AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **69(1)** (2017), Pages 169–173

Optimal 1-planar graphs which quadrangulate other surfaces

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Abstract

It is known that for any orientable surface $\mathbb{S}_g$ other than the sphere, there exists an optimal 1-planar graph which can be embedded on $\mathbb{S}_g$ as a triangulation. In this paper, we prove that for any orientable surface $\mathbb{S}_g$ with genus $g \geq 3$ and any non-orientable surface $\mathbb{N}_k$ with genus $k \geq 6$ $(k \neq 7)$, there exists an optimal 1-planar graph which can be embedded on the surface as a quadrangulation. Furthermore, every optimal 1-planar graph can quadrangulate a surface.

1 Introduction

We deal with finite and simple graphs unless otherwise noted. For the terminology and notation used but undefined in this paper, see [2]. A *surface* is a compact connected 2-manifold without boundary. The well-known classification theorem of surfaces states that any surface is homeomorphic to one of the following: the sphere with $g \geq 0$ handles (the *orientable surface* $\mathbb{S}_g$ *of genus* g) or the sphere with $k \geq 1$ crosscaps (the *non-orientable surface* $\mathbb{N}_k$ *of genus* k). A *map* on a surface F^2 is a 2-cell embedding of a graph on F^2. A map G on a surface F^2 is a *triangulation* if every face of G is bounded by a 3-cycle (a k-*cycle* is one of length k). A map G on a surface F^2 is a *quadrangulation* if every face of G is bounded by a 4-cycle. We focus on graphs which allow embeddings of specific types on specific surfaces. Lawrencenko and Negami [4, 5] completely determined the graphs which triangulate both the torus and the Klein bottle. Nakamoto et al. [7] showed that for any non-spherical surface F^2, there exists a graph which triangulates the sphere and which quadrangulates F^2. Suzuki [10, 11] investigated the existence of graphs which triangulate a non-spherical surface F_1^2 and which quadrangulate another surface F_2^2.

A graph G drawn on the sphere (possibly with edge crossings) is 1-*planar* if every edge of G crosses other edges at most once. A 1-planar graph G is *optimal* if G satisfies $|E(G)| = 4|V(G)| - 8$. Suzuki [8] proved that for any positive integer g, there exists an optimal 1-planar graph which triangulates the orientable surface $\mathbb{S}_g$ of genus g. Nagasawa, the author and Suzuki [6] recently proved that for any positive integer k, there exists no optimal 1-planar graph which triangulates the non-orientable surface $\mathbb{N}_k$ of genus k.

In this paper, we consider optimal 1-planar graphs which quadrangulate a non-spherical surface. We not only completely determine the surfaces which can be quadrangulated by an optimal 1-planar graph, but also show that every optimal 1-planar graph can quadrangulate a surface.

Theorem 1 *For an orientable surface $\mathbb{S}_g$ of genus g, there exists an optimal 1-planar graph which quadrangulates $\mathbb{S}_g$ if and only if $g \geq 3$. For a non-orientable surface $\mathbb{N}_k$ of genus k, there exists an optimal 1-planar graph which quadrangulates $\mathbb{N}_k$ if and only if $k = 6$ or $k \geq 8$.*

Theorem 2 *For any $g \geq 3$, every optimal 1-planar graph with $2g + 2$ vertices quadrangulates the orientable surface $\mathbb{S}_g$ of genus g.*

Theorem 3 *For any $k \geq 6$ other than 7, every optimal 1-planar graph with $k + 2$ vertices quadrangulates the non-orientable surface $\mathbb{N}_k$ of genus k.*

The related topic can be found in [3]. This paper is organized as follows. In Section 2, we introduce a notion of 4-cycle double covers, which corresponds to a face set of a quadrangulation on a surface. In Section 3, we prove Theorems 1, 2 and 3.

2 Optimal 1-planar graphs and 4-cycle double covers

Let G be a graph. Assume that G has a family of cycles, denoted by $\mathcal{C}$, such that each edge of G is contained in exactly two cycles of $\mathcal{C}$. Then we call $\mathcal{C}$ a *cycle double cover* of G. In particular, $\mathcal{C}$ is a 4-*cycle double cover* if every cycle in $\mathcal{C}$ is of length 4. Let v be a vertex of G. We denote a subset of a 4-cycle double cover $\mathcal{C}$ including v by $\mathcal{C}_v$. For any 4-cycle $vwxy$ of $\mathcal{C}_v$, add the edge wy (multiple edges are allowed) and call the resulting multigraph H_v. Then $\mathcal{C}_v$ is *good* if $E(H_v \setminus G)$ induces a cycle of length $\deg_G(v)$. The following proposition is easy to show.

Proposition 4 *Let G be a graph. G has an embedding on a surface F^2 as a quadrangulation if and only if G has a 4-cycle double cover such that $\mathcal{C}_v$ is good for every vertex v of G.*

To construct optimal 1-planar graphs which quadrangulate other surfaces, the following lemma is crucial.

Lemma 5 *If an optimal* 1*-planar graph* G *quadrangulates an orientable surface* $\mathbb{S}_g$ *of genus* g *(respectively, a non-orientable surface* $\mathbb{N}_k$ *of genus* k*), then* $n = |V(G)|$ *is* $2g+2$ *(respectively,* $k+2$*).*

Proof. Let G' be the resulting quadrangulation on the surface F^2. Then Euler's formula

$$|V(G')| - |E(G')| + |F(G')| = \chi(F^2)$$

holds, where $\chi(F^2)$ is the Euler characteristic of F^2. Now $|V(G')| = |V(G)|$ and $|E(G')| = |E(G)| = 4n-8$. Since every face of G' is bounded by a 4-cycle, $|F(G')| = \frac{1}{2}|E(G')|$ holds. Then $\chi(F^2) = -n+4$. When $F^2 \simeq \mathbb{S}_g$, $\chi(\mathbb{S}_g) = 2-2g$ and we obtain $n = 2g+2$. When $F^2 \simeq \mathbb{N}_k$, $\chi(\mathbb{N}_k) = 2-k$ and we obtain $n = k+2$. ■

Let G be an optimal 1-planar graph. It is known (see [9, Theorem 11] for example) that all non-crossing edges of G induce a 3-connected plane quadrangulation, denoted by $Q(G)$. By using this construction, the following lemma follows.

Lemma 6 (Suzuki [9]) *There exists an optimal* 1*-planar graph with* n *vertices if and only if* $n = 8$ *or* $n \geq 10$*.*

A *k-edge cut* S of a connected graph G is a subset of $E(G)$ with cardinality k such that $G - S$ is disconnected. A k-edge cut S is *proper* if S does not contain a $(k-1)$-edge cut of G.

Lemma 7 *Let* G *be a quadrangulation on a surface with an even number of faces. Then the dual* G^* *of* G *has a perfect matching.*

Proof. Since G^* is 4-regular, there is no proper 1- or 3-edge cut of G^*. (Otherwise there must be a vertex with odd degree.) Furthermore, G^* does not have either a proper 2-edge cut or a loop since G is simple. Then G^* is 4-edge-connected. It is known (see [1, Theorem 2.37] for example) that $(k-1)$-edge-connected k-regular graph of even order (possibly with multiple edges) has a perfect matching for $k \geq 2$. Then the lemma follows. ■

3 Proofs of the main theorems

Proof of Theorem 2. For a given positive integer g, we shall construct a 4-cycle double cover of an optimal 1-planar graph, which corresponds to a quadrangulation on $\mathbb{S}_g$. By Lemma 5, we must prepare an optimal 1-planar graph G with $2g+2$ vertices. By Lemma 6, g must be greater than 2.

Since the number of faces of $Q(G)$ is $2g$ by Euler's formula, there exists a perfect matching M of the dual $(Q(G))^*$ of $Q(G)$ by Lemma 7. For each edge wx of the dual M^* of M, let $f_1 = wxuv$ and $f_2 = xwyz$ be two faces of $Q(G)$ so that the orders to be clockwise on the sphere. Taking four 4-cycles $uvxw, uwyx, vwzx$ and $wxyz$, one can see that these $2|F(Q(G))|$ 4-cycles form a 4-cycle double cover $\mathcal{C}$ of G. The order of neighbors around u on the resulting surface F^2 is v, w, x, one around x is u, y, w, v, z, respectively. By symmetry and the construction of $\mathcal{C}$, $\mathcal{C}_v$ is good for every vertex v of G. Now we check the orientability of F^2. Those four 4-cycles cover the edges uw, vx, wx, wz and xy twice with opposite directions, respectively, and make a clockwise orientation of $uvwyzx$ on F^2. Then $\mathcal{C}$ corresponds to a quadrangulation on $\mathbb{S}_g$ by Proposition 4. ∎

Proof of Theorem 3. For a given positive integer k, we shall construct a 4-cycle double cover of an optimal 1-planar graph, which corresponds to a quadrangulation on $\mathbb{N}_k$. By Lemma 5, we must prepare an optimal 1-planar graph G with $k+2$ vertices. By Lemma 6, k must be 6 or greater than 7.

For each face $f = uvwx$ of $Q(G)$, take two 4-cycles $uvxw$ and $uxvw$. One can see that these $2|F(Q(G))|$ 4-cycles form a 4-cycle double cover $\mathcal{C}$ of G. By the construction of $\mathcal{C}$, $\mathcal{C}_v$ is good for every vertex v of G; the order of neighbors around v on the resulting surface F^2 coincides with one on the sphere. Now we check the orientability of F^2. Those two 4-cycles $uvxw$ and $uxvw$ form a Möbius band on F^2, and then $\mathcal{C}$ corresponds to a quadrangulation on $\mathbb{N}_k$ by Proposition 4. ∎

Proof of Theorem 1. This follows from the proofs of Theorems 2 and 3. ∎

Remark Roughly speaking, the main idea of the proof of Theorem 2 is as follows:

(i) partition the faces of $Q(G)$ into pairs of adjacent faces,

(ii) for every pair f_1 and f_2 of adjacent faces, attach a handle (one end of the handle is attached to f_1, and another end of the handle is attached to f_2), and

(iii) embed the pairs of the diagonals of f_1 and f_2 in the two faces with the attached handle such that the four added edges do not cross and all obtained faces are quadrangular.

And then the main idea of the proof of Theorem 3 is as follows:

(i) for every face of $Q(G)$, insert a crosscap in the face, and

(ii) embed the two diagonals of the face in the face with the inserted crosscap such that the two added edges do not cross and all obtained faces are quadrangular.

Acknowledgments

The author is grateful to the anonymous referees for their helpful comments.

References

[1] J. Akiyama and M. Kano, Factors and factorizations of graphs: proof techniques in factor theory, Lec. Notes in Math. **2031**, Springer, Heidelberg, 2011.

[2] R. Diestel, Graph Theory, 4th edition, Graduate Texts in Mathematics, **173**, Springer, Heidelberg, 2010.

[3] S. Lawrencenko, Realizing the chromatic numbers and orders of spinal quadrangulations of surfaces, *J. Combin. Math. Combin. Comput.* **87** (2013), 303–308.

[4] S. Lawrencenko and S. Negami, Constructing the graphs that triangulate both the torus and the Klein bottle, *J. Combin. Theory Ser. B* **77** (1999), 211–218.

[5] S. Lawrencenko and S. Negami, Irreducible triangulations of the Klein bottle, *J. Combin. Theory Ser. B* **70** (1997), 265–291.

[6] T. Nagasawa, K. Noguchi and Y. Suzuki, No optimal 1-planar graph triangulates any non-orientable closed surface, (submitted).

[7] A. Nakamoto, S. Negami, K. Ota and J. Širáň, Planar triangulations which quadrangulate other surfaces, *European J. Combin.* **25** (2004), 817–833.

[8] Y. Suzuki, Optimal 1-planar graphs which triangulate other surfaces, *Discrete Math.* **310** (2010), 6–11.

[9] Y. Suzuki, Re-embeddings of maximum 1-planar graphs, *SIAM J. Discrete Math.* **24** (2010), 1527–1540.

[10] Y. Suzuki, Graphs that triangulate a given surface and quadrangulate another surface, *J. Combin. Theory Ser. B.* **97** (2007), 237–244.

[11] Y. Suzuki, Triangulations on closed surfaces which quadrangulate other surfaces II, *Discrete Math.* **303** (2005), 235–243.

(Received 15 Feb 2017; revised 15 June 2017)