AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **69(3)** (2017), Pages 402–409

On the beta-number of the joins of graphs

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In memory of Mirka Miller

Abstract

The beta-number of a graph G is the smallest positive integer n for which there exists an injective function $f : V(G) \to \{0, 1, \dots, n\}$ such that each $uv \in E(G)$ is labeled $|f(u) - f(v)|$ and the resulting set of edge labels is $\{c, c+1, \dots, c + |E(G)| - 1\}$ for some positive integer c. The beta-number of G is $+\infty$, otherwise. If $c = 1$, then the resulting beta-number is called the strong beta-number of G. In this paper, we determine formulas for the (strong) beta-numbers of the joins of certain graphs and either the empty graph of order n or the star with $n+1$ vertices. The work of this paper extends the known classes of graceful graphs.

1 Introduction

All graphs considered in this paper are finite and undirected without loops or multiple edges. The *vertex set* of a graph G is denoted by $V(G)$, while the *edge set* is denoted by $E(G)$. The graph with n vertices and no edges is referred to as the *empty graph* of order n. Let G and H be vertex-disjoint graphs. Then the *union* of G and H, denoted by $G \cup H$, is the graph having $V(G \cup H) = V(G) \cup V(H)$ and $E(G \cup H) = E(G) \cup E(H)$. If G and H are vertex-disjoint graphs, then the *join* of

G and H, written $G+H$, is that graph consisting of the union $G \cup H$, together with all edges of the type uv, where $u \in V(G)$ and $v \in V(H)$.

For integers a and b with $a \leq b$, the set $\{x \in \mathbb{Z} : a \leq x \leq b\}$ will be denoted by writing $[a, b]$, where $\mathbb{Z}$ denotes the set of integers.

Among all labelings of graphs, graceful labelings are probably the best known and most studied. Graceful labelings originated with a paper published in 1967 by Rosa [10] who used the term β-valuations. For a graph G of size q, an injective function $f : V(G) \to [0, q]$ is called a *β-valuation* if each $uv \in E(G)$ is labeled $|f(u) - f(v)|$ and the resulting edge labels are distinct. Golomb [4] subsequently called these labelings *graceful* and this is now the popular term. A *graceful graph* is a graph that admits a graceful labeling. Graceful labelings have been the focus of many papers. For recent contributions to this subject and other types of labelings, the authors refer the reader to the survey by Gallian [3].

The authors initiated the study of the beta-number and strong beta-number in [7]. The *beta-number*, denoted by $\beta(G)$, of a graph G with q edges is the smallest positive integer n for which there exists an injective function $f : V(G) \to [0, n]$ such that each $uv \in E(G)$ is labeled $|f(u) - f(v)|$ and the resulting set of edge labels is $[c, c+q-1]$ for some positive integer c. The beta-number of G is $+\infty$, otherwise. If $c = 1$, then the resulting beta-number is called the *strong beta-number* of G and is denoted by $\beta_s(G)$. These parameters can be regarded as measures of how close a graph is to being graceful. It is an immediate consequence of the definitions of two parameters that if G is a graceful graph, then $\beta(G) = \beta_s(G)$.

The following lemma found in [7] indicates how the two parameters discussed above are related.

Lemma 1 *For every graph G of order p and size q,*

$$\max\{p-1, q\} \leq \beta(G) \leq \beta_s(G).$$

Let G be a graph of order p and size q. It is clear that if $\beta_s(G) = p-1$, then $q \leq p-1$. It is also true that if $\beta_s(G) = q$, then G is graceful, which implies that $q \geq p-1$. From these observations, we have the following two immediate consequences of Lemma 1, which concern with the graphs that are sparse and dense, respectively.

Lemma 2 *If G is a graph of order p and size q with $\beta_s(G) = p-1$, then $q \leq p-1$ and $\beta(G) = p-1$.*

Lemma 3 *If G is a graph of order p and size q with $\beta_s(G) = q$, then $q \geq p-1$ and $\beta(G) = q$.*

Consider a graph G of order p and size q with $q = p-1$. If $\beta(G) = p-1$, then there exists an injective function $f : V(G) \to [0, p-1]$ such that each $uv \in E(G)$ is labeled $|f(u) - f(v)|$ and the resulting set of edge labels is $[c, c+q-1]$ for some positive integer c. It follows that

$$\begin{aligned} c+q-1 &\leq \max\{|f(u) - f(v)| : uv \in E(G)\} \\ &= p-1 = q. \end{aligned}$$

This together with $c \geq 1$ implies that $c = 1$ so that $\beta_s(G) = p - 1$. It is also immediate from Lemma 1 that if $\beta_s(G) = p-1$, then $\beta(G) = p-1$. Therefore, we have the following result.

Lemma 4 *Let G be a graph of order p and size q with $q = p-1$. Then $\beta(G) = p-1$ if and only if $\beta_s(G) = p-1$.*

In this paper, we provide lower and upper bounds for $\beta(G+H)$ if G is a graph satisfying the condition $\beta(G) = |V(G)|-1$, and H is isomorphic to the empty graph of order n. This leads us to formulas for $\beta(G+H)$ and $\beta_s(G+H)$ if G is a graph with the property that $\beta_s(G) = |V(G)|-1$, and H is isomorphic to the empty graph of order n. We also determine formulas for $\beta(G+H)$ and $\beta_s(G+H)$ if G is a graph with the same property as the previous one, and H is isomorphic to the star with $n+1$ vertices. As corollaries of these results, we obtain some classes of graceful graphs. Thus, the work of this paper extends the known classes of graceful graphs.

There are other graph labeling parameters that measure how close a graph is to being graceful. For further knowledge on the (strong) beta-number of graphs and related concepts, the authors suggest that the reader consults the results in [2, 4, 6, 8, 9, 11].

2 Results on (Strong) Beta-Numbers

We begin this section with the following result, which supplies lower and upper bounds for the beta-number of the joins of a graph G with $\beta(G) = |V(G)| - 1$ and the empty graph of order n.

Theorem 1 *If G is a graph of order p and size q with $\beta(G) = p-1$, then there exists some positive integer c such that*

$$q + np \leq \beta(G + nK_1) \leq c + q + np - 1$$

for every positive integer n.

Proof: Let G be a graph that satisfies our hypothesis. Then there exists an injective function $f : V(G) \to [0, p-1]$ such that each $uv \in E(G)$ is labeled $|f(u) - f(v)|$ and the resulting set of edge labels is $[c, c+q-1]$ for some positive integer c. Let n be a positive integer, and define the graph $H \cong G + nK_1$ with

$$V(H) = V(G) \cup \{w_i : i \in [1, n]\}$$

and

$$E(H) = E(G) \cup \{vw_i : v \in V(G) \text{ and } i \in [1, n]\}.$$

Now, consider the function $g : V(H) \to [0, c+q+np-1]$ such that

$$g(v) = \begin{cases} f(v) & \text{if } v \in V(G), \\ c + q + ip - 1 & \text{if } v = w_i \text{ and } i \in [1, n]. \end{cases}$$

Notice that $\{g(v) : v \in V(G)\} = [0, p-1]$ and

$$\{g(w_i) : i \in [1, n]\} = \{c + q + ip - 1 : i \in [1, n]\}.$$

This shows that g is an injective function. Notice also that

$$\{|g(u) - g(v)| : uv \in E(G)\} = [c, c + q - 1],$$

and

$$\{|g(v) - g(w_i)| : v \in V(G) \text{ and } i \in [1, n]\} = [c + q, c + q + np - 1],$$

which is a set of np consecutive integers. Consequently,

$$\{|g(u) - g(v)| : uv \in E(H)\} = [c, c + |E(H)| - 1],$$

which implies that $\beta(H) \leq c + q + np - 1$. Finally, notice that H is connected; so $|E(H)| \geq |V(H)| - 1$. It is now immediate from Lemma 1 that $\beta(H) \geq |E(H)| = q + np$. □

For a graph G with $\beta_s(G) = |V(G)| - 1$, we have the following result, which shows that the lower bound given in Theorem 1 is sharp.

Theorem 2 *If G is a graph of order p and size q with $\beta_s(G) = p - 1$, then*

$$\beta(G + nK_1) = \beta_s(G + nK_1) = q + np$$

for every positive integer n.

Proof: Let n be a positive integer, and assume that G is a graph of order p and size q with $\beta_s(G) = p - 1$. In light of Lemma 3, it suffices to establish that $\beta_s(G + nK_1) = q + np$. Since G satisfies the condition $\beta_s(G) = p - 1$, it follows from Lemma 2 that $\beta(G) = p - 1$. Thus, by Theorem 1 and Lemma 1, we obtain

$$\beta_s(G + nK_1) \geq \beta(G + nK_1) \geq q + np.$$

On the other hand, if $c = 1$, then $\beta(G) = \beta_s(G)$ by the definitions of the two parameters. This together with our assumption implies that $\beta(G) = p - 1$. Thus, applying Theorem 2 with $c = 1$, we have

$$\beta_s(G + nK_1) = \beta(G + nK_1) \leq q + np.$$

□

The preceding result is of particular interest, since there are infinitely many graphs G for which $\beta_s(G) = |V(G)| - 1$ (see Table 1 which summarizes what has been known about such graphs). In this table, the star with $n + 1$ vertices and the path with n vertices are denoted by S_n and P_n, respectively.

Table 1: Summary of strong beta-numbers of graphs

G	$\beta_s(G)$	notes
$S_m \cup S_n$	$m+n+1$	if mn is even [7]
$P_m \cup S_n$	$m+n$	if $m=2$ and n is even, or $m \geq 3$ and $n \geq 1$ [7]
mP_2	$2m-1$	if $m \equiv 0$ or $1 \pmod 4$ [5]
$4S_n$	$4n+3$	if $n \geq 1$ [5]
$P_m \cup P_n$	$m+n-1$	if $2 \leq m \leq n$ and $(m,n) \neq (2,2)$ [8]
$S_l \cup S_m \cup S_n$	$l+m+n+2$	if lmn is even [6]
$S_k \cup S_l \cup S_m \cup S_n$	$k+l+m+n+3$	for all positive integers k, l, m and n [6]

The converse of Theorem 2 is not true. To see this, consider the result found by Acharya [1] that if G is a connected graph with a graceful labeling, then $G+nK_1$ is graceful for every positive integer n. It is clear that $K_{1,1}$ is a connected graph with a graceful labeling. Thus, applying the mentioned result with $G \cong K_{1,1}$ successively, we conclude that the graphs

$$K_{1,1,m} \cong K_{1,1} + mK_1 \text{ and } K_{1,1,m,n} \cong K_{1,1,m} + nK_1$$

are graceful for all positive integers m and n. Consequently,

$$\beta(K_{1,1,m,n}) = \beta_s(K_{1,1,m,n}) = |E(K_{1,1,m})| + n\,|V(K_{1,1,m})|\,.$$

However, it follows from Lemma 2 that $\beta_s(K_{1,1,m}) \neq |V(K_{1,1,m})| - 1$, since

$$|E(K_{1,1,m})| = 2m+1 > m+1 = |V(K_{1,1,m})| - 1.$$

If T is a graceful tree of order p, then T has size $p-1$ and satisfies the hypothesis of Theorem 2. Therefore, we have the following result.

Corollary 1 *If T is a graceful tree of order p, then*

$$\beta(T+nK_1) = \beta_s(T+nK_1) = (n+1)\,p-1$$

for every positive integer n.

The above result is relatively important, since various classes of trees have been proved to admit graceful labelings (see [3] for a detailed list of results).

For a graph G with $|E(G)| = |V(G)| - 1$, we have another result on the (strong) beta-numbers involving the join of graphs.

Theorem 3 *Let G be a graph of order p and size q with $q = p-1$. If $\beta_s(G) = p-1$, then*

$$\beta(G + S_n) = \beta_s(G + S_n) = (n+2)p + n - 1$$

for every positive integer n.

PROOF: By assumption, there exists an injective function $f : V(G) \to [0, p-1]$ such that each $uv \in E(G)$ is labeled $|f(u) - f(v)|$ and the resulting set of edge labels is $[1, p-1]$. Let n be a positive integer, and define the graph $H \cong G + S_n$ with

$$V(H) = V(G) \cup \{x\} \cup \{y_i : i \in [1, n]\}$$

and

$$\begin{aligned} E(H) &= E(G) \cup \{xy_i : i \in [1, n]\} \\ &\quad \cup \{vx : v \in V(G)\} \cup \{vy_i : v \in V(G) \text{ and } i \in [1, n]\}. \end{aligned}$$

Then $|V(H)| = p + n + 1$ and $|E(H)| = (n+2)p + n - 1$. In light of Lemma 3, it suffices to prove the theorem for $\beta_s(H)$. The lower bound follows from Lemma 1, since H is connected, that is, $|E(H)| \geq |V(H)| - 1$.

To show the upper bound, consider the function $g : V(H) \to [0, (n+2)p + n - 1]$ such that

$$g(v) = \begin{cases} f(v) & \text{if } v \in V(G), \\ (n+2)p + n - 1 & \text{if } v = x, \\ (i+1)p + i - 1 & \text{if } v = y_i \text{ and } i \in [1, n]. \end{cases}$$

This leads us to conclude that $\beta_s(G + S_n) \leq (n+2)p + n - 1$. To see this, notice that

$$\begin{aligned} \{g(v) : v \in V(G)\} &= [0, p-1], \\ \{g(y_i) : i \in [1, n]\} &= \{(i+1)p + i - 1 : i \in [1, n]\}, \end{aligned}$$

and $g(x) = (n+2)p + n - 1$. This verifies that g is an injective function. Notice also that

$$\begin{aligned} \{|g(u) - g(v)| : uv \in E(G)\} &= [1, p-1], \\ \{|g(x) - g(y_i)| : i \in [1, n]\} &= \{i(p+1) - 1 : i \in [1, n]\}, \\ \{|g(v) - g(y_i)| : v \in V(G) \text{ and } i \in [1, n]\} &= \bigcup_{i=1}^{n} [i(p+1), i(p+1) + p - 1)], \\ \{|g(x) - g(v)| : v \in V(G)\} &= [n(p+1) + p, n(p+1) + 2p - 1]. \end{aligned}$$

Consequently,

$$\{|g(u) - g(v)| : uv \in E(H)\} = [1, |E(H)|],$$

which implies that $\beta_s(H) \leq (n+2)p + n - 1$. □

In light of Lemma 4, the preceding result has an immediate consequence.

Corollary 2 *Let G be a graph of order p and size q with $q = p - 1$. If $\beta(G) = p - 1$, then*

$$\beta(G + S_n) = \beta_s(G + S_n) = (n + 2)p + n - 1$$

for every positive integer n.

Recall that if G is a graph with $\beta_s(G) = |E(G)|$, then G is graceful. Combining this with Theorem 2, we have the following result.

Corollary 3 *If G is a graph of order p with $\beta_s(G) = p - 1$, then $G + nK_1$ is graceful for every positive integer n.*

Restating Corollary 1 and Theorem 3 for graceful graphs, we have the following two results.

Corollary 4 *If T is a graceful tree, then $T + nK_1$ is graceful for every positive integer n.*

Corollary 5 *If G is a graceful graph of order p and size q with $q = p - 1$, then $G + S_n$ is graceful for every positive integer n.*

From the above result, we particularly have the following corollary.

Corollary 6 *If T is a graceful tree, then $T + S_n$ is graceful for every positive integer n.*

It is interesting to note that the above four corollaries considerably extend the known classes of graceful graphs. It is also worth to mention that the truth of the logically equivalent contrapositive of either Corollary 4 or Corollary 6 implies the falsehood of the well-known conjecture by Kotzig (see Rosa [10]) that every nontrivial tree is graceful. This may provide a viable approach to show that not all trees are graceful.

3 Conclusions

In this paper, we present some results on $\beta(G + H)$ and $\beta_s(G + H)$ when G is a graph such that $\beta_s(G) = |V(G)| - 1 \geq |E(G)|$, and H is isomorphic to either nK_1 or S_n. In light of Lemma 1, it seems natural to explore bounds and formulas for $\beta(G + H)$ and $\beta_s(G + H)$ when G is a graph such that $\beta_s(G) = |E(G)| > |V(G)| - 1$, and H is some class of graphs. Thus, we propose the following two problems.

Problem 1 *For some classes of graphs H, find bounds for $\beta(G + H)$ and $\beta_s(G + H)$ when G is a graph such that $\beta_s(G) = |E(G)| > |V(G)| - 1$.*

Problem 2 *For some classes of graphs H, find formulas for $\beta(G+H)$ and $\beta_s(G+H)$ when G is a graph such that $\beta_s(G) = |E(G)| > |V(G)| - 1$.*

Acknowledgments

The authors are very grateful to the anonymous referees for their careful reading, and for the many comments and suggestions that improved the readability of this paper.

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(Received 15 Nov 2016; revised 29 May 2017)