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Broadcasts in graphs: diametrical trees*

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Abstract

A dominating broadcast on a graph $G = (V, E)$ is a function $f : V \to \{0, 1, \dots, \mathrm{diam}(G)\}$ such that $f(v) \leq e(v)$ (the eccentricity of v) for all $v \in V$, and each $u \in V$ is at distance at most $f(v)$ from a vertex v with $f(v) \geq 1$. The upper broadcast domination number of G is $\Gamma_b(G) = \max\{\sum_{v \in V} f(v) : f$ is a minimal dominating broadcast on $G\}$. As shown by Erwin in [D. Erwin, Cost domination in graphs, Doctoral dissertation, Western Michigan University, 2001], $\Gamma_b(G) \geq \mathrm{diam}(G)$ for any graph G. We investigate trees whose upper broadcast domination number equals their diameter and, among more general results, characterize caterpillars with this property.

1 Introduction

Suppose a telecommunications company has to provide radio coverage to a collection of geographic regions. A single tower transmitting with a strength (or cost) of one unit can provide coverage to the region it is located in and all regions immediately adjacent to it. The company aims to minimize its expenses by erecting as few towers as possible. If we consider each region as a vertex of a graph G, where two vertices are adjacent if their corresponding geographic regions are adjacent, then any *dominating set* S (i.e. each vertex of G belongs to S or is adjacent to a vertex in S) represents a suitable arrangement of radio towers, and a dominating set of minimum cardinality represents a minimum cost arrangement. However, if the company is able to build its towers with varying signal strength so that a tower may transmit its signal a greater distance, but at a proportionally greater cost, the total cost could be significantly less than for the former arrangement. This situation can be modelled with a *broadcast* on G, as defined below.

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Unless stated otherwise, all graphs considered here are assumed to be simple, nontrivial and connected. For undefined graph theoretic concepts and terminology we refer the reader to [7] and [13].

A *caterpillar* is a tree of order at least three, the removal of whose leaves produces a path. We use standard notation for functions and write $f : A \to B$ to denote the fact that f is a function from A to B; we also write $f = \{(a, f(a)) : a \in A\}$. If f and g are functions with the same domain A such that $g(a) \leq f(a)$ for each $a \in A$, we write $g \leq f$. If in addition $g(a) < f(a)$ for at least one $a \in A$, we write $g < f$.

As usual we denote the domination and upper domination numbers of a graph G by $\gamma(G)$ and $\Gamma(G)$, respectively. A *broadcast* on a graph $G = (V, E)$ is a function $f : V \to \{0, 1, \ldots, \operatorname{diam}(G)\}$ such that $f(v) \leq e(v)$ (the eccentricity of v) for all $v \in V$. A broadcast f on G is *dominating* if each $u \in V$ is at distance at most $f(v)$ from a vertex v with $f(v) \geq 1$, and *minimal dominating* if no broadcast f' on G with $f' < f$ is dominating. The *cost* of a broadcast f is $\sigma(f) = \sum_{v \in V} f(v)$. The *broadcast domination number* of G is

$$\gamma_b(G) = \min\{\sigma(f) : f \text{ is a dominating broadcast on } G\},$$

and the *upper broadcast domination number* of G is

$$\Gamma_b(G) = \max\{\sigma(f) : f \text{ is a minimal dominating broadcast on } G\}.$$

Broadcast domination was introduced by Erwin [11, 12], who proved the bounds

$$\gamma_b(G) \leq \min\{\gamma(G), \operatorname{rad}(G)\} \leq \max\{\Gamma(G), \operatorname{diam}(G)\} \leq \Gamma_b(G) \tag{1}$$

for any graph G. Graphs for which $\gamma_b(G) = \operatorname{rad}(G)$ are called *radial graphs*. Radial trees are characterized in [16, 17]. The upper broadcast domination number $\Gamma_b(G)$ is also studied in [1, 2, 10, 21]. Other studies of broadcast domination can be found in [3, 4, 5, 6, 8, 9, 14, 15, 18, 19, 20, 22, 23, 24, 25].

Our purpose is to investigate trees whose upper broadcast domination number equals their diameter. Following the terminology for broadcast domination numbers, we call such trees *diametrical trees*. The characterization of diametrical trees is listed as an open problem in [21].

After presenting further definitions and known results in Section 2, we state a number of lemmas concerning properties of non-diametrical trees in Section 3. To avoid interrupting the flow of the proof of our main theorem, we defer the proofs of all lemmas to Section 5. A consequence of these lemmas is that a tree containing a path of length at least three, internally disjoint from a diametrical path, is non-diametrical. This result hints that the caterpillars may contain classes of diametrical trees, which is indeed the case. Our goal is to prove the characterization of diametrical caterpillars stated in Theorem 1.1 below, which we do in Section 4. We conclude with open problems in Section 6.

Theorem 1.1 *A caterpillar T with diametrical path $P : v_0, v_1, \ldots, v_d$ is diametrical if and only if* (*i*) *each* v_i, $i \in \{1, \ldots, d-1\}$, *is adjacent to at most two leaves,*

(*ii*) *for any* $i \in \{1, \ldots, d-2\}$, $\min\{\deg_T(v_i), \deg_T(v_{i+1})\} = 2$,

(*iii*) *whenever* v_i *and* v_j, $i < j$, *are adjacent to at least two leaves each, there exists an index* k, $i < k < j$, *such that* $\deg_T(v_k) = \deg_T(v_{k+1}) = 2$.

2 Definitions and Known Results

For a broadcast f on a graph $G = (V, E)$, define $V_f^+ = \{v \in V : f(v) > 0\}$. The vertices in V_f^+ are called *broadcast vertices*. A vertex u *hears* the broadcast f from some vertex $v \in V_f^+$, and v f*-dominates* u, if the distance $d(u, v) \leq f(v)$. An edge uw *hears* f if both u and w hear f from the same vertex $v \in V_f^+$. A vertex $v \in V_f^+$ *overdominates* a vertex u if $d(u, v) < f(v)$. For $v \in V_f^+$, define the

- f*-neighbourhood* of v as $N_f[v] = \{u \in V(G) : d(u, v) \leq f(v)\}$,
- f*-boundary* of v as $B_f(v) = \{u \in V(G) : d(u, v) = f(v)\}$,
- f*-private neighbourhood* of v as $\mathrm{PN}_f(v) = \{u \in N_f[v] : u \notin N_f[w]$ for all $w \in V^+ - \{v\}\}$,
- f*-private boundary* of v as $\mathrm{PB}_f(v) = \{u \in N_f[v] : u$ is not dominated by $(f - \{(v, f(v))\}) \cup \{(v, f(v) - 1)\}\}$.

Note that if $f(v) = 1$, then $\mathrm{PB}_f(v) = \mathrm{PN}_f(v)$, and if $f(v) \geq 2$, then $\mathrm{PB}_f(v) = B_f(v) \cap \mathrm{PN}_f(v)$. For example, consider the tree T in Figure 1. The broadcast f defined by $f(u) = 4$, $f(v) = 2$, $f(w) = 3$, $f(z) = 1$ and $f(x) = 0$ otherwise is a dominating broadcast such that $\mathrm{PB}_f(x) = \{x'\}$ for each $x \in \{u, w, z\}$, and $\mathrm{PB}_f(v) = \varnothing$.

The property that makes a dominating broadcast minimal dominating, determined in [11] and stated in [21] in terms of private boundaries, is essential in the study of upper broadcast numbers. We state it again here.

Proposition 2.1 [11] *A dominating broadcast* f *is a minimal dominating broadcast if and only if* $\mathrm{PB}_f(v) \neq \varnothing$ *for each* $v \in V_f^+$.

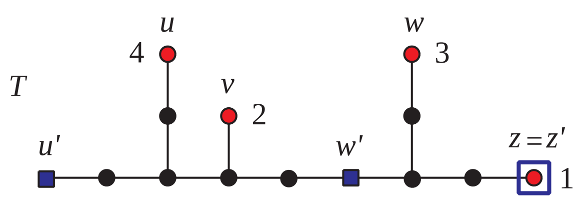

Figure 1: A tree T with a dominating broadcast f such that $\mathrm{PB}_f(x) = \{x'\}$ for each $x \in \{u, w, z\}$, and $\mathrm{PB}_f(v) = \varnothing$.

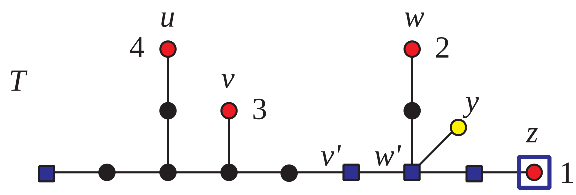

Figure 2: A tree T with a broadcast f such that $\mathrm{PB}_f(x) \neq \varnothing$ for each $x \in V_f^+$, but f cannot be extended to a minimal dominating broadcast.

By Proposition 2.1 the broadcast f in Figure 1, although dominating, is not minimal dominating. The broadcast $f' = (f - \{(v,2)\}) \cup \{(v,0)\}$ is a minimal dominating broadcast on T. In general it is not true that if f is a dominating broadcast on a graph G, then some broadcast f' with $f' \leq f$ is a minimal dominating broadcast on G, nor is it necessarily true that if f is a broadcast on G such that $\mathrm{PB}_f(v) \neq \varnothing$ for each $v \in V_f^+$, then some broadcast f' with $f \leq f'$ is a minimal dominating broadcast on G. Consider the tree T and broadcast f shown in Figure 2. Here, $\mathrm{PB}_f(x) = \{x'\}$ for each $x \in \{v, w\}$ and y is not f-dominated. Moreover, f cannot be extended to a broadcast that dominates y without leaving v or w with an empty private boundary.

It is well known that any independent set of vertices in a graph G can be extended to a maximal (but not necessarily maximum) independent set of G, and that a maximal independent set is also a minimal dominating set (cf. [13, pp. 70 – 71]). Denoting the cardinality of a maximum independent set of G by $\alpha(G)$, it follows that $\alpha(G) \leq \Gamma(G)$ for all graphs G.

Remark 2.2 [11] *The characteristic function of a minimal dominating set in a graph G is a minimal dominating broadcast on G. Hence $\Gamma_b(G) \geq \Gamma(G) \geq \alpha(G)$ for any graph G.*

Proposition 2.3 [11] *If f is a broadcast on a graph G and for each $i \in \{1, 2\}$ we have $u_i \in V_f^+$, $u_i' \in \mathrm{PB}_f(u_i)$, where $u_1 \neq u_2$, and P_i is a u_i – u_i' geodesic, then P_1 and P_2 are disjoint.*

Using Proposition 2.3, Erwin [11] shows that $\Gamma_b(G) \leq |E(G)|$ for any graph G, and together with the lower bound (1) this implies that $\Gamma_b(P_n) = n - 1$ for each $n \geq 2$. Proposition 2.3 is used frequently in the proofs in Section 5.

3 Non-Diametrical Trees

In this section we state a number of sufficient conditions for a tree T to be non-diametrical. The proofs are given in Section 5. We assume throughout that T has diameter d and a diametrical path $P : v_0, v_1, \ldots, v_d$. For each $i \in \{0, \ldots, d\}$, let T_i be the subtree of T induced by all vertices that are connected to v_i by paths

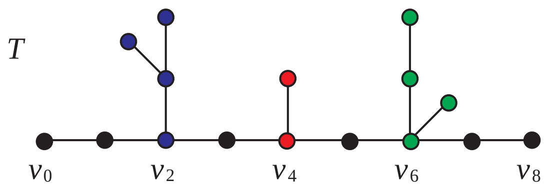

Figure 3: A tree T and diametrical path $P : v_0, v_1, ..., v_8$ such that $T_2 \cong K_{1,3},\ T_4 \cong K_2,\ T_6 \cong P_4$ and $T_i = K_1$ otherwise.

that are internally disjoint from P. Note that $T_i = K_1$ if and only if $i \in \{0, d\}$, or $i \in \{1, \ldots, d-1\}$ and $\deg(v_i) = 2$. For example, in the tree T in Figure 3, $T_2 \cong K_{1,3},\ T_4 \cong K_2,\ T_6 \cong P_4$ and $T_i = K_1$ for $i \in \{0, 1, 3, 5, 7, 8\}$.

A *stem* of a tree $T \ncong K_2$ is a vertex adjacent to a leaf and a *strong stem* is a stem that is adjacent to at least two leaves; in Figure 3, v_1, v_4 and v_6 are (not the only) examples of stems. The complete bipartite graph $K_{1,t},\ t \geq 1$, is also called a *star.* Thus a tree T with diametrical path P as above is a caterpillar if each T_i is either a star with v_i as centre, or K_1.

Lemma 3.1 *Let T be a tree with diameter $d \geq 3$ and diametrical path $P : v_0, v_1, \ldots, v_d$. If there exists an $i \in \{1, \ldots, d-2\}$ such that each of v_i and v_{i+1} is adjacent to a leaf other than v_0 (if $i = 1$) or v_d (if $i + 1 = d - 1$), then $\Gamma_b(T) > \operatorname{diam}(T)$.*

Lemma 3.2 *If there exists a subscript $i \in \{2, \ldots, d-2\}$ such that T_i has an independent set of cardinality 3 that dominates but does not contain v_i, or if $\max\{\deg(v_1), \deg(v_{d-1})\} \geq 4$, then $\Gamma_b(T) > \operatorname{diam}(T)$.*

Lemma 3.3 *If there exists a subscript $i \in \{2, \ldots, d-2\}$ such that T_i has an independent set of cardinality 2 that does not dominate v_i, then $\Gamma_b(T) > \operatorname{diam}(T)$.*

Lemma 3.4 *If $\operatorname{diam}(T_i) \geq 4$ for some i, or if $\operatorname{diam}(T_i) = 3$ and v_i is a peripheral vertex of T_i, then $\Gamma_b(T) > \operatorname{diam}(T)$.*

By Lemmas 3.2 – 3.4, if T is a diametrical tree, then each T_i is isomorphic to either K_1, K_2, P_3 with v_i either a leaf or the stem of P_3, or P_4 with v_i being a stem of P_4. Thus, diametrical trees are "nearly" caterpillars. We henceforth restrict our investigation to caterpillars. By Lemma 3.1, if $T_i \cong K_2$, we may assume that neither T_{i-1} nor T_{i+1} is isomorphic to K_2. If $T_i \cong P_3$ with v_i being a leaf of P_3, or if $T_i \cong P_4$, then T is not a caterpillar and we ignore these cases. We give one more sufficient condition for a caterpillar to be non-diametrical.

Lemma 3.5 *Let T be a caterpillar with diametrical path $P : v_0, v_1, \ldots, v_d$. If two vertices v_i, v_{i+2k} are strong stems, for some $i \geq 1$ and some integer k such that $i + 2k \leq d - 1$, and v_{i+2r} is a stem for each $r \in \{1, \ldots, k-1\}$, then $\Gamma_b(T) > d$.*

4 Diametrical Caterpillars

If T is a diametrical caterpillar, then T does not satisfy the hypothesis of any of Lemmas 3.1 – 3.5. In this section we show that the converse is also true: If the caterpillar T does not satisfy the hypothesis of any of Lemmas 3.1 – 3.5, then T is diametrical. The negation of these hypotheses, applied to caterpillars, gives the characterization of diametrical caterpillars stated in Theorem 1.1, which we restate here for convenience.

Theorem 1.1 *A caterpillar T with diametrical path $P : v_0, v_1, \ldots, v_d$ is diametrical if and only if*

(*i*) *each v_i, $i \in \{1, \ldots, d-1\}$, is adjacent to at most two leaves,*

(*ii*) *for any $i \in \{1, \ldots, d-2\}$, $\min\{\deg_T(v_i), \deg_T(v_{i+1})\} = 2$,*

(*iii*) *whenever v_i and v_j, $i < j$, are strong stems, there exists an index k, $i < k < j$, such that $\deg_T(v_k) = \deg_T(v_{k+1}) = 2$.*

Proof. Suppose T is a diametrical caterpillar. By Lemma 3.2, each v_i, $i \in \{2, \ldots, d-2\}$, is adjacent to at most two leaves, while v_1 and v_{d-1} are adjacent to at most one leaf other than v_0 and v_d, respectively, hence (*i*) holds. Similarly, condition (*ii*) follows directly from Lemma 3.1. For (*iii*), condition (*ii*) already implies that of any two consecutive internal vertices of P, at least one has degree 2. Lemma 3.5 now implies that if v_i and v_j are both strong stems, then some pair of consecutive strong stems between v_i and v_j (inclusive) are separated by at least two vertices of degree 2. Hence (*iii*) holds.

For the converse, note that the only caterpillars of diameter three or less that satisfy conditions (*i*) – (*iii*) are P_3, P_4 and the tree obtained by joining a new leaf to a stem of P_4. It is easy to verify that they are diametric. Assume that Theorem 1.1 is false and let T be a smallest non-diametrical caterpillar that satisfies (*i*) – (*iii*). Then T has diameter at least four. We state two more lemmas, the proofs of which are also given in Section 5.

Lemma 4.1 *No vertex of T is a strong stem.*

Lemma 4.2 *No vertex v_i, $i \in \{2, \ldots, d-2\}$, is adjacent to a leaf.*

By Lemmas 4.1 and 4.2, $T = P_{d+1}$, which is impossible because Erwin [11] showed that $\Gamma_b(P_n) = n - 1 = \mathrm{diam}(P_n)$ for all $n \geq 2$. ■

5 Proofs of Lemmas

This section contains the proofs of Lemmas 3.1 – 4.2, restated here for convenience.

Lemma 3.1 *Let T be a tree with diameter $d \geq 3$ and diametrical path $P : v_0, v_1, \dots, v_d$. If some v_i and v_{i+1}, $i \in \{1, \dots, d-2\}$, are adjacent to leaves other than v_0 or v_d, then $\Gamma_b(T) > \operatorname{diam}(T)$.*

Proof. Suppose the hypothesis of the lemma is satisfied. Say v_i is adjacent to the leaf ℓ and v_{i+1} is adjacent to the leaf ℓ'. Define the broadcast g by $g(v_0) = i+1$, $g(v_d) = d - i$ and $g(x) = 0$ otherwise. Then $\ell \in \mathrm{PB}_g(v_0)$ and $\ell' \in \mathrm{PB}_g(v_d)$, hence $\mathrm{PB}_g(x) \neq \varnothing$ for all $x \in V_g^+$. If g is also dominating, let $f = g$; otherwise, let T' be the subgraph of T induced by all vertices that are not g-dominated, let S be a maximal independent set of T' and define the broadcast f by $f(x) = g(x)$ if $x \in V(T) - V(T')$, $f(x) = 1$ if $x \in S$ and $f(x) = 0$ if $x \in V(T') - S$. By definition, f is a dominating broadcast on T. Since ℓ and ℓ' are leaves, no vertex in S is adjacent to ℓ or ℓ', hence $\ell \in \mathrm{PB}_f(v_0)$ and $\ell' \in \mathrm{PB}_f(v_d)$. Since no vertex in S hears the broadcast g, $x \in \mathrm{PB}_f(x)$ for each $x \in S$. Hence, by Proposition 2.1, f is a minimal dominating broadcast. Moreover, $\sigma(f) \geq i + 1 + d - i = d + 1$ and the result follows. ■

The proof of the next lemma is illustrated in Figure 4.

Lemma 3.2 *If there exists a subscript $i \in \{2, \dots, d-2\}$ such that T_i has an independent set of cardinality 3 that dominates T_i but does not contain v_i, or if $\max\{\alpha(T_1), \alpha(T_{d-1})\} \geq 2$, then $\Gamma_b(T) > \operatorname{diam}(T)$.*

Proof. We may assume that T does not satisfy the hypothesis of Lemma 3.1, otherwise we are done. Suppose $\alpha(T_1) = t \geq 2$. See Figure 4(a). Since v_0 is a peripheral vertex of T, no vertex of T_1 is at distance greater than one from v_1. Hence $T_1 = K_{1,t}$ and, by Lemma 3.1, v_2 is not adjacent to a leaf. Let S be the set consisting of v_0 and the t leaves of T_1, and define the broadcast g by $g(v_d) = d - 2$, $d(x) = 1$ if $x \in S$ and $g(x) = 0$ otherwise. Then $v_2 \in \mathrm{PB}_g(v_d)$ and $x \in \mathrm{PB}_g(x)$ for each $x \in S$, hence $\mathrm{PB}_g(x) \neq \varnothing$ for all $x \in V_g^+$. If g is dominating, let $f = g$, otherwise let T' be the subgraph of T induced by all vertices that are not dominated by g. Since v_2 is not adjacent to a leaf, there exists a maximal independent set X of T' that does not contain a vertex adjacent to v_2. Define the broadcast f by $f(x) = g(x)$ if $x \in V(T) - V(T')$, $f(x) = 1$ if $x \in X$ and $f(x) = 0$ if $x \in V(T') - X$. Then $v_2 \in \mathrm{PB}_f(v_d)$ and $x \in \mathrm{PB}_f(x)$ for each $x \in S \cup X$, so f is a minimal dominating broadcast on T with $\sigma(f) \geq t + 1 + d - 2 > d$. Hence $\Gamma_b(T) > d$.

If $\alpha(T_{d-1}) \geq 2$ the result follows similarly. Hence assume some T_i, $i \in \{2, \dots, d-2\}$, has an independent set of cardinality 3 that dominates T_i but does not contain v_i. Then T_i has a maximal independent set S of cardinality $c \geq 3$ such that $v_i \notin S$. Define the broadcast g by $g(v_0) = i - 1$, $g(v_d) = d - i - 1$, $g(x) = 1$ if $x \in S$ and $g(x) = 0$ otherwise. Since $v_i \notin S$, $v_{i-1} \in \mathrm{PB}_g(v_0)$ and $v_{i+1} \in \mathrm{PB}_g(v_d)$. In addition, $x \in \mathrm{PB}_g(x)$ for each $x \in S$. If $i \geq 3$ and v_{i-1} is adjacent to a leaf, then we may assume, by Lemma 3.1, that v_{i-2} is not adjacent to a leaf (other than v_0 if $i = 3$). Similarly, if $i \leq d - 3$ and v_{i+1} is adjacent to a leaf, we may assume that v_{i+2} is not adjacent to a leaf (other than v_d if $i = d - 3$). Let T' be the subgraph of T induced by the vertices that are not dominated by g and choose a maximal independent set X of T' as follows.

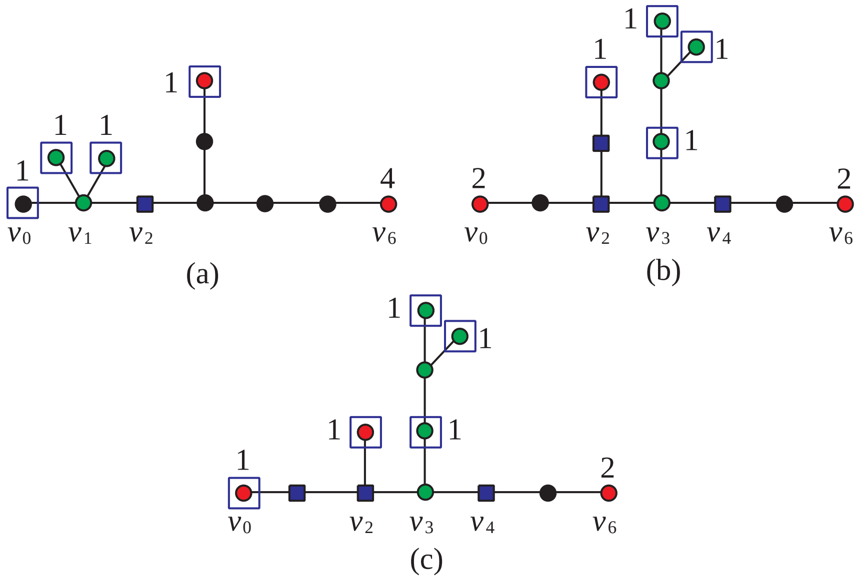

Figure 4: An illustration of the proof of Lemma 3.2.

- If T' has a maximal independent set that does not contain a vertex adjacent to v_{i-1} or to v_{i+1}, let X be such a set. See Figure 4(b).
- If each maximal independent set of T' contains a vertex adjacent to v_{i-1} (or v_{i+1} or both), then v_{i-1} (or v_{i+1}) is adjacent to a leaf. Then v_{i-2} (or v_{i+2}) is not adjacent to a leaf, and there exists a maximal independent set of T' that contains no vertex adjacent to v_{i-2} (or v_{i+2}); let X be such a set. See Figure 4(c).

Define the broadcast f on T as follows. If neither v_{i-1} nor v_{i+1} is adjacent to a leaf, let

$$f(x) = \begin{cases} g(x) & \text{if } x \in V(T) - V(T') \\ 1 & \text{if } x \in X \\ 0 & \text{otherwise.} \end{cases}$$

Then f is a dominating broadcast such that $\sigma(f) \geq i - 1 + d - i - 1 + c > d$, $v_{i-1} \in \mathrm{PB}_f(v_0)$, $v_{i+1} \in \mathrm{PB}_f(v_d)$ and $x \in \mathrm{PB}_f(x)$ for each $x \in S \cup X$.

If v_{i-1} is adjacent to a leaf and v_{i+1} is not, let

$$f(x) = \begin{cases} i-2 & \text{if } x = v_0 \\ g(x) & \text{if } x \in V(T - v_0) - V(T') \\ 1 & \text{if } x \in X \\ 0 & \text{otherwise.} \end{cases}$$

Then $|X| \geq 1$ and v_{i-1} hears f from an adjacent leaf. Hence f is a dominating broadcast such that $\sigma(f) \geq i - 2 + d - i - 1 + c + |X| > d$, $v_{i-2} \in \mathrm{PB}_f(v_0)$, $v_{i+1} \in \mathrm{PB}_f(v_d)$ and $x \in \mathrm{PB}_f(x)$ for each $x \in S \cup X$.

Similarly, if v_{i+1} is adjacent to a leaf and v_{i-1} is not, let

$$f(x) = \begin{cases} d-i-2 & \text{if } x = v_d \\ g(x) & \text{if } x \in V(T - v_d) - V(T') \\ 1 & \text{if } x \in X \\ 0 & \text{otherwise.} \end{cases}$$

Finally, if both v_{i-1} and v_{i+1} are adjacent to leaves, define f by

$$f(x) = \begin{cases} i-2 & \text{if } x = v_0 \\ d-i-2 & \text{if } x = v_d \\ g(x) & \text{if } x \in V(T - \{v_0, v_d\}) - V(T') \\ 1 & \text{if } x \in X \\ 0 & \text{otherwise.} \end{cases}$$

Now $|X| \geq 2$ and f is a dominating broadcast such that $\sigma(f) \geq i-2+d-i-2+c+|X| \geq d-4+c+|X| > d$, $v_{i-2} \in \mathrm{PB}_f(v_0)$, $v_{i+2} \in \mathrm{PB}_f(v_d)$ and $x \in \mathrm{PB}_f(x)$ if $x \in S \cup X$.

Hence in each case f is a minimal dominating broadcast such that $\sigma(f) > d$, which implies that $\Gamma_b(T) > \mathrm{diam}(T)$. ■

Lemma 3.3 *If there exists a subscript $i \in \{2, \ldots, d-2\}$ such that T_i has an independent set of cardinality 2 that does not dominate v_i, then $\Gamma_b(T) > \mathrm{diam}(T)$.*

Proof. Suppose T_i has an independent set D of cardinality 2 that does not dominate v_i. If every maximal independent set of T_i that contains D, but not v_i, dominates v_i, the result follows from Lemma 3.2. Hence assume this is not the case (in particular, v_i is not a stem) and let S be a maximal independent set of cardinality $c \geq 2$ of $T_i - v_i$ containing no vertex adjacent to v_i. Define the broadcast g on T by $g(v_0) = i$, $g(v_d) = d-i-1$, $g(x) = 1$ for each $x \in S$ and $g(x) = 0$ otherwise. Note that $v_i \in \mathrm{PB}_g(v_0)$, $v_{i+1} \in \mathrm{PB}_g(v_d)$, $x \in \mathrm{PB}_g(x)$ for each $x \in X$ and $\sigma(g) \geq i+d-i-1+c > d$. We can now proceed as in the proof of Lemma 3.2 to construct a minimal dominating broadcast f on T such that $\sigma(f) \geq \sigma(g) > d$ to obtain that $\Gamma_b(T) > d$. The details are omitted. ■

Lemma 3.4 *If $\mathrm{diam}(T_i) \geq 4$ for some i, or if $\mathrm{diam}(T_i) = 3$ and v_i is a peripheral vertex of T_i, then $\Gamma_b(T) > \mathrm{diam}(T)$.*

Proof. If $\mathrm{diam}(T_i) \geq 5$, then T_i contains a subgraph isomorphic to P_6, which, regardless of which vertex of P_6 corresponds to v_i, has an independent set of cardinality 3 that dominates but does not contain v_i, and the result follows from Lemma 3.2. If $\mathrm{diam}(T_i) = 4$ and v_i corresponds to a stem of a subgraph isomorphic to P_5, the result follows similarly.

Suppose $\mathrm{diam}(T_i) = k \in \{3, 4\}$ and v_i is a peripheral vertex of T_i. Then v_i is not a stem. Let ℓ be a vertex of T_i at distance k from v_i. Define the broadcast g on T by $g(\ell) = k$, $g(v_0) = i-1$, $g(v_d) = d-i-1$ and $g(x) = 0$ otherwise. Then $v_i \in \mathrm{PB}_g(\ell)$, $v_{i-1} \in \mathrm{PB}_g(v_0)$ and $v_{i+1} \in \mathrm{PB}_g(v_d)$, while $\sigma(g) = i-1+d-i-1+k > d$. Possibly

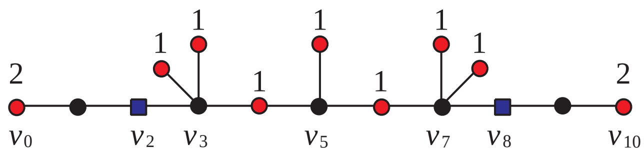

Figure 5: An illustration of the proof of Lemma 3.5.

v_{i-1} or v_{i+1} is a stem, or both are. We proceed as in the proof of Lemma 3.2 to show that $\Gamma_b(T) > d$.

Finally, suppose $\mathrm{diam}(T_i) = 4$ and v_i is the central vertex of a subgraph $H \cong P_5$ of T_i. Let ℓ_1 and ℓ_2 be the leaves of H and let w be the stem of H adjacent to ℓ_2. If v_i is a stem of T_i the result again follows from Lemma 3.2, hence assume v_i is not a stem. Define the broadcast g by $g(\ell_1) = 2,\ g(\ell_2) = 1,\ g(v_0) = i-1,\ g(v_d) = d-i-1$ and $g(x) = 0$ otherwise. Then $v_i \in \mathrm{PB}_g(\ell_1),\ w \in \mathrm{PB}_g(\ell_2),\ v_{i-1} \in \mathrm{PB}_g(v_0)$ and $v_{i+1} \in \mathrm{PB}_g(v_d)$, while $\sigma(g) = i-1+d-i-1+3 > d$. As before it (eventually) follows that $\Gamma_b(T) > d$. ■

Lemma 3.5 *Let T be a caterpillar with diametrical path $P : v_0, v_1, \ldots, v_d$. If two vertices v_i, v_{i+2k} are strong stems, for some $i \geq 1$ and some integer k such that $i + 2k \leq d-1$, and v_{i+2r} is a stem for each $r \in \{1, \ldots, k-1\}$, then $\Gamma_b(T) > d$.*

Proof. Let S be the set of leaves adjacent to $v_{i+2t},\ t \in \{0, 1, \ldots, k\}$, and $X = \{v_{i+1}, v_{i+3}, \ldots, v_{i+2k-1}\}$. Then $S \cup X$ is independent. By the hypothesis, $|S| \geq k+3$ and so $|S \cup X| \geq 2k+3$. By Lemma 3.1 we may assume that $\deg_T(x) = 2$ for each $x \in X$, otherwise the result follows.

If $i = 1$ and $i + 2k = d-1$, then $S \cup X$ is a maximal independent set of T of cardinality at least $d+1$. Let f be the characteristic function of $S \cup X$.

If $i = 1$ and $i+2k < d-1$, define the broadcast f on T by $f(x) = 1$ if $x \in S \cup X$, $f(v_d) = d-i-2k-1$ and $f(x) = 0$ otherwise. Then $x \in \mathrm{PB}_f(x)$ for each $x \in S \cup X$ and $v_{i+2k+1} \in \mathrm{PB}_f(v_d)$. Since v_{i+2k} is a stem, we may assume that $\deg(v_{i+2k+1}) = 2$, otherwise the result holds by Lemma 3.1. Therefore f is a dominating broadcast, thus a minimal dominating broadcast, and $\sigma(f) = |S \cup X| + d - i - 2k - 1 \geq d+1$.

If $i > 1$ and $i + 2k = d-1$, reverse the direction of P and proceed as above. Hence assume $1 < i < i+2k < d-1$. See Figure 5, where $d = 10$, $i = 3$ and $k = 2$. As above we may assume that $\deg(v_{i-1}) = \deg(v_{i+2k+1}) = 2$. Define the broadcast f by $f(v_0) = i-1,\ f(v_d) = d-i-2k-1,\ f(x) = 1$ for each $x \in S \cup X$ and $f(x) = 0$ otherwise. Then f is a dominating broadcast such that $\sigma(f) \geq d-2k-2+2k+3 > d,\ v_{i-1} \in \mathrm{PB}_f(v_0),\ v_{i+2k+1} \in \mathrm{PB}_f(v_d)$ and $x \in \mathrm{PB}_f(x)$ for each $x \in S \cup X$. Hence f is a minimal dominating broadcast of T such that $\sigma(f) > d$. The result now follows. ■

Before proving Lemmas 4.1 and 4.2 we state and prove two additional lemmas. If f is a broadcast on T and T' is a subtree of T, we define the *restriction of f to T'* to be the broadcast $f' = f \restriction T'$ with $V_{f'}^+ = V_f^+ \cap V(T')$ and $f'(x) = f(x)$ for all $x \in V(T')$.

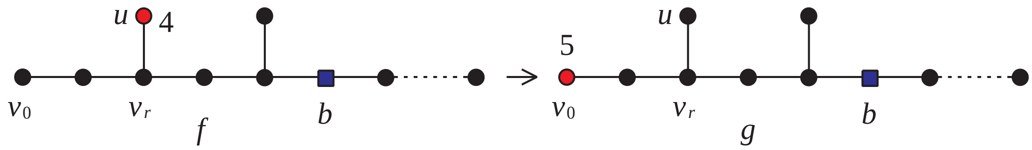

Figure 6: A step in the proof of Lemma 5.1.

Lemma 5.1 *Suppose T is a smallest non-diametrical caterpillar that satisfies Theorem* 1.1(*i*) – (*iii*). *Let f be a minimal dominating broadcast on T such that $\sigma(f) >$* $\operatorname{diam}(T)$. *Then $v_0 \in V_f^+$ or $\{v_0\} = \mathrm{PB}_f(x)$ for some $x \in V_f^+$, and a similar result holds for v_d.*

Proof. Suppose the conclusion is false and say $u \in V_f^+$ broadcasts to v_0, where $u \neq v_0$. Since $\{v_0\} \neq \mathrm{PB}_f(u)$, there exists $b \in \mathrm{PB}_f(u) - \{v_0\}$. Possibly b is a leaf adjacent to v_1, in which case $v_0 \in \mathrm{PB}_f(u)$, $\operatorname{diam}(T-b) = \operatorname{diam}(T)$ and f is a minimal dominating broadcast on $T - b$. But then $T - b$ satisfies Theorem 1.1(i) – (iii) and $\Gamma_b(T-b) > \operatorname{diam}(T-b)$, contradicting the choice of T. Hence assume b is not a leaf adjacent to v_1.

Let $r \geq 1$ be the largest index such that v_r lies on the $u - v_0$ path in T. Possibly $v_r = u$, otherwise u is a leaf adjacent to v_r. Since v_0 is a peripheral vertex, u broadcasts to all vertices of T_i for each $i = 0, \ldots, r$, and each vertex x in each such T_i is overdominated by u. Therefore $b \in V(T_t)$ for some $t > r$. In addition, if b lies on P, then b is not a stem, otherwise the leaves adjacent to b are not f-dominated. Therefore u also broadcasts to each vertex of each T_i for $r \leq i \leq t$. See Figure 6. But then the broadcast g defined by $g(v_0) = f(u) - d(u, v_r) + d(v_0, v_r)$, $g(u) = 0$ and $g(x) = f(x)$ otherwise is also a dominating broadcast such that $b \in \mathrm{PB}_g(v_0)$ and $\mathrm{PB}_g(x) = \mathrm{PB}_f(x)$ for all $x \in V_g^+ - \{v_0\}$, that is, g is a minimal dominating broadcast. Now $\sigma(g) \geq \sigma(f) - d(u, v_r) + d(v_0, v_r) \geq \sigma(f) - 1 + 1 = \sigma(f)$. Hence $\sigma(g) = \sigma(f)$ if and only if $r = 1$ and u is a leaf adjacent to v_1. In this case, $T - v_0$ also satisfies (i) – (iii), $\operatorname{diam}(T - v_0) = \operatorname{diam}(T)$, and f is also a minimal dominating broadcast on $T - v_0$, contradicting the choice of T. Hence $\sigma(g) > \sigma(f)$ and we again have a contradiction, because $\sigma(f) = \Gamma_b(T)$ and no minimal dominating broadcast has cost greater than $\Gamma_b(T)$. This proves the lemma for v_0. The result for v_d follows by symmetry. ■

Lemma 5.2 *Let T be a smallest non-diametrical caterpillar that satisfies Theorem* 1.1(*i*) – (*iii*) *and f be a minimal dominating broadcast on T such that $\sigma(f) >$* $\operatorname{diam}(T)$. *Then each leaf $w \notin \{v_0, v_d\}$ of T is either a broadcast vertex or* $\mathrm{PB}_f(u) =$ *$\{w\}$ for some $u \in V_f^+$.*

Proof. Suppose the conclusion is false and $w \notin \{v_0, v_d\}$ is a leaf of T that is neither a broadcast vertex nor the only vertex in the private boundary of some $u \in V_f^+$. Then $T - w$ is a tree with diameter d that satisfies (i) – (iii), and f is a minimal dominating broadcast on $T - w$ as well, contrary to the choice of T. ■

We now return to Lemmas 4.1 and 4.2.

Lemma 4.1 *If T is a smallest non-diametrical caterpillar that satisfies Theorem 1.1(i) – (iii), then no vertex of T is a strong stem.*

Proof. Suppose, to the contrary, that some vertex v of T is a strong stem. Then $v = v_i$ for some i, since T is a caterpillar. Say v_i is adjacent to the leaves ℓ and ℓ'. Let f be a minimal dominating broadcast on T such that $\sigma(f) > \mathrm{diam}(T)$. By Lemmas 5.1 and 5.2 we may assume that each leaf of T is either a broadcast vertex, or the only vertex in the f-private boundary of some vertex in V_f^+. Let u be the vertex that broadcasts to ℓ.

Suppose $u \neq \ell$. Then $\mathrm{PB}_f(u) = \{\ell\}$. If $u \neq \ell'$, then $d(u, \ell) = d(u, \ell')$ and we also have $\ell' \in \mathrm{PB}_f(u)$, contrary to Lemma 5.2. Hence $u = \ell'$, $f(\ell') = 2$ and $\mathrm{PB}_f(\ell') = \{\ell\}$. Let H_1 and H_2 be the subtrees of $T - v_i$ that contain v_0 and v_d, respectively. If $i \in \{1, d-1\}$, assume without loss of generality that $i = d - 1$ and ignore H_2. Since $\mathrm{diam}(T) \geq 4$, H_1 is nontrivial. By Theorem 1.1(ii), v_{i-1} is not a stem of T, hence $\mathrm{diam}(H_1) = i-1$. Since ℓ' broadcasts to v_{i-1} and $\mathrm{PB}_f(\ell') = \{\ell\}$, v_{i-1} also hears f from some vertex $w \in V_f^+ - \{\ell'\}$. Since $\ell \in \mathrm{PB}_f(\ell')$, $w \notin \{v_i, \ell\} \cup V(H_2)$, hence $w \in V(H_1)$. By Proposition 2.3 applied to $w, \ell' \in V_f^+$, $\mathrm{PB}_f(w) \subseteq V(H_1)$. Therefore $f \upharpoonright H_1$ is a minimal dominating broadcast on H_1.

- If v_{i-2} is not a stem of T, then either v_{i-2} is adjacent to only one leaf in H_1, namely v_{i-1}, in which case H_1 satisfies Theorem 1.1(i) – (iii), or v_{i-2} is adjacent to the two leaves v_{i-1} and v_0 in H_1, in which case $H_1 \cong P_3$.

- On the other hand, if v_{i-2} is a stem of T, then by Theorem 1.1(iii) and the fact that v_i is adjacent to two leaves, v_{i-2} is adjacent to exactly one leaf in T, so that it is adjacent to two leaves in H_1. If v_{i-2} is the only strong stem of H_1, then H_1 satisfies Theorem 1.1(i) – (iii). Hence suppose that for some $i' < i - 2$, $v_{i'}$ is a strong stem (of H_1 and of T). Since (iii) holds for T, and $\deg_T(v_{i-2}), \deg_T(v_i) > 2$, there exists an index k, $i' < k < i - 2$, such that $\deg_T(v_k) = \deg_T(v_{k+1}) = 2$. Therefore H_1 satisfies Theorem 1.1(i) – (iii) in this case as well.

By the choice of T, $\Gamma_b(H_1) = \mathrm{diam}(H_1) = i - 1$ in all cases. Since $f \upharpoonright H_1$ is a minimal dominating broadcast on H_1, $\sigma(f \upharpoonright H_1) \leq i - 1$. Similarly, if $1 < i < d$, H_2 satisfies Theorem 1.1(i) – (iii) and $f \upharpoonright H_2$ is a minimal dominating function of H_2 such that $\sigma(f \upharpoonright H_2) \leq \mathrm{diam}(H_2) = d - i - 1$. But then $\sigma(f) = \sigma(f \upharpoonright H_1) + \sigma(f \upharpoonright H_2) + 2 \leq d$ (or $\sigma(f) = \sigma(f \upharpoonright H_1) + 2 \leq d - 2 + 2 = d$, if $i = d - 1$), which is a contradiction because T is non-diametrical.

Hence we may assume that $u = \ell$; that is, ℓ is a broadcast vertex. If ℓ broadcasts to ℓ', we get a contradiction as above. Hence $f(\ell) = 1 = f(\ell')$ (since no other vertex can broadcast to ℓ' without broadcasting to ℓ). Then $v_i \notin \mathrm{PB}_f(x)$ for each $x \in V_f^+$. We may now define H_1 and H_2 as above and proceed as before to obtain a contradiction. ■

Lemma 4.2 *If T is a smallest non-diametrical caterpillar that satisfies Theorem 1.1(i) – (iii), then no vertex v_i, $i \in \{2, \dots, d-2\}$, is adjacent to a leaf.*

Proof. Suppose, to the contrary, that some v_i, $i \in \{2, \dots, d-2\}$, is adjacent to a leaf and let k be the largest index in $\{2, \dots, d-2\}$ such that v_k is a stem. By Lemma 4.1 we may assume that T has no strong stems. By Theorem 1.1(ii), $\deg_T(v_{k-1}) = \deg_T(v_{k+1}) = 2$. Let ℓ be the leaf adjacent to v_k and let f be a minimal dominating broadcast on T such that $\sigma(f) > \mathrm{diam}(T)$. By Lemmas 5.1 and 5.2 we may assume that each of ℓ and v_d is either a broadcast vertex or the only vertex in the f-private boundary of some vertex in V_f^+. We consider several cases. In each case we delete an edge to obtain subtrees of T, each of which contains at most one strong stem. Since T satisfies Theorem 1.1(i) – (iii), so do the subtrees. By the choice of T, each subtree thus obtained is diametrical. We omit these details in the cases for the sake of brevity.

Case 1 ℓ belongs to a private boundary and $v_d \in V_f^+$. Then either $\ell \in V_f^+$ and $\ell \in \mathrm{PB}_f(\ell)$, or $\mathrm{PB}_f(u) = \{\ell\}$ for a vertex $u \neq \ell$.

Case 1(a) $\{\ell\} = \mathrm{PB}_f(v_d)$. Then $f(v_d) = d-k+1$ and v_d broadcasts to v_{k-1}. Hence v_{k-1} does not belong to the private boundary of any vertex in V_f^+. Therefore v_{k-1} also hears f from a vertex in $V_f^+ - \{v_d\}$. Also, $\{v_k, \dots, v_{d-1}\} \cap V_f^+ = \varnothing$. Let T' be the subtree of $T - v_{k-1}v_k$ that contains v_0. For each vertex $u \in V_f^+ \cap V(T')$, Proposition 2.3 applied to u and v_d implies that $\mathrm{PB}_f(u) \subseteq V(T')$. Therefore $f \upharpoonright T'$ is a minimal dominating broadcast on T'. By the choice of T, $\sigma(f \upharpoonright T') \leq \mathrm{diam}(T') = k-1$. But now $\sigma(f) = \sigma(f \upharpoonright T') + f(v_d) \leq k - 1 + d - k + 1 = d$, a contradiction.

Case 1(b) $\ell \in \mathrm{PB}_f(u)$, $u \neq v_d$ (possibly $u = \ell$). Then u broadcasts to v_k, hence $v_k \notin \mathrm{PB}_f(v_d)$. By Proposition 2.3 and the choice of k as the largest index such that $v_k \neq v_{d-1}$ is a stem, there exists an index $j > k$ such that $v_j \in \mathrm{PB}_f(v_d)$ (and thus $f(v_d) = d - j$). Evidently, then, the edge $v_{j-1}v_j$ does not hear f from any vertex in V_f^+. Let T' be the subtree of $T - v_{j-1}v_j$ that contains v_0. As in Case 1(a) we see that $f \upharpoonright T'$ is a minimal dominating broadcast on T'.

If $j = k+1$, then u broadcasts to v_k and ℓ but not to v_{k+1}. (This is only possible if $u = \ell$ and $f(\ell) = 1$.) In this case, $\mathrm{diam}(T') = k+1$ and $f(v_d) = d - j = d - k - 1$.

If $j > k+1$, i.e., $j - 1 \geq k+1$, then $\mathrm{diam}(T') = j - 1$. In either case we obtain a contradiction as before as in Case 1(a).

Case 2 $\ell \in \mathrm{PB}_f(u)$ and $v_d \in \mathrm{PB}_f(w)$. By Lemma 5.1, $u \neq w$.

Case 2(a) $v_d \in \mathrm{PB}_f(v_d)$. Then $f(v_d) = 1$. If $\mathrm{PB}_f(v_d) = \{v_{d-1}, v_d\}$, delete the edge $v_{d-2}v_{d-1}$ and proceed as in Case 1(b) to get a contradiction. Thus, assume $\mathrm{PB}_f(v_d) = \{v_d\}$. Then v_{d-1} hears f from some other vertex as well, hence $f \upharpoonright (T - v_d)$ is a minimal dominating broadcast on $T - v_d$. By the choice of T, $\sigma(f \upharpoonright (T - v_d)) \leq d-1$ and so $\sigma(f) \leq d$, a contradiction.

Case 2(b) $\{v_d\} = \mathrm{PB}_f(w)$ for some $w \neq v_d$. Since w does not broadcast to ℓ, Proposition 2.3 and the choice of k imply that $w = v_i$ for some $i \geq k+1$. Since

$v_d \in \mathrm{PB}_f(v_i)$, $f(v_i) = d-i$. Let $r = \min\{i, \min\{j : v_j \in \mathrm{PN}_f(v_i)\}$. Since u broadcasts to v_k and $i \geq k+1$, $r \geq k+1$. Let T' and T'' be the subtrees of $T - v_{r-1}v_r$ that contain v_0 and v_d, respectively. Then $\mathrm{diam}(T'') = d - r$ and $V_f^+ \cap V(T'') = \{v_i\}$. Since $r \leq i$, $f(v_i) = d - i \leq d - r$.

If $r = i$, then $v_r, \ldots, v_d$ is a path from v_r to $v_d \in \mathrm{PB}_f(v_r)$. Otherwise, $r < i$ and, by definition of r, $\{v_r, \ldots, v_i, \ldots, v_d\} \subseteq \mathrm{PN}_f(v_i)$. In either case, Proposition 2.3 again implies that $\mathrm{PB}_f(x) \subseteq V(T')$ for each $x \in V_f^+ - \{v_i\}$. Hence $f \restriction T'$ is a minimal dominating broadcast on T', so that by the choice of T, $\sigma(f \restriction T') \leq \mathrm{diam}(T')$. If $r = k+1$, then $\mathrm{diam}(T') = r$, while if $r > k+1$, then $\mathrm{diam}(T') = r - 1$. In either case $\sigma(f) \leq r + f(v_i) \leq d$, a contradiction.

Case 3 ℓ is a broadcast vertex and $\mathrm{PB}_f(u) = \{v_d\}$ for some vertex $u \neq v_d$.

Case 3(a) $\mathrm{PB}_f(\ell) = \{v_d\}$. Then $f(\ell) = d - k + 1 \geq 3$. Let P be the $\ell - v_d$ path in T and let $w \in V_f^+ - \{\ell\}$. Then $P \cong P_{f(\ell)+1}$. By Proposition 2.3, $w \in V(T_i)$ for some $i \leq k-1$. Also, $\mathrm{PB}_f(w) \cap V(P) = \varnothing$. Thus, if T' is the subtree of $T - v_{k-1}v_k$ that contains v_0, then $\mathrm{diam}(T') = k-1$ and $f \restriction T'$ is a minimal dominating broadcast on T', which is a diametrical tree. Now $\sigma(f) = \sigma(f \restriction T') + f(\ell) \leq k - 1 + d - k + 1 = d$, a contradiction.

Case 3(b) $w \in \mathrm{PB}_f(\ell)$ and $\mathrm{PB}_f(u) = \{v_d\}$, where $u \notin \{\ell, v_d\}$ and $w \neq v_d$. By Proposition 2.3, $u = v_i$ for some $i \geq k+1$. We now proceed as in Case 2(b) to obtain a contradiction.

Case 4 ℓ and v_d are both broadcast vertices. If $f(v_d) = 1$, then $v_d \in \mathrm{PB}_f(v_d)$. This is Case 2(a), hence assume $f(v_d) \geq 2$. Then $\mathrm{PB}_f(v_d) = \{v_i\}$ for some i such that $k+1 \leq i \leq d-2$. Evidently, then, the edge $e = v_{i-1}v_i$ does not hear f from any vertex. By deleting e we proceed as before to obtain a contradiction.

Since Cases 1 – 4 and their subcases cover all possibilities for ℓ and v_d, the lemma follows. ■

This concludes the proofs of Lemmas 3.1 – 4.2, hence the proof of Theorem 1.1 is complete.

6 Open Problems

A characterization of diametrical caterpillars is presented in Theorem 1.1. In general, diametrical trees can have paths of length one or two, but not longer paths, that are internally disjoint from a diametrical path.

Problem 1 *Characterize diametrical trees that contain at least one path of length two internally disjoint from a diametrical path.*

Problem 2 *Characterize trees T with (i) $\Gamma_b(T) = \alpha(T)$, (ii) $\Gamma_b(T) = \Gamma(T)$.*

Problem 3 *Study other classes of graphs G such that $\Gamma_b(G) = \operatorname{diam}(G)$.*

Problem 4 [21] *Determine the maximum ratio $\Gamma_b(G)/\Gamma(G)$ for (i) general graphs, (ii) trees.*

The stars $K_{1,n}$ satisfy $\operatorname{diam}(K_{1,n}) = 2$ and $\Gamma_b(K_{1,n}) = n$, and hence the ratio $\Gamma_b(G)/\operatorname{diam}(G)$ is unbounded.

The proof of Lemma 3.1 suggests the following problem.

Problem 5 *If G and H are graphs and G is an isometric subgraph of H, is it true that $\Gamma_b(G) \leq \Gamma_b(H)$?*

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