2, \ldots, 4k+3\}$. We make a case distinction with respect to the possibilities for the set $S = \{0, w_2, w_3\}$. By Lemma 2 we may assume that $w_2 \geqslant 4$, $w_3 \geqslant 4$, and $|w_2 - w_3| \geqslant 4$.

Case 1 $w_2 \equiv w_3 \pmod 4$, that is $S = \{0,\ 4m + i,\ 4m' + i\}$ with $i \in \{0, 1, 2, 3\}$ and $1 \leqslant m < m' \leqslant k$. The vertex set $A = \{4k+i+1,\ 4k+i+2,\ 4k+i+3,\ 4k+i+4\}$ is an S-block with representing vector $r(A|S) = (k + 1, k - m + 1, k - m' + 1)$. Since $B - S$ resolves A, Lemma 1 implies $|B - S| \geqslant 3$.

Case 2 $S = \{0,\ 4m,\ 4m' + i\}$ with $m \in \{1, \ldots, k+1\}$, $m' \in \{1, \ldots, k\}$, $m \neq m'$ and $i \in \{1, 2, 3\}$. In this case $A = \{4(m' + k) + \ell\ :\ \ell \in \{5, 6, 7, 8\}\}$ is an S-block with

$$r(A|S) = \begin{cases} (k - m' + 1,\ k - m' + m + 1,\ k + 1) & \text{if } m < m', \\ (k - m' + 1,\ k - (m - m') + 2,\ k + 1) & \text{if } m > m'. \end{cases}$$

By Lemma 1, $|B - S| \geqslant 3$.

Case 3 $S = \{0,\ 4m + 1,\ 4m' + i\}$ with $m, m' \in \{1, \ldots, k\}$, $m \neq m'$ and $i \in \{2, 3\}$. Let $A = (A_1, A_2)$ with $A_1 = \{4k + 4, 4k + 5\}$ and $A_2 = \{4k + 4 + j + 4m'\ :\ j \in \{2, 3, 4\}\}$. Then A is an S-cluster with $r(A_1|S) = (k+1, k-m+1, k-m'+1)$ and

$$r(A_2|S) = \begin{cases} (k - m' + 1,\ k - m' + m + 1,\ k + 1) & \text{if } m < m', \\ (k - m' + 1,\ k - (m - m') + 2,\ k + 1) & \text{if } m > m'. \end{cases}$$

By Lemma 3, $|B - S| \geqslant 3$.

Case 4 $S = \{0, 4m+2, 4m'+3\}$ with $m, m' \in \{1, \ldots, k\}$, $m \neq m'$. Let $A = (A_1, A_2)$ with $A_1 = \{3, 4\}$ and $A_2 = \{4(m+k)+\ell\ :\ \ell \in \{5, 6, 7\}\}$. Then A is an S-cluster with $r(A_1|S) = (1, m, m')$ and

$$r(A_2|S) = \begin{cases} (k - m + 1,\ k + 1,\ k - (m' - m) + 1) & \text{if } m < m', \\ (k - m + 1,\ k + 1,\ k - m + m' + 1) & \text{if } m > m'. \end{cases}$$

By Lemma 3, $|B - S| \geqslant 3$.

In any case, we conclude $|B| \geqslant 6$, which is the required contradiction.

4.2 $n \equiv 0 \pmod 8$

We write $n = 8k + 8$, and prove $\dim(G) \geqslant 6$. Similar to the previous subsection we assume that there is a resolving set $B = \{w_1, w_2, w_3, w_4, w_5\}$ such that $w_1 < w_2 < w_3 < w_4 < w_5$. Without loss of generality $w_1 = 0$ and $w_2, w_3 \in \{3, 4, \dots, 4k + 3\}$, and we make a case distinction with respect to the set $S = \{0,\, w_2,\, w_3\}$.

Case 1 $w_2 \equiv w_3 \pmod 4$, that is $S = \{0,\, 4m + i,\, 4m' + i\}$ with $i \in \{0, 1, 2, 3\}$ and $1 \leqslant m < m' \leqslant k$. The vertex set $A = \{4k+i+1,\, 4k+i+2,\, 4k+i+3,\, 4k+i+4\}$ is an S-block with representing vector $r(A|S) = (k+1,\, k-m+1,\, k-m'+1)$. Since $B - S$ resolves A, Lemma 1 implies $|B - S| \geqslant 3$.

Case 2 $S = \{0,\, 4m,\, 4m'+i\}$, $i \in \{1, 2, 3\}$, $1 \leqslant m < m' \leqslant k$. Then $A = \{4(m'+k)+\ell \,:\, \ell \in \{4, 5, 6, 7\}\}$ is an S-block with $r(A|S) = (k-m'+1, k+m-m'+1, k+1)$, and by Lemma 1, $|B - S| \geqslant 3$.

Case 3 $S = \{0,\, 4m + 1,\, 4m' + i\}$, $i \in \{0, 2\}$, $1 \leqslant m < m' \leqslant k$. Let $A = (A_1,\, A_2)$ with $A_1 = \{4k + 3, 4k + 4\}$ and $A_2 = \{4k + 3 + 4m' + \ell \,:\, \ell \in \{2, 3, 4\}$. Then A is an S-cluster with $r(A_1|S) = (k + 1, k - m + 1, k - m' + 1)$ and $r(A_2|S) = (k - m' + 1, k - m' + m + 1, k + 1)$. By Lemma 5, this implies $|B - S| \geqslant 3$.

Case 4 $S = \{0, 4m+1, 4m'+3\}$, $1 \leqslant m < m' \leqslant k$. In this case $A = \{4m+4k+j \,:\, j \in \{4, 5, 6, 7\}$ is an S-block with $r(A|S) = (k-m+1,\, k+1,\, k-m'+m+1)$. Hence, by Lemma 1, $|B - S| \geqslant 3$.

Case 5 $S = \{0,\, 4m + 2,\, 4m'\}$, $1 \leqslant m < m' \leqslant k$. Let $A = (A_1,\, A_2)$ with $A_1 = \{4k + 3, 4k + 4\}$ and $A_2 = \{4k + 3 + 4m + \ell \,:\, \ell \in \{2, 3, 4\}\}$. Then A is an S-cluster with $r(A_1|S) = (k + 1,\, k - m + 1,\, k - m' + 1)$ and $r(A_2|S) = (k - m + 1,\, k + 1,\, k + m - m' + 2)$. By Lemma 5, this implies $|B - S| \geqslant 3$.

Case 6 $S = \{0,\, 4m + 2,\, 4m' + 1\}$, $1 \leqslant m < m' \leqslant k$. Let $A = (A_1,\, A_2)$ with $A_1 = \{4k + 3,\, 4k + 4,\, 4k + 5\}$ and $A_2 = \{4k + 5 + 4m + j \,:\, j \in \{1, 2\}\}$. Then A is an S-cluster as $r(A_1|S) = (k + 1,\, k - m + 1,\, k - m' + 1)$ and $r(A_2|S) = (k-m+1,\, k+1,\, k+m-m'+2)$. Hence, by Lemma 5, $|B-S| \geqslant 3$.

Case 7 $S = \{0,\, 4m+2,\, 4m'+3\}$, $1 \leqslant m < m' \leqslant k$. In this case $A = \{4k+4m+3+j \,:\, j \in \{1, 2, 3, 4\}\}$ is an S-block with $r(A|S) = (k-m+1,\, k+1,\, k+m-m'+1)$. Hence, by Lemma 1, $|B - S| \geqslant 3$.

Case 8 $S = \{0,\, 4m + 3,\, 4m' + i\}$, $i \in \{0, 1\}$, $0 \leqslant m < m' \leqslant k$ (Note that in this case m can be 0 as well). Then $A = (A_1,\, A_2)$ with $A_1 = \{4k + 2,\, 4k + 3\}$ and $A_2 = \{4k + 4m' + 3 + j \,:\, j \in \{1, 2, 3\}\}$ is an S-cluster with $r(A_1|S) = (k + 1,\, k - m,\, k - m' + 1)$. Hence, by Lemma 5, $|B - S| \geqslant 3$.

Case 9 $S = \{0,\, 4m+3,\, 4m'+2\}$, $0 \leqslant m < m' \leqslant k$. Let $A = (A_0,\, A_1, \ldots, A_k,\, B_1,\, B_2,\, B_3)$ where

$$A_0 = \{4k+4, 4k+5, 4k+6\}, \quad A_i = \{4k+4+4i, a+5+4i\} \text{ for } 1 \leqslant i \leqslant k,$$
$$B_1 = \{8k+7, 1\}, \qquad B_2 = \{2, 4\}, \qquad B_3 = \{4m'+1, 4m'+3\}.$$

Then A is an S-cluster with the following representations:

$$\begin{aligned}
r(A_0 \,|\, S) &= (k+1,\, k-m+1,\, k-m'+1),\\
r(A_i \,|\, S) &= \begin{cases} (k-i+1,\, k-m+i+1,\, k-m'+i+1) & \text{if } 1 \leqslant i \leqslant m,\\ (k-i+1,\, k+m-i+2,\, k+m'-i+2) & \text{if } m < i, \end{cases}\\
r(B_1 \,|\, S) &= (1, m+1, m'+1),\\
r(B_2 \,|\, S) &= (1, m+1, m'),\\
r(B_3 \,|\, S) &= (m'+1, m'-m, 1).
\end{aligned}$$

Hence, by Lemma 6, $|B-S| \geqslant 3$.

4.3 $n \equiv 7 \pmod 8$

We write $n = 8k+7$ and prove $\dim(G) \geqslant 6$. Similar to the previous subsection we assume that there is a resolving set $B = \{w_1, w_2, w_3, w_4, w_5\}$ such that $w_1 < w_2 < w_3 < w_4 < w_5$. Without loss of generality $w_1 = 0$ and $w_2, w_3 \in \{2, 3 \ldots, 4k+3\}$, and we make a case distinction with respect to the set $S = \{0, w_2, w_3\}$.

Case 1 $w_2 \equiv w_3 \pmod 4$, that is $S = \{0,\, 4m+i,\, 4m'+i\}$ with $i \in \{0,1,2\}$ and $1 \leqslant m < m' \leqslant k$. Then $A = \{4k+i+1,\, 4k+i+2,\, 4k+i+3,\, 4k+i+4\}$ is an S-block with representing vector $r(A|S) = (k+1, k-m+1, k-m'+1)$. Since $B-S$ resolves A, Lemma 1 implies $|B-S| \geqslant 3$.

Case 2 $S = \{0,\, 4m,\, 4m'+i\}$, $i \in \{1,2\}$, $1 \leqslant m < m' \leqslant k$. Then $A = \{4(m'+k)+j \,:\, j \in \{3,4,5,6\}\}$ is an S-block with $r(A|S) = (k-m'+1,\, k+m-m'+1, k+1)$, and by Lemma 1, $|B-S| \geqslant 3$.

Case 3 $S = \{0,\, 4m,\, 4m'+3\}$, for $1 \leqslant m \leqslant m' \leqslant k-1$. Let $A = (A_1,\, A_2)$ with $A_1 = \{4k+2,\, 4k+3\}$ and $A_2 = \{4k+2+4m+j \,:\, j \in \{2,3,4\}\}$. Then A is an S-cluster with $r(A_1|S) = (k+1,\, k-m+1,\, k-m')$ and $r(A_2|S) = (k-m+1,\, k+1,\, k+m-m'+1)$. Hence, by Lemma 5, $|B-S| \geqslant 3$.

Case 4 $S = \{0,\, 4m+1,\, 4m'\}$, for $1 \leqslant m < m' \leqslant k$. Let $A = (A_1,\, A_2)$ with $A_1 = \{4k+2,\, 4k+3\}$ and $A_2 = \{4k+2+4m'+j \,:\, j \in \{2,3,4\}\}$. Then A is an S-cluster with $r(A_1|S) = (k+1,\, k-m+1,\, k-m'+1)$ and $r(A_2|S) = (k-m'+1,\, k+m-m'+1,\, k+1)$. Hence, by Lemma 5, $|B-S| \geqslant 3$.

Case 5 $S = \{0,\, 4m+1,\, 4m'+3\}$, for $1 \leqslant m < m' \leqslant k-1$. Let $A = (A_1,\, A_2)$ with $A_1 = \{4k+2,\, 4k+3\}$ and $A_2 = \{4k+2+4m'+j \,:\, j \in \{2,3,4\}\}$. Then A is an S-cluster with $r(A_1|S) = (k+1,\, k-m+1,\, k-m'+1)$ and $r(A_2|S) = (k-m'+1,\, k+m-m'+1,\, k+1)$. Hence, by Lemma 5, $|B-S| \geqslant 3$.

Case 6 $S = \{0,\, 4m+1,\, 4m'+2\}$, for $1 \leqslant m < m' \leqslant k$. Then $A = \{4k+4m+2+j : j \in \{1,2,3,4\}\}$ is an S-block with representation $r(A|S) = (k-m+1,\, k+1,\, k+m-m'+1)$. By Lemma 1, $|B-S| \geqslant 3$.

Case 7 $S = \{0,\, 4m+2,\, 4m'\}$, for $0 \leqslant m < m' \leqslant k$. Let $A = (A_0, B_0, A_1, B_1, \ldots, A_k, B_k, A_{k+1})$ with $A_i = \{4k+1+4i,\, 4k+2+4i\}$ and $B_i = \{4k+3+4i,\, 4k+4+4i\}$. Then A is an S-cluster with the following representations:

$$r(A_0|S) = (k+1,\, k-m,\, k-m'+1),$$
$$r(A_i|S) = \begin{cases} (k-i+2,\, k+i-m,\, k+i-m'+1) & \text{for } 1 \leqslant i \leqslant m+1, \\ (k-i+2,\, k+m-i+2,\, k+i-m'+1) & \text{for } m+1 < i \leqslant m', \\ (k-i+2,\, k+m-i+2,\, k+m'-i+2) & \text{for } m' < i \leqslant k+1, \end{cases}$$
$$r(B_i|S) = \begin{cases} (k-i+1,\, k+i-m+1,\, k+i-m'+1) & \text{for } 0 \leqslant i \leqslant m, \\ (k-i+1,\, k+m-i+2,\, k+i-m'+1) & \text{for } m < i \leqslant m', \\ (k-i+1,\, k+m-i+2,\, k+m'-i+1) & \text{for } m' < i \leqslant k. \end{cases}$$

Hence, by Lemma 7, $|B-S| \geqslant 3$.

Case 8 $S = \{0,\, 4m+2,\, 4m'+1\}$, $0 \leqslant m < m' \leqslant k$. Let $A = (A_0,\, A_1, \ldots A_k,\, B_1,\, B_2)$, where

$$A_i = \{3+4i,\, 4+4i\}, \text{ for } 0 \leqslant i \leqslant k-1, \quad A_k = \{3+4k,\, 4+4k,\, 5+4k\},$$
$$B_1 = \{3+4(k+m'),\, 4+4(k+m')\}, \qquad B_2 = \{5+4(k+m'), 6+4(k+m')\}.$$

Then A is an S-cluster with with the following representations.

$$r(A_i|S) = \begin{cases} (i+1,\, m-i,\, m'-i), & \text{for } 0 \leqslant i < m, \\ (i+1,\, i-m+1,\, m'-i), & \text{for } m \leqslant i < m', \\ (i+1,\, i-m+1,\, i-m'+1), & \text{for } m' \leqslant i \leqslant k, \end{cases}$$
$$r(B_1|S) = (k-m'+1,\, k+m-m'+2,\, k+1),$$
$$r(B_2|S) = (k-m'+1,\, k+m-m'+1,\, k+1).$$

By Lemma 8, $|B-S| \geqslant 3$.

Case 9 $S = \{0,\, 4m+2,\, 4m'+3\}$, for $0 \leqslant m < m' \leqslant k-1$. Let $A = (A_1,\, A_2)$, where $A_1 = \{4k+4m+j : j \in \{4,5,6\}\}$ and $A_2 = \{4k+4m'+j : j \in \{7,8\}\}$. Then A is an S-cluster with the representations $r(A_1|S) = (k-m+1,\, k+1,\, k+m-m'+1)$ and $r(A_2|S) = (k-m',\, k+m-m'+1,\, k+1)$. Hence, by Lemma 5, $|B-S| \geqslant 3$.

Case 10 $S = \{0,\, 4m+3,\, 4m'\}$, for $0 \leqslant m < m' \leqslant k$. Let $A = (A_1,\, A_2)$, where $A_1 = \{4k+5,\, 4k+6\}$ and $A_2 = \{4k+4m'+j : j \in \{3,4,5\}\}$. Then A is an S-cluster with representations $r(A_1|S) = (k+1,\, k-m+1,\, k-m'+2)$ and $r(A_2|S) = (k-m'+1,\, k+m-m'+2,\, k+1)$. By Lemma 5, $|B-S| \geqslant 3$.

Case 11 $S = \{0,\, 4m+3,\, 4m'+1\}$,for $0 \leqslant m < m' \leqslant k$. Let $A = (A_1, A_2, A_3, A_4)$, where $A_1 = \{1, 2\}$, $A_2 = \{4k+2,\, 4k+3\}$, $A_3 = \{4k+4,\, 4k+5\}, A_4 = \{4k+4m+j \,:\, j \in \{7, 8, 9\}\}$. Then A is an S-cluster with representations $r(A_1|S) = (1,\, m+1,\, m')$, $r(A_2|S) = (k+1,\, k-m,\, k-m'+1)$, $r(A_3|S) = (k+1,\, k-m+1,\, k-m'+1)$, $r(A_4|S) = (k-m,\, k+1,\, k+m-m'+2)$. Hence, by Lemma 9, $|B - S| \geqslant 3$.

Case 12 $S = \{0,\, 4m+3,\, 4m'+2\}$, for $0 \leqslant m < m' \leqslant k$. Let $A = (A_1,\, A_2)$, where $A_1 = \{4k+5,\, 4k+6\}$ and $A_2 = \{4k+4m+j \,:\, j \in \{7, 8, 9\}\}$. Then A is an S-cluster with representations $r(A_1|S) = (k+1,\, k-m+1,\, k-m'+1)$ and $r(A_2|S) = (k-m,\, k+1,\, k+m-m'+2)$. By Lemma 5, this implies $|B-S| \geqslant 3$.

Case 13 $S = \{0,\, 4m+3,\, 4m'+3\}$, for $0 \leqslant m < m' \leqslant k$. Let $A = (A_1,\, A_2)$, where $A_1 = \{4k+5,\, 4k+6\}$ and $A_2 = \{4k+4m+j \,:\, j \in \{7, 8, 9\}\}$. Then A is an S-cluster with representations $r(A_1|S) = (k+1,\, k-m+1,\, k-m'+1)$ and $r(A_2|S) = (k-m',\, k+m-m'+1,\, k+1)$. By Lemma 5, this implies $|B - S| \geqslant 3$.

4.4 $n \equiv 5 \pmod 8$

We write $n = 8k+5$ and prove $\dim(G) \geqslant 5$. Assume that there is a resolving set $B = \{w_1, w_2, w_3, w_4\}$ such that $w_1 < w_2 < w_3 < w_4$. Without loss of generality $w_1 = 0$ and $w_2 \in \{1, 2, \ldots, 4k+2\}$, and we make a case distinction with respect to the set $S = \{0, w_2\}$.

Case 1 $S = \{0,\, 4m\}$, for $1 \leqslant m \leqslant k$. Let $A = \{8k+4,\, 8k+3,\, 8k+2,\, 8k+1\}$. Then A is an S-block with the representation $r(A|S) = (1,\, m+1)$. By Lemma 1, this implies $|B - S| \geqslant 3$.

Case 2 $S = \{0, 4m+1\}$, for $1 \leqslant m \leqslant k$. Let $A = \{1, 2, 3, 4\}$. Then A is an S-block with the representation $r(A|S) = (1, m)$. This implies that by Lemma 1, $|B - S| \geqslant 3$.

Case 3 $S = \{0, 4m+2\}$, for $1 \leqslant m \leqslant k$. Let $A = (A_1, A_2)$, where $A_1 = \{8k+1, 8k+2\}$ and $A_2 = \{2, 3, 4\}$. Then A is an S-cluster with representations $r(A_1|S) = (1, m+1)$, for $1 \leqslant m < k$, $r(A_1|S) = (1, k)$ for $m = k$ and $r(A_2|S) = (1, m)$. This implies that $|B - S| \geqslant 3$ by Lemma 10.

Case 4 $S = \{0, 4m+3\}$, for $1 \leqslant m < k$. Let $A = (A_1, A_2)$, where $A_1 = \{8k+1, 8k+2, 8k+3\}$ and $A_2 = \{3, 4\}$. Then A is an S-cluster with representations $r(A_1|S) = (1, m+2)$ and $r(A_2|S) = (1, m)$. Hence, by Lemma 10, we have $|B - S| \geqslant 3$.

Case 5 $S = \{0, 1\}$. Consider $A = \{8k+2, 8k+3, 8k+4, 2, 3, 4\}$. Clearly A is an S-block with the representation $r(A|S) = (1, 1)$. Hence, by Lemma 11, $|B - S| \geqslant 3$.

Case 6 $S = \{0, 2\}$. Let $A = (A_1, A_2)$, where $A_1 = \{8k+1, 8k+2\}$ and $A_2 = \{8k+3, 8k+4, 1, 3, 4\}$. Then A is an S-cluster with representations $r(A_1|S) = (1, 2)$ and $r(A_2|S) = (1, 1)$. This implies that by Lemma 12, $|B - S| \geqslant 3$.

Case 7 $S = \{0, 3\}$. Let $A = (A_1, A_2)$, where $A_1 = \{8k+1, 8k+2, 8k+3\}$, and $A_2 = \{8k+4, 1, 2, 4\}$. Then A is an S-cluster with representations as $r(A_1|S) = (1, 2)$ and $r(A_2|S) = (1, 1)$. Hence by Lemma 13, $|B - S| \geqslant 3$.

The above cases are summarized in the table below. The first column has the different choices for w_2, the second column has the S-cluster generated by $S = \{0, w_2\}$ and the last column gives the Lemma which gives the contradiction.

w_2	$S-cluster$	Lemma
$4m,\ 1 \leqslant m \leqslant k$	$\{8k+4,\ 8k+3,\ 8k+2,\ 8k+1\}$	1
$4m+1,\ 1 \leqslant m \leqslant k$	$\{1,\ 2,\ 3,\ 4\}$	1
$4m+2,\ 1 \leqslant m \leqslant k$	$(\{8k+1,\ 8k+2\}, \{2,\ 3,\ 4\})$	10
$4m+3,\ 1 \leqslant m \leqslant k$	$(\{8k+1,\ 8k+2,\ 8k+3\}, \{3,\ 4\})$	10
1	$\{8k+2,\ 8k+3,\ 8k+4,\ 2,\ 3,\ 4\}$	11
2	$(\{8k+1,\ 8k+2\}, \{8k+3,\ 8k+4,\ 1,\ 3,\ 4\})$	12
3	$(\{8k+1,\ 8k+2,\ 8k+3\}, \{8k+4,\ 1,\ 2,\ 4\})$	13

4.5 $n \equiv 2 \pmod 8$

We write $n = 8k+2$ and prove that this implies $\dim(G) = 5$. Suppose $B = \{w_1, w_2, w_3, w_4\}$ be a resolving set of G with $w_1 < w_2 < w_3 < w_4$. Without loss of generality, $w_1 = 0$ and $w_2 \in \{1, 2, \ldots, 4k\}$. We make a case distinction with respect to the possibilities for the set $S = \{0,\ w_2\}$.

Case 1 $S = \{0,\ 4m\}$, for $1 \leqslant m \leqslant k-1$. Then $A = \{8k-2,\ 8k-1,\ 8k,\ 8k+1\}$ is an S-block with the representation $r(A|S) = (1,\ m+1)$. By Lemma 1, this implies $|B - S| \geqslant 3$.

Case 2 $S = \{0,\ 4m+1\}$, for $1 \leqslant m \leqslant k-1$. Then $A = \{1, 2, 3, 4\}$ is an S-block with representation $r(A|S) = (1,\ m)$. By Lemma 1, this implies $|B - S| \geqslant 3$.

Case 3 $S = \{0,\ 4m+2\}$, for $1 \leqslant m \leqslant k-2$. Then $A = (A_1,\ A_2)$ with $A_1 = \{8k-2, 8k-1\}$ and $A_2 = \{2, 3, 4\}$ is an S-cluster with representations $r(A_1|S) = (1, m+1)$ and $r(A_2|S) = (1, m)$. This implies that $|B - S| \geqslant 3$ by Lemma 10.

Case 4 $S = \{0,\ 4m+3\}$, for $1 \leqslant m < k-2$. Then $A = (A_1,\ A_2)$ with $A_1 = \{8k-2,\ 8k-1,\ 8k\}$ and $A_2 = \{3,\ 4\}$ is an S-cluster with representations $r(A_1|S) = (1,\ m+2)$ and $r(A_2|S) = (1,\ m)$. Hence, by Lemma 10, $|B-S| \geqslant 3$.

Case 5 $S = \{0,\ 1\}$. Then $A = \{8k-1,\ 8k,\ 8k+1,\ 2,\ 3,\ 4\}$ is an S-block with representation $r(A|S) = (1, 1)$. By Lemma 11, $|B - S| \geqslant 3$.

Case 6 $S = \{0, 2\}$. Then $A = (A_1, A_2)$ with $A_1 = \{8k-2, 8k-1\}$ and $A_2 = \{8k, 8k+1, 1, 3, 4\}$ is an S-cluster with representations $r(A_1|S) = (1, 2)$ and $r(A_2|S) = (1, 1)$. By Lemma 12, $|B - S| \geqslant 3$.

Case 7 $S = \{0, 3\}$. Then $A = (A_1, A_2)$ with $A_1 = \{8k-2, 8k-1, 8k\}$ and $A_2 = \{8k+1, 1, 2, 4\}$ is an S-cluster with representations $r(A_1|S) = (1, 2)$ and $r(A_2|S) = (1, 1)$. Hence, by Lemma 13, $|B - S| \geqslant 3$.

Case 8 $S = \{0, 4k\}$. Then $A = \{8k-2, 8k-1, 8k, 1, 2, 3\}$ is an S-cluster with $r(A|S) = (1, k)$. Hence, by Lemma 11, $|B - S| \geqslant 3$.

Case 9 $S = \{0, 4k-1\}$. Then $A = (A_1, A_2)$ with $A_1 = \{4, 3\}$ and $A_2 = \{2, 1, 8k+1, 8k-1, 8k-2\}$ is an S-cluster with representations $r(A_1|S) = (1, k-1)$ and $r(A_2|S) = (1, k)$. Hence, by Lemma 12, $|B - S| \geqslant 3$.

Case 10 $S = \{0, 4k-2\}$. Then $A = (A_1, A_2)$ with $A_1 = \{4, 3, 2\}$ and $A_2 = \{1, 8k+1, 8k, 8k-2\}$ is an S-cluster with representations $r(A_1|S) = (1, k-1)$ and $r(A_2|S) = (1, k)$. Hence, by Lemma 13, $|B - S| \geqslant 3$.

The above cases are summarised in the table below.

w_2	S-cluster	Lemma
$4m,\ 1 \leqslant m \leqslant k-1$	$\{-1, -2, -3, -4\}$	1
$4m+1,\ 1 \leqslant m \leqslant k-1$	$\{1, 2, 3, 4\}$	1
$4m+2,\ 1 \leqslant m \leqslant k-2$	$\{-4, -3\}, \{2, 3, 4\}$	10
$4m+3,\ 1 \leqslant m \leqslant k-2$	$\{-4, -3, -2\}, \{3, 4\}$	10
1	$\{-3, -2, -1, 2, 3, 4\}$	11
2	$\{-4, -3\}, \{-2, -1, 1, 3, 4\}$	12
3	$\{-4, -3, -2\}\{-1, 1, 2, 4\}$	13
$4k$	$\{-4, -3, -2, 1, 2, 3\}$	11
$4k-1$	$\{4, 3\}, \{2, 1, -1, -3, -4\}$	12
$4k-2$	$\{4, 3, 2\}, \{1, -1, -2, -4\}$	13

4.6 $n \equiv 3 \pmod 8$

We write $n = 8k+3$ with $k \geqslant 3$ and prove $\dim(G) \geqslant 5$. Suppose $B = \{w_1, w_2, w_3, w_4\}$ is a resolving set of G with $w_1 < w_2 < w_3 < w_4$. Without loss of generality, $w_1 = 0$ and $w_2, w_3 \in \{1, 2, \ldots, 4k+1\}$. We make a case distinction with respect to the possibilities for the set S.

Case 1 $S = \{0, 4m\}$, $1 \leqslant m \leqslant k-1$. Then $A = \{8k+2, 8k+1, 8k, 8k-1\}$ is an S-block with the representation $r(A|S) = (1, m+1)$. Hence, by Lemma 1, $|B - S| \geqslant 3$.

Case 2 $S = \{0, 4m+1\}$, $1 \leqslant m \leqslant k$. Then $A = \{1, 2, 3, 4\}$ is an S-block with the representation $r(A|S) = (1, m)$. Hence, by Lemma 1, $|B - S| \geqslant 3$.

Case 3 $S = \{0, 4m+2\}$, $1 \leqslant m \leqslant k$. Then $A = \{4k+3, 4k+4, 4k+5, 4k+6\}$ is an S-block with the representation $r(A|S) = (k, k-m+1)$. Hence, by Lemma 1, $|B - S| \geqslant 3$.

Case 4 $S = \{0, 4m+3, w_3\}$, $1 \leqslant m \leqslant k-2$ and $w_3 = 4m' + i$ $i \in \{0,1,2,3\}$ and $m < m'$. Now depending on the choice of w_3, we have the following possibilities as given in the table below. The first column gives the possibilities of w_3, the second column has the set A which is an S-cluster, the third column gives the representation $r(A|S)$, and the last column contains the lemma by which we have $|B - S| \geqslant 2$.

w_3	A (S-cluster)	$r(A\|S)$	Lemma
$4m',\ m < m' \leqslant k$	$\{4k-1, 4k-2, 4k-3\}$	$(k, k-m-1, k-m')$	1
$4m'+1,\ m < m' \leqslant k$	$\{4m+1, 4m+2, 4m+4\}$	$(m+1, 1, m'-m)$	1
$4m'+2,\ m < m' \leqslant k-1$	$\{4k+4, 4k+5, 4k+6\}$	$(k, k-m+1, k-m'+1)$	1
$4m'+3,\ m < m' \leqslant k-1$	$\{4k+4, 4k+5, 4k+6\}$	$(k, k-m+1, k-m'+1)$	1

Case 5 $S = \{0, 1, w_3\}$, where $w_3 \in \{2, 3, \ldots, 4k+1\}$.

w_3	A (S-cluster)	$r(A\|S)$	Lemma
$4m',\ 1 \leqslant m' \leqslant k-1$	$\{8k+2, 8k+1, 8k\}$	$(1, 1, m'+1)$	1
$4m'+1,\ 1 \leqslant m' \leqslant k$	$\{2, 3, 4\}$	$(1, 1, m')$	1
$4m'+2,\ 1 \leqslant m' \leqslant k-1$	$\{4k+4, 4k+5, 4k+6\}$	$(k, k, k-m'+1)$	1
$4m'+3,\ 1 \leqslant m' \leqslant k-1$	$\{4k+4, 4k+5, 4k+6\}$	$(k, k, k-m'+1)$	1
2	$\{8k+1, 8k+2, 3, 4\}$	$(1, 1, 1)$	14
3	$(\{8k, 8k+1\}, \{8k+2, 2, 4\}, \{6, 7\})$	$((1,1,2), (1,1,1), (2,2,1))$	15
$4k$	$(\{4k-2, 4k-1\}, \{4k+5, 4k+6\})$	$((k, k, 1), (k, k, 2))$	16

The assumption $k \geqslant 3$ is used for $S = \{0, 1, 3\}$: for $k = 1$ the set $\{6, 7\}$ is not an S-block because $d(0,6) \neq d(0,7)$.

Case 6 $S = \{0, 2, w_3\}$, where $w_3 \in \{3, 4, \ldots, 4k+1\}$.

w_3	A (S-cluster)	$r(A\|S)$	Lemma
$4m',\ 1 \leqslant m' \leqslant k$	$(\{4k+1, 4k+2\}, \{4k+3, 4k+4\}, \{4k+5, 4k+6\})$	ρ_1	17
$4m'+1,\ 1 \leqslant m' \leqslant k$	$\{1, 3, 4\}$	$(1, 1, m')$	1
$4m'+2,\ 1 \leqslant m' \leqslant k-1$	$\{4k+1, 4k+2\}, \{4k+3, 4k+4\}, \{4k+5, 4k+6\}$	ρ_2	17
$4m'+3,\ 1 \leqslant m' \leqslant k-2$	$\{8k-1, 8k\}, \{3, 4\}$	$((1, 2, m'+2), (1, 1, m'))$	16
3	$(\{8k-1, 8k\}, \{8k+2, 1, 4\})$	$((1, 2, 2), (1, 1, 1))$	18
$4k-1$	$(\{4k-2, 4k-3\}, \{4k-4, 4k-5\}, \{4k-6, 4k-7\})$	ρ_3	17

The representations ρ_1, ρ_2 and ρ_3 in the table are

$$\rho_1 = ((k+1, k, k-m'+1), (k, k+1, k-m'+1), (k, k, k-m'+2)),$$
$$\rho_2 = ((k+1, k, k-m'), (k, k+1, k-m'+1), (k, k, k-m'+1)),$$
$$\rho_3 = ((k, k-1, 1), (k-1, k-1, 1), (k-1, k-2, 2)).$$

The assumption $k \geqslant 3$ is used for $S = \{0, 1, 4k-1\}$.

Case 7 $S = \{0, 3, w_3\}$, where $w_3 \in \{4, 5, \ldots, 4k+1\}$.

w_3	A (S-cluster)	r(A—S)	Lemma
4	$\{5, 6, 7\}$	$(2, 1, 1)$	1
$4m',\ 2 \leqslant m' \leqslant k$	$\{5, 6, 7\}$	$(2, 1, m'-1)$	1
$4m'+1,\ 1 \leqslant m' \leqslant k$	$\{1, 2, 4\}$	$(1, 1, m')$	1
$4m'+2,\ 1 \leqslant m' \leqslant k-1$	$(\{4k+1, 4k+2\}, \{4k+4, 4k+5\})$	$((k+1, k, k-m'), (k, k+1, k-m'+1))$	19
$4m'+3,\ 1 \leqslant m' \leqslant k-1$	$\{-1, 1, 2\}$	$(1, 1, m'+1)$	1

Case 8 $S = \{0, 4k, 4k+1\}$. Then $A = \{4k-1, 4k-2, 4k-3\}$ is an S-block with the representation $r(A|S) = (k, 1, 1)$. Hence, by Lemma 1, $|B - S| \geqslant 2$.

Case 9 $S = \{0,\, 4k-1,\, w_3\}$, where $w_3 \in \{4k,\, 4k+1\}$.

w_3	A (S-cluster)	$r(A\|S)$	Lemma
$4k$	$(\{8k-3,\, 8k-2\},\, \{1,\, 2\})$	$((2,\, k,\, k),\, (1,\, k,\, k))$	16
$4k+1$	$\{4k,\, 4k-2,\, 4k-3\}$	$(k,\, 1,\, 1)$	1

5 Upper bounds

The upper bound for the cases which can not be derived from [10], are proved in this section.

Lemma 20. *Let $G = C(n, \pm\{1,2,3,4\})$ be a circulant graph with $n = 8k+9$. Then $\dim(G) \leqslant 6$.*

Proof. We show that the set $X = \{0, 1, 4, 7, 4k+6, 4k+7\}$ is a metric basis for G. For any two vertices $a, b \in V - X$, we need to show that $d(a,x) \neq d(b,x)$ for some $x \in X$. Writing $a = 4m_1 + r_1$ and $b = 4m_2 + r_2$ with $m_1, m_2 \in \{0, 1, \ldots, 2k+1\}$ and $r_1, r_2 \in \{1, 2, 3, 4\}$, we have the following cases.

Case 1 $m_1, m_2 \leqslant k$. If $m_1 \neq m_2$, then $d(0,a) \neq d(0,b)$, as $d(0,a) = m_1 + 1$ and $d(0,b) = m_2 + 1$. If $m_1 = m_2$, without loss of generality $r_1 < r_2$. The following list describes how a and b are resolved for each of the possible values of a.

$$\begin{aligned}
a = 4m_1 + 1,\ 0 \leqslant m_1 \leqslant k &\implies d(1,a) = m_1,\ d(1,b) = m_1 + 1,\\
a = 4m_1 + 2,\ 1 \leqslant m_1 \leqslant k &\implies d(4k+7,a) = k - m_1 + 2,\\
i \qquad & d(4k+7,b) = k - m_1 + 1,\\
a = 4m_1 + 3,\ 1 \leqslant m_1 \leqslant k &\implies d(7,a) = m_1 - 1,\ d(7,b) = m_1,\\
a = 2 &\implies d(7,a) = 2,\ d(7,b) = 1,\\
a = 3 &\implies d(4,a) = 1,\ d(4,b) = 0.
\end{aligned}$$

Case 2 $m_1, m_2 \geqslant k+1$. If $m_1 \neq m_2$, then $d(0,a) \neq d(0,b)$, as $d(0,a) = (2k+2) - m_1$ and $d(0,b) = (2k+2) - m_2$. If $m_1 = m_2$, without loss of generality $r_1 < r_2$. The following list describes how a and b are resolved for each of the possible values of a.

$$\begin{aligned}
a = 4m_1{+}1, k+2 \leqslant m_1 \leqslant 2k+1 &\implies d(1,a) = 2k{-}m_1{+}3, d(1,b) = 2k - m_1{+}2,\\
a = 4m_1{+}2, k+1 \leqslant m_1 \leqslant 2k+1 &\implies d(4k{+}6,a) = m_1{-}k{-}1, d(4k{+}6,b) = m_1{-}k,\\
a = 4m_1{+}3, k+1 \leqslant m_1 \leqslant 2k+1 &\implies d(4k{+}7,a) = m_1{-}k{-}1, d(4k{+}7,b) = m_1{-}k,\\
a = 4k+5 &\implies d(7,a) = k+1,\ d(7,b) = k.
\end{aligned}$$

Case 3 $m_1 \leqslant k$ and $m_2 \geqslant k+1$. If $m_1 + m_2 \neq 2k+1$, then $d(0,a) \neq d(0,b)$, as $d(0,a) = m_1 + 1$ and $d(0,b) = 2k + 2 - m_2$. If $m_1 + m_2 = 2k+1$, then we have the following possibilities.

$$\begin{aligned}
a = 4m_1 + r_1,\ 1 \leqslant m_1 \leqslant k-1 &\implies d(4,a) = m_1,\ d(4,b) = m_1 + 2,\\
a = 4k + r_1 &\implies d(4,a) = k, d(4,b) = k+1,\\
a = r_1 &\implies d(4,a) \in \{1,0\},\ d(4,b) = 2. \qquad \square
\end{aligned}$$

Lemma 21. *Let* $G = C(n, \pm\{1, 2, 3, 4\})$ *with* $n = 8k + 7$. *Then* $\dim(G) \leqslant 6$.

Proof. We show that $X = \{0, 1, 2, 3, 4, 5\}$ is a metric basis for G. For any two vertices $a, b \in V = \{0, \pm 1, \pm 2, \dots, \pm(4k + 3)\}$, we need to show that $d(a, x) \neq d(b, x)$ for some $x \in X$. Let $a = \pm(4m_1 + r_1)$ and $b = \pm(4m_2 + r_2)$, where $0 \leqslant m_1, m_2 \leqslant k$, $1 \leqslant r_1, r_2 \leqslant 4$ (if $m_i = k$, then $1 \leqslant r_i \leqslant 3$). Note that $d(0, a) = m_1 + 1$ and $d(0, b) = m_2 + 1$, so we may assume $m_1 = m_2 = m$.

Case 1 $a = 4m + r_1$ and $b = 4m + r_2$. Without loss of generality $r_1 < r_2$, hence $r_1 \in \{1, 2, 3\}$, and $d(r_1, a) = m_1$, $d(r_1, b) = m_1 + 1$.

Case 2 $a = -(4m + r_1)$ and $b = -(4m + r_2)$. If $m \leqslant k - 1$, without loss of generality $r_1 < r_2$, hence $r_1 \in \{1, 2, 3\}$, and $d(4 - r_1, a) = m_1$, $d(4 - r_1, b) = m_1 + 1$. If $m_1 = m_2 = k$, without loss of generality $r_1 < r_2$, hence $r_1 \in \{1, 2\}$, and $d(6 - r_1, a) = k + 1$, $d(6 - r_1, b) = k$.

Case 3 $a = 4m + r_1$ and $b = -(4m + r_2)$. If $m = 0$ then $d(4, a) \in \{0, 1\}$, $d(4, b) = 2$. If $1 \leqslant m \leqslant k - 1$ then $d(4, a) = m$, $d(4, b) = m + 2$. If $m = k$ then $d(4, a) = k$, $d(4, b) = k + 1$. □

References

[1] A. Borchert and S. Gosselin, The metric dimension of circulant graphs and Cayley hypergraphs, *Util. Math.* (to appear), available online:
ion.uwinnipeg.ca/ sgosseli/Borchert Gosselin posted.pdf.

[2] G. Chartrand, L. Eroh, M. A. Johnson and O. R. Oellermann, Resolvability in graphs and the metric dimension of a graph, *Discrete Appl. Math.* **105** no. 1-3 (2000), 99–113.

[3] K. Chau and S. Gosselin, The metric dimension of circulant graphs and their cartesian products, *Opuscula Math.* **37** no. 4 (2017), 509–534.

[4] C. Grigorious, P. Manuel, M. Miller, B. Rajan and S. Stephen, On the metric dimension of circulant and Harary graphs, *Appl. Math. Comput.* **248** (2014), 47–54.

[5] F. Harary and R. A. Melter, On the metric dimension of a graph, *Ars Combin.* **2** (1976), 191–195.

[6] M. Imran, A. Q. Baig, S. A. U. Bokhary and I. Javaid, On the metric dimension of circulant graphs, *Appl. Math. Lett.* **25** no. 3 (2012), 320–325.

[7] S. Khuller, B. Raghavachari and A. Rosenfeld, Landmarks in graphs, *Discrete Appl. Math.* **70** no. 3 (1996), 217–229.

[8] R. A. Melter and I. Tomescu, Metric bases in digital geometry, *Computer Vision, Graphics, and Image Proc.* **25**i no. 1 (1984), 113–121.

[9] P. J. Slater, Leaves of trees, *Congr. Numer.* **14** (1975), 549–559.

[10] T. Vetrik, The metric dimension of circulant graphs, *Canad. Math. Bull.* **60** (2017), 206–216.

(Received 19 Apr 2017; revised 18 June 2017)