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Classification of groups with toroidal coprime graphs

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Abstract

The coprime graph of a group G, denoted by Γ_G, is a graph whose vertices are elements of G and two distinct vertices x and y are adjacent if and only if $(|x|, |y|) = 1$. In this paper we discuss some basic properties of Γ_G and try to classify all finite groups whose coprime graphs are toroidal and projective.

1 Introduction

In order to get a better understanding of a given algebraic structure A, one can associate to it a graph G and study an interplay of algebraic properties of A and combinatorial properties of G. In particular there are many ways to associate a graph to a group. For example, commuting graph of groups, non-commuting graph of groups, non-cyclic graph of groups and generating graph of groups have attracted many researchers towards this dimension. One can refer to [1, 2, 3, 6, 14, 15] for such studies. Let G be a finite group. One can associate a graph to G in many different ways. Since the order of an element is one of the most basic concepts of group theory, X. Ma et al. [13] defined the *coprime graph* of a group G, denoted by Γ_G, as follows: take G as the vertices of Γ_G and two distinct vertices x and y are adjacent if and only if $(|x|, |y|) = 1$. In this paper, we discuss some basic properties of Γ_G and try to classify all finite groups whose coprime graphs are toroidal and projective.

Now recall some definitions of graph theory which are necessary in this paper. A graph G in which each pair of distinct vertices is joined by an edge is called a complete graph. We use K_n to denote the complete graph with n vertices. A subset Ω of $V(G)$ is called a clique if the induced subgraph of Ω is complete. The order of the largest clique in G is its clique number, which is denoted by $\omega(G)$. The chromatic number of G, $\chi(G)$, is the minimum k for which there is an assignment of k colors, $1, \dots k$, to the vertices of G such that adjacent vertices have different colors. An

r-partite graph is one whose vertex set can be partitioned into r subsets so that no edge has both ends in any one subset. A complete r-partite graph is one in which each vertex is joined to every vertex that is not in the same subset. A complete bipartite graph (2-partite graph) with part sizes m and n is denoted by $K_{m,n}$. If $G = K_{1,n}$ where $n \geq 1$, then G is a star graph. A *split graph* is a simple graph in which the vertices can be partitioned into a clique and an independent set. A graph G is said to be unicycle graph if it contains a unique cycle. A tree is a connected acyclic graph. A graph is *outerplanar* if it can be embedded into the plane so that all its vertices lie on the same face.

The genus of a graph is the minimal integer t such that the graph can be drawn without crossing itself on a sphere with t handles (that is an oriented surface of genus t). Thus a planar graph has genus zero, because it can be drawn on a sphere without self-crossing. A genus one graph is called a toroidal graph. In other words, a graph G is toroidal if it can be embedded on a torus, this means that, the graph's vertices can be placed on a torus such that no edges cross. Usually it is assumed that G is also non-planar. A subdivision of a graph is any graph that can be obtained from the original graph by replacing edges by paths. A remarkable characterization of planar graphs was given by Kuratowski in 1930. Kuratowski's Theorem says that a graph G is *planar* if and only if it contains no subdivision of K_5 or $K_{3,3}$.

Let G be a group and N and H be subgroups of G with N normal (but H is not necessarily normal in G) and $N \cap H = \{e\}$ and every $g \in G$ can be written uniquely as $g = nh$, where $n \in N, h \in H$. Define $\phi : H \to \frac{G}{N}$, by $\phi(h) = \overline{h} = hN$, which is an isomorphism. Then one can construct $G = N \rtimes_\phi H$ a new group called the semidirect product of N and H. Let $\pi(n)$ be the set of prime divisors of a natural number n, and denote $\pi(|G|)$ by $\pi(G)$. Throughout this paper, we assume that G is a finite group. We denote the group of integers addition modulo n by $\mathbb{Z}_n$. For basic definitions on groups, one may refer to [9].

In this paper, we will prove the following main results.

Theorem 1.1. *Let G be a finite group. Then $\gamma(\Gamma_G) = 1$ if and only if G is isomorphic to one of the following groups: S_3, $\mathbb{Z}_2 \times \mathbb{Z}_6$, $\mathbb{Z}_{12}$, $\mathbb{Z}_{15}$, $\mathbb{Z}_{20}$, $\mathbb{Z}_2 \times \mathbb{Z}_{10}$, or $\mathbb{Z}_{21}$.*

Theorem 1.2. *Let G be a finite group. Then $\overline{\gamma}(\Gamma_G) = 1$ if and only if G is isomorphic to S_3, $\mathbb{Z}_{12}$, $\mathbb{Z}_6 \times \mathbb{Z}_2$, or $\mathbb{Z}_{15}$.*

2 Some basic properties of Γ_G

In this section, we study some fundamental properties of the coprime graph.

Remark 2.1. Let G be a p-group, where p is prime. Then for any two non-identity elements $x, y \in G$, $(|x|, |y|) > 1$ and so the subgraph induced by $G - \{e\}$ in Γ_G is isomorphic to $\overline{K}_{|G|-1}$. Hence $\Gamma_G \cong K_{1,|G|-1}$.

The characterization for split graphs was given by S. Földers et al. [11]. Using this characterization, we charcterize all finite groups G whose Γ_G is a split graph.

Theorem 2.2. [11, Theorem 6.3] *Let G be a connected graph. Then G is a split graph if and only if G contains no induced subgraph isomorphic to $2K_2$, C_4, C_5.*

Theorem 2.3. *Let G be a finite group. Then Γ_G is a split graph if and only if G is a p-group or $G \cong \mathbb{Z}_2 \times Q$ where Q is a q-group with q odd.*

Proof. Assume that Γ_G is a split graph. If G is a p-group, then clearly Γ_G is a split graph. Suppose G is not a p-group. Then $|G| = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ where p_i's are distinct primes, $p_1 < p_2 < \cdots < p_k$, $k \geq 2$ and $\alpha_i \geq 1$ for all i. Let S_{p_i} be the union of Sylow p_i-subgroups of G. If $k \geq 3$, then every vertex in $S_{p_2} - \{e\}$ is adjacent to every vertex in $S_{p_3} - \{e\}$ and so Γ_G contains C_4 as a subgraph, a contradiction. Hence $k = 2$. If $|S_{p_1}| - 1 \geq 2$, then every vertex in $S_{p_1} - \{e\}$ is adjacent to every vertex in $S_{p_2} - \{e\}$ so that C_4 as a induced subgraph of Γ_G. Thus $p_1^{\alpha_1} - 1 \leq |S_{p_1}| - 1 < 2$, $p_1^{\alpha_1} = 2$ and so $|G| = 2p_2^{\alpha_2}$. Thus the Sylow 2-subgroup and Sylow p_2-subgroup are normal and hence $G \cong \mathbb{Z}_2 \times Q$, where Q is a p_2-group.

The converse is clear. □

The following theorem is used in the subsequent theorem.

Theorem 2.4. [10] $\chi(\Gamma_G) = \omega(\Gamma_G) = \pi(G) + 1$.

Theorem 2.5. *Let G be a finite group. Then Γ_G is not a unicycle graph.*

Proof. Assume that Γ_G is unicycle. Clearly G is not a p-group. Therefore $|G| = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_n}$, where $p_1, p_2, \ldots, p_n$ are distinct prime integers, $n \geq 2$, $p_i < p_j$ for $i < j$ and $\alpha_i \geq 1$ for $i \in \{1, 2, \ldots, n\}$. Let a_i be an element of order p_i. Then Γ_G contains two cycles $e - a_1 - a_2 - e$ and $e - a_1 - a_2^{-1} - e$, which is a contradiction. □

Theorem 2.6. *Let G be a finite group. Then Γ_G is a tree if and only if G is isomorphic to a p-group.*

Proof. Assume that Γ_G is a tree. Suppose G is not a p-group. Then at least two prime integers divides $|G|$. By Theorem 2.4, $\omega(\Gamma_G) \geq 3$ and so Γ_G contains a cycle, a contradiction. Thus G is isomorphic to a p-group.

Conversely, if G is a p-group, then $\Gamma_G \cong K_{1,|G|-1}$ and hence Γ_G is tree. □

The following characterization of outerplanar graphs was given by Chartrand and Harary [8]. Using this characterization, we charcterize all finite groups G whose Γ_G is outerplanar.

Theorem 2.7. [8] A graph G is outerplanar if and only if it contains no subdivision of K_4 or $K_{2,3}$.

Theorem 2.8. *Let G be a finite group. Then Γ_G is outerplanar if and only if G is a p-group or $G \cong \mathbb{Z}_6$.*

Proof. Assume that Γ_G is outerplanar. Suppose G is not a p-group. Let $|G| = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, where p_i's are distinct primes with $p_1 < p_2 < \cdots < p_k$, $k \geq 2$ and $\alpha_i \geq 1$. If $k \geq 3$, then $K_{2,3}$ is a subgraph of Γ_G, a contradiction. Therefore $k = 2$.

Suppose $\alpha_i > 1$ for some i, let it be α_1, then Γ_G contains $K_{2,3}$ as a subgraph, a contradiction. Hence $\alpha_1 = \alpha_2 = 1$. If $p_i \geq 5$ for some i, then $K_{2,3}$ is a subgraph of Γ_G, which is again a contradiction. Thus $|G| = 6$ and so $G \cong \mathbb{Z}_6$ or S_3. But $\Gamma_{S_3} \cong K_{1,2,3}$, which is not possible. Thus $G \cong \mathbb{Z}_6$.

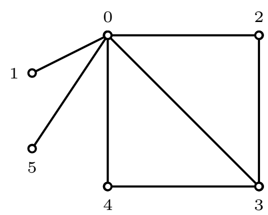

Fig. 1: $\Gamma_{\mathbb{Z}_6}$

Conversely, if G is a p-group or $\mathbb{Z}_6$, then Γ_G is isomorphic to either $K_{1,|G|-1}$ or the graph as given in Fig. 1. $\square$

Lemma 2.9. *Let G be a group of order $p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ where p_i's are distinct prime integers for $i \in \{1, \ldots, k\}$ and $k \geq 2$, then Γ_G has a subgraph isomorphic to $K_{1, p_1^{\alpha_1}-1, p_2^{\alpha_2}-1, \ldots, p_k^{\alpha_k}-1}$.*

Proof. Note that G has a Sylow p_i-subgroup of order $p_i^{\alpha_i}$ for every $i \in \{1, 2, \ldots, k\}$. Also every element of Sylow p_i-subgroup is adjacent to every element of Sylow p_j-subgroup for all $i \neq j$, which completes the proof. $\square$

Lemma 2.10. *Let G be a finite cyclic group and not a p-group. Then Γ_G is the union of $K_{\phi(d_1), \phi(d_2), \ldots, \phi(d_k)}$ where d_i's are divisors of $|G|$ and $(d_i, d_j) = 1$ for all $i \neq j$.*

Proof. It is straight forward. $\square$

A subset S of a graph G is said to be an *independent set* if no two vertices in S are adjacent. The *independent number* $\alpha(G)$ is the number of vertices in the largest independent set in G.

Theorem 2.11. *Let G be a group of order $p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_n}$, where p_i's are distinct primes, $n \geq 1$. Then $\alpha(\Gamma_G) \geq \max\{|n_{p_i}| : i = 1, \ldots, n\}$ where $n_{p_i} = \{x \in G : p_i | |x|\}$. Moreover if G is cyclic, then $\alpha(\Gamma_G) = \max\{|n_{p_i}| : i = 1, \ldots, n\}$.*

Proof. Consider the set $n_{p_i} = \{x \in G : p_i | |x|\}$, $1 \leq i \leq n$. Since $(|x|, |y|) \neq 1$ for every $x, y \in n_{p_i}$, x and y are not adjacent in Γ_G. Therefore n_{p_i}'s are independent sets of Γ_G and by defintion of independent number, $\alpha(\Gamma_G) \geq \max\{|n_{p_i}| : i = 1, \ldots, n\}$. $\square$

Remark 2.12. Let $G = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_5$. Let S be the collection of set of all $x \in G$ such that $|x| = 6, 10, 15$ or 30. In Γ_G, $|n_{p_5}| = 48$, which is maximum. But $|S| = 50 > |n_{p_5}|$.

3 Proof of Theorem 1.1

The main goal of this section is to determine all finite groups G whose coprime graph has genus one. Dorbidi [10] determine the finite groups G for which Γ_G is planar. The following observation proved by Dorbidi [10] is used frequently in this article and hence given below.

Theorem 3.1. [10, Theorem 3.6] *Let G be a finite group. Then Γ_G is a planar graph if and only if G is a p-group or $G \cong \mathbb{Z}_2 \times Q$ where Q is a q-group.*

It is well known that any compact surface is either homeomorphic to a sphere, or to a connected sum of g tori, or to a connected sum of k projective planes (see [12, Theorem 5.1]). We denote by $\mathbb{S}_g$ the surface formed by a connected sum of g tori, and by $\mathbb{N}_k$ the one formed by a connected sum of k projective planes. The number g is called the genus of the surface $\mathbb{S}_g$ and k is called the crosscap of $\mathbb{N}_k$. When considering the orientability, the surfaces $\mathbb{S}_g$ and sphere are among the orientable class and the surfaces $\mathbb{N}_k$ are among the non-orientable one.

A simple graph which can be embedded in $\mathbb{S}_g$ but not in $\mathbb{S}_{g-1}$ is called a graph of *genus* g. Similarly, if it can be embedded in $\mathbb{N}_k$ but not in $\mathbb{N}_{k-1}$, then we call it a graph of *crosscap* k. The notations $\gamma(G)$ and $\overline{\gamma}(G)$ are denoted for the genus and crosscap of a graph G, respectively. It is easy to see that $\gamma(H) \leq \gamma(G)$ and $\overline{\gamma}(H) \leq \overline{\gamma}(G)$ for all subgraph H of G. Also a graph G is called planar if $\gamma(G) = 0$, and it is called toroidal if $\gamma(G) = 1$.

For a rational number q, $\lceil q \rceil$ is the first integer number greater or equal than q. In the following lemma we bring some well-known formulas for genus of a graph (see [5]).

Lemma 3.2. *The following statements hold:*
(i) $\gamma(K_n) = \left\lceil \frac{1}{12}(n-3)(n-4) \right\rceil$ *if* $n \geq 3$*;*
(ii) $\gamma(K_{m,n}) = \left\lceil \frac{1}{4}(m-2)(n-2) \right\rceil$ *if* $m, n \geq 2$.

If G is a graph and $V'(G) = \{x \in V(G) : \deg(x) = 1\}$, then we use G' for the subgraph $G - V'$ and call it the *reduction* of G. Then we can easily observe that $\gamma(G) = \gamma(G')$.

Proof of Theorem 1.1. Assume that $\gamma(\Gamma_G) = 1$. Then by Theorem 3.1, G is not a p-group. Let $|G| = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, where p_i's are distinct primes with $p_1 < p_2 < \cdots < p_k$, $k \geq 2$ and $\alpha_i \geq 1$. If $k \geq 4$, then by Lemma 2.9, $K_{1,1,2,4,6}$ is a subgraph of Γ_G and so by Lemma 3.2, $\gamma(\Gamma_G) > 2$, which is a contradiction. Therefore $k \leq 3$.

Case 1. $k = 2$. Then $|G| = p_1^{\alpha_1} p_2^{\alpha_2}$.

Suppose $|G|$ is an odd integer. If $\alpha_i > 1$ for some i, let it be α_1, then by Lemma 2.9, $K_{1,8,4}$ is a subgraph of Γ_G and hence $\gamma(\Gamma_G) \geq 3$. Thus $\alpha_i = 1$ for $i = 1, 2$. If $p_i > 7$ for some i, then by Lemma 2.9, Γ_G contains a subgraph isomorphic to $K_{1,2,10}$ and hence $\gamma(\Gamma_G) \geq 2$, a contradiction. Therefore $p_i \leq 7$ and so $|G| = 15, 21$ or 35.

If $|G| = 35$, then by Lemma 2.9, $K_{1,4,6}$ is a subgraph of Γ_G, which is a contradiction.

If $|G| = 15$, then G is isomorphic to $\mathbb{Z}_{15}$.

If $|G| = 21$, then G is isomorphic to $\mathbb{Z}_{21}$ or $\mathbb{Z}_7 \rtimes \mathbb{Z}_3$.

Suppose $G \cong \mathbb{Z}_7 \rtimes \mathbb{Z}_3 = \langle x, y \mid x^3 = y^7 = 1, x^{-1}yx = y^2 \rangle$. Consider the vertex sets $W_1' = \{x^i y^j : 1 \leq i < 3, 0 \leq j < 7\}$ and $W_2' = \langle y \rangle - \{1\}$. Since the order of every element in W_1' is 3 and the order of every element in W_2' is 7, every element in W_1' is adjacent to every element in W_2' and so Γ_G contains a subgraph isomorphic to $K_{14,6}$. Thus $\gamma(\Gamma_{\mathbb{Z}_7 \rtimes \mathbb{Z}_3}) \geq 12$, which is a contradiction. Hence $G \cong \mathbb{Z}_{21}$.

Suppose $|G|$ is an even integer. Then $|G| = 2^{\alpha_1} p_2^{\alpha_2}$. If $\alpha_1 = 1$, then the Sylow 2-subgroup is not normal because if it is normal then Γ_G is planar. Therefore in this case, consider the Sylow 2-subgroup is not normal.

If $\alpha_i \geq 3$ for some i, then $K_{3,7}$ is a subgraph of Γ_G and so $\gamma(\Gamma_G) \geq 2$, which is a contradiction and hence $\alpha_i \leq 2$ for $i = 1, 2$. Suppose $\alpha_1 = \alpha_2 = 2$. Then by Lemma 2.9, Γ_G contains $K_{1,3,7}$ as a subgraph, a contradiction.

(i) Consider $\alpha_1 = 2$ and $\alpha_2 = 1$. Suppose that $p_2 \geq 7$. Then by Lemma 2.9, $K_{1,3,6}$ is a subgraph of Γ_G, a contradiction. Therefore $p_2 < 7$ and so $|G| = 12$ or 20.

If $|G| = 12$, then G is isomorphic to one of the following groups:

$$\mathbb{Z}_3 \rtimes \mathbb{Z}_4,\ \mathbb{Z}_{12},\ A_4,\ D_{12},\ \mathbb{Z}_6 \times \mathbb{Z}_2.$$

Suppose $G \cong \mathbb{Z}_3 \rtimes \mathbb{Z}_4 = \langle x, y \mid x^4 = y^3 = 1, x^{-1}yx = y^{-1} \rangle$. Consider the vertex sets $V_1'' = \{x^i y^j : 1 \leq i \leq 3, 0 \leq j \leq 2\} - \{x^2 y, x^2 y^2\}$ and $V_2'' = \langle y \rangle$. Since every element of V_1'' is adjacent to every element of V_2'', $K_{3,7}$ is a subgraph of Γ_G and so $\gamma(\Gamma_G) \geq 2$.

Consider $G \cong A_4$. Since A_4 contains 3 elements of order 2 and 8 elements of order 3, Γ_{A_4} contains an induced subgraph isomorphic to $K_{3,8}$ induced by these elements and hence $\gamma(\Gamma_{A_4}) \geq 2$.

Suppose $G \cong D_{12} = \langle r, s \mid r^6 = s^2 = 1, sr = r^{-1}s \rangle$. Let us consider the vertex sets $S_1 = \{r^2, r^4, 1\}$ and $S_2 = \{r^3\} \cup \{sr^i : 0 \leq i \leq 5\}$. Since every vertex of S_1 is adjacent to every vertex of S_2, Γ_G contains a subgraph isomorphic to $K_{3,7}$ so that $\gamma(\Gamma_{D_{12}}) \geq \gamma(K_{3,7}) = 2$. Hence $G \cong \mathbb{Z}_{12}$ or $\mathbb{Z}_6 \times \mathbb{Z}_2$.

If $|G| = 20$. Then G is isomorphic to one of the following groups:

$$\mathbb{Z}_5 \rtimes_1 \mathbb{Z}_4,\ \mathbb{Z}_{20},\ \mathbb{Z}_5 \rtimes_2 \mathbb{Z}_4,\ D_{20}, \text{ or } \mathbb{Z}_{10} \times \mathbb{Z}_2.$$

If $G \cong \mathbb{Z}_5 \rtimes_1 \mathbb{Z}_4 = \langle x, y \mid x^4 = y^5 = 1, x^{-1}yx = y^{-1} \rangle$, then consider the vertex sets $A_1 = (\langle x \rangle - \{1\}) \cup \{xy, xy^2\}$ and $A_2 = \langle y \rangle - \{1\}$. It is easily seen that A_1 and A_2 induces a subgraph isomorphic to $K_{4,5}$ and so $\gamma(\Gamma_{\mathbb{Z}_5 \rtimes_1 \mathbb{Z}_4}) \geq 2$.

If $G \cong \mathbb{Z}_5 \rtimes_2 \mathbb{Z}_4 = \langle x, y \mid x^4 = y^5 = 1, x^{-1}yx = y^2 \rangle$, then $\Gamma_{\mathbb{Z}_5 \rtimes_2 \mathbb{Z}_4}$ contains a subgraph isomorphic to $K_{4,5}$ induced by the vertex sets A_1' and A_2' where $A_1' = \{x^2y^i : 0 \leq i \leq 4\}$ and $A_2' = \langle y \rangle - \{1\}$. Therefore $\gamma(\Gamma_{\mathbb{Z}_5 \rtimes_2 \mathbb{Z}_4}) \geq 2$.

Suppose $G \cong D_{20} = \langle r, s \mid r^{10} = s^2 = 1,\ sr = r^{-1}s \rangle$. It is easily seen that $\Gamma_{D_{12}}$ is a subgraph of $\Gamma_{D_{20}}$. Therefore $\gamma(\Gamma_G) \geq 12$, a contradiction. Hence $G \cong \mathbb{Z}_{20}$ or $\mathbb{Z}_2 \times \mathbb{Z}_{10}$.

(ii) Suppose $\alpha_1 = 1$ and $\alpha_2 = 2$. Then clearly $K_{3,8}$ is a subgraph of Γ_G, a contradiction.

(iii) Suppose $\alpha_1 = \alpha_2 = 1$. If $p_i \geq 11$, then Γ_G contains a copy of $K_{3,10}$. Hence $\gamma(\Gamma_G) \geq 2$, a contradiction and so $p_i < 11$ for $i = 1, 2$. In this case the possible orders of G are $6, 10$ and 14.

If $|G| = 10$, then $G \cong \mathbb{Z}_{10}$ or D_{10}. By Theorem 3.1, $G \ncong \mathbb{Z}_{10}$. If $G \cong D_{10} = \langle r, s \mid r^5 = s^2 = 1,\ sr = r^{-1}s \rangle$, then $K_{4,5}$ is a subgraph of $\Gamma_{D_{10}}$ induced by the vertex sets S_1'' and S_2'' where $S_1'' = \langle r \rangle - \{1\}$ and $S_2'' = \{sr^i \mid 0 \leq i \leq 4\}$. Thus $\gamma(\Gamma_G) \geq 2$, a contradiction.

If $|G| = 14$, then $G \cong \mathbb{Z}_{14}$ or D_{14}. By Theorem 3.1, $\Gamma_{\mathbb{Z}_{14}}$ is planar. Suppose $G \cong D_{14} = \langle r, s \mid r^7 = s^2 = 1,\ sr = r^{-1}s \rangle$. It is clear that $\Gamma_{D_{10}}$ is a subgraph of $\Gamma_{D_{14}}$. Therefore $\gamma(\Gamma_{D_{14}}) \geq 2$, which is a contradiction. Hence $G \cong S_3$.

Case 2. Suppose $k = 3$. If $p_i \geq 7$ for some i, then by Lemma 2.9, Γ_G contains a induced subgraph isomorphic to $K_{1,1,2,6}$. So $\gamma(\Gamma_G) \geq 2$ and hence $p_i \leq 5$ for all i. Since $p_i < p_j$ for $i < j$, $|G| = 2^{\alpha_1}3^{\alpha_2}5^{\alpha_3}$. If $\alpha_i \geq 2$ for some i, let it be α_1, then Γ_G contains $K_{1,3,2,4}$ as a subgraph, a contradiction. Thus $\alpha_i = 1$ for all i and so $|G| = 30$. Hence G is isomorphic to one of the following groups:

$$\mathbb{Z}_{30},\ \mathbb{Z}_3 \times D_{10},\ D_{30},\ \text{or } \mathbb{Z}_5 \times S_3.$$

Consider $G \cong \mathbb{Z}_{30}$. Since G is cyclic and by Lemma 2.10, $2K_{3,4}$ as a subgraph of Γ_G formed by the vertex sets $\{V_1, V_2\}$ and $\{U_1, U_2\}$ where V_1 contains elements of order 5, V_2 contains elements of order 2 and 6 and U_1 contains elements of order 10, U_2 contains elements of order 3 and identity. Hence $\gamma(\Gamma_G) \geq 2$.

Consider the graph $\Gamma_{\mathbb{Z}_3 \times D_{10}}$. Since $\Gamma_{D_{10}}$ is a subgraph of $\Gamma_{\mathbb{Z}_3 \times D_{10}}$, $\gamma(\Gamma_{\mathbb{Z}_3 \times D_{10}}) \geq \gamma(\Gamma_{D_{10}}) \geq 2$.

Suppose $G \cong D_{30}$. It is clear that $\Gamma_{D_{10}}$ is a subgraph of $\Gamma_{D_{30}}$. Therefore $\gamma(\Gamma_G) \geq 2$, a contradiction.

Consider the group $G \cong \mathbb{Z}_5 \times S_3$. Let us consider the vertex sets $W_1 = \{a_i \in G : |a_i| = 15\}$ and $W_2 = \{b_i \in G : |b_i| = 2\}$. Then it is easily seen that $|W_1| = 8$ and $|W_2| = 3$ and every vertex in W_1 is adjacent to every vertex in W_2. Thus $\{W_1, W_2\}$ induced $K_{3,8}$ as a subgraph of Γ_G and hence $\gamma(\Gamma_{\mathbb{Z}_5 \times S_3}) \geq 2$, a contradiction.

Conversely, suppose G is isomorphic to one of the following groups: S_3, $\mathbb{Z}_2 \times \mathbb{Z}_6$, $\mathbb{Z}_{12}$, $\mathbb{Z}_{15}$, $\mathbb{Z}_{20}$, $\mathbb{Z}_2 \times \mathbb{Z}_{10}$ and $\mathbb{Z}_{21}$.

If $G \cong \mathbb{Z}_{15}$, then by Lemma 2.10, $K_{3,4}$ is a subgraph of the graph induced by the vertex sets V_1' whose elements of order 3 and identity and V_2' whose elements of order 5 and hence $\gamma(\Gamma_{\mathbb{Z}_{15}}) \geq 1$. Consider $\Gamma'_{\mathbb{Z}_{15}} = \Gamma_{\mathbb{Z}_{15}} - \{x \in \mathbb{Z}_{15} : |x| = 15\}$. Then the embedding in Fig. 2 explicitly shows that $\gamma(\Gamma_{\mathbb{Z}_{15}}) = 1$.

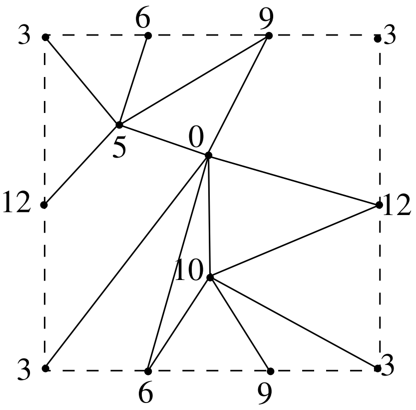

Fig. 2: Embedding of $\Gamma'_{\mathbb{Z}_{15}}$

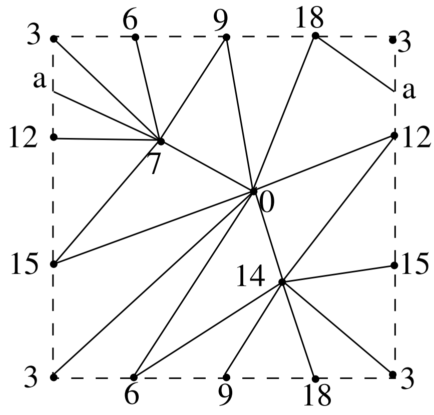

Fig. 3: Embedding of $\Gamma'_{\mathbb{Z}_{21}}$

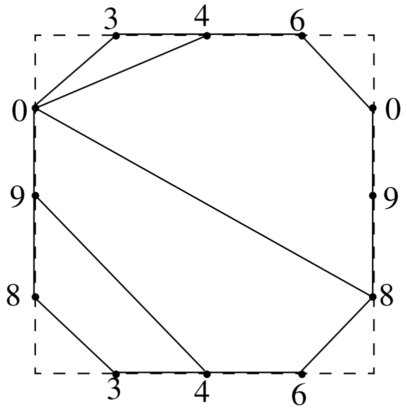

Fig. 4: Embedding of $\Gamma'_{\mathbb{Z}_{12}}$

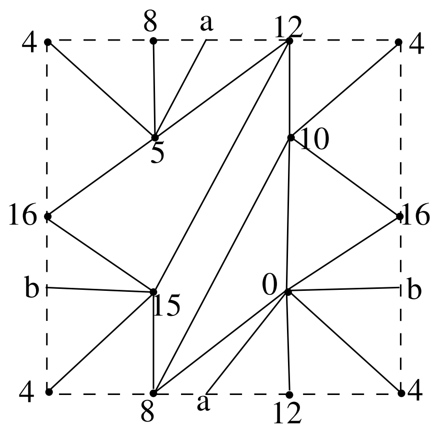

Fig. 5: Embedding of $\Gamma'_{\mathbb{Z}_{20}}$

Suppose $G \cong \mathbb{Z}_{21}$, then by Lemma 2.9, $K_{1,2,6}$ is a subgraph of Γ_G and hence $\gamma(\Gamma_{\mathbb{Z}_{21}}) \geq 1$. Let $\Gamma'_{\mathbb{Z}_{21}} = \Gamma_{\mathbb{Z}_{21}} - \{x \in \mathbb{Z}_{21} : |x| = 21\}$ and the embedding in Fig. 3 explicitly shows that $\gamma(\Gamma_{\mathbb{Z}_{15}}) = 1$.

Consider the graph $\Gamma_{\mathbb{Z}_{12}}$. Since $\mathbb{Z}_{12}$ is cyclic, by Lemma 2.10, $\Gamma_{\mathbb{Z}_{12}}$ contains a subgraph isomorphic to $K_{3,3}$ and hence $\gamma(\Gamma_{\mathbb{Z}_{12}}) \geq 1$. Here $\Gamma'_{\mathbb{Z}_{12}} = \Gamma_{\mathbb{Z}_{12}} - \{x \in \mathbb{Z}_{12} : |x| = 12 \text{ or } 6\}$. Then the embedding in Fig. 4 explicitly shows that $\gamma(\Gamma_{\mathbb{Z}_{12}}) = 1$.

If $G \cong \mathbb{Z}_2 \times \mathbb{Z}_6$, then it is easily seen that $\Gamma_{\mathbb{Z}_{12}} \cong \Gamma_{\mathbb{Z}_2 \times \mathbb{Z}_6}$. Therefore $\gamma(\Gamma_{\mathbb{Z}_2 \times \mathbb{Z}_6}) = 1$.

Consider $G \cong \mathbb{Z}_{20}$. Since G is cyclic, by Lemma 2.10, $K_{4,4}$ as a subgraph induced by the vertex sets Ω_1 and Ω_2 where Ω_1 contains elements of order 5 and Ω_2 contains elements of order 2, 4 and identity. Hence $\gamma(\Gamma_G) \geq 1$. Consider $\Gamma'_G = \Gamma_G - \{x \in G : |x| = 20 \text{ or } 10\}$. Then the embedding in Fig. 5 explicitly shows that $\gamma(\Gamma_G) = 1$.

Suppose $G \cong \mathbb{Z}_2 \times \mathbb{Z}_{10}$, then it is easily seen that $\Gamma_{\mathbb{Z}_{20}} \cong \Gamma_{\mathbb{Z}_2 \times \mathbb{Z}_{10}}$. Hence $\gamma(\Gamma_{\mathbb{Z}_2 \times \mathbb{Z}_{10}}) = 1$.

If $G \cong S_3$, then $\Gamma_{S_3} \cong K_{1,2,3}$. Also Γ_{S_3} is a subgraph of $\Gamma_{\mathbb{Z}_{12}}$, then $\gamma(\Gamma_{S_3}) = 1$.$\square$

4 Proof of Theorem 1.2.

The main goal of this section is to determine all finite groups G whose coprime graph has crosscap one. The following two results about the crosscap formulae of a complete graph and a complete bipartite graph are very useful in the proof of Theorem 1.2.

Lemma 4.1. *[16] The following statements hold:*
(i) $\overline{\gamma}(K_n) = \begin{cases} \left\lceil \frac{1}{6}(n-3)(n-4) \right\rceil & \text{if } n \geq 3 \text{ and } n \neq 7; \\ 3 & \text{if } n = 7 \end{cases}$
(ii) $\overline{\gamma}(K_{m,n}) = \left\lceil \frac{1}{2}(m-2)(n-2) \right\rceil$, *where* $m, n \geq 2$.

By slight modifications in the proof of Theorem 1.1 with Lemma 4.1, and using Figs. 6 and 7, one can prove Theorem 1.2.

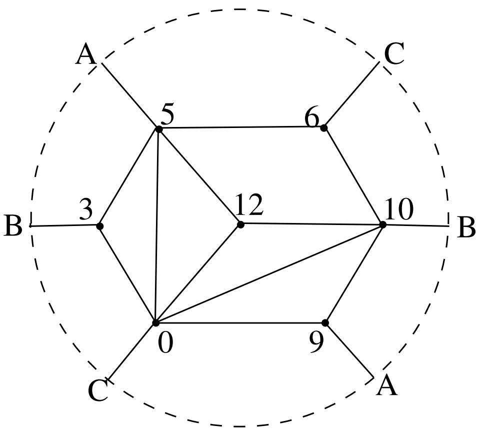

Fig. 6: Embedding of $\Gamma'_{\mathbb{Z}_{15}}$ in $\mathbb{N}_1$

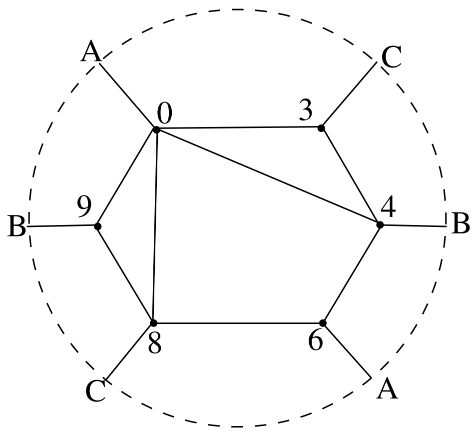

Fig. 7: Embedding of $\Gamma'_{\mathbb{Z}_{12}}$ in $\mathbb{N}_1$

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