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The chromatic index of block intersection graphs of Kirkman triple systems and cyclic Steiner triple systems

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Abstract

The block intersection graph of a combinatorial design with block set $\mathcal{B}$ is the graph with $\mathcal{B}$ as its vertex set such that two vertices are adjacent if and only if their associated blocks are not disjoint. The chromatic index of a graph G is the least number of colours that enable each edge of G to be assigned a single colour such that adjacent edges never have the same colour. A graph G for which the chromatic index equals the maximum degree is called Class 1; otherwise the chromatic index exceeds the maximum degree by one and G is called Class 2. We conjecture that whenever a Steiner triple system has a block intersection graph with an even number vertices, the graph is Class 1. We prove this to be true for Kirkman triple systems and cyclic Steiner triple systems of order $v \equiv 9 \pmod{12}$. We also prove that the conjecture holds for cyclic Steiner triple systems of order $v \equiv 1 \pmod{12}$ for which $\frac{\varphi(v)}{v-1} \geqslant \frac{2}{3}$, where φ is Euler's totient function.

1 Introduction

A *balanced incomplete block design* of order v, block size k and index λ, denoted as a BIBD(v, k, λ), consists of a v-set V accompanied by a block set $\mathcal{B}$ which itself is a set (or multiset) of k-subsets of V such that each 2-subset of V is contained by exactly λ of the blocks of $\mathcal{B}$. A *Steiner triple system* of order v is a BIBD$(v, 3, 1)$, and is typically denoted by the notation STS(v). It is a classic result of Kirkman that a STS(v) exists if and only if $v \equiv 1$ or $3 \pmod 6$ [16]. For more information on Steiner triple systems and their properties, see [10].

Given a block design $(V, \mathcal{B})$, its associated *block intersection graph* is the graph on vertex set $\mathcal{B}$ for which two vertices B_1 and B_2 are adjacent if and only if $|B_1 \cap B_2| \geqslant 1$. In [5] Graham is attributed with having asked whether STS block intersection graphs have Hamilton cycles, a question that was subsequently affirmatively answered (see [5] and [14]). In addition to being Hamiltonian, block intersection graphs of BIBDs are also pancyclic [4, 19]. Even more recently they have been shown to be cycle extendable, which in turn has enabled a polynomial-time algorithm to be developed for finding cycles of arbitrary specified length in them [1].

The *chromatic index* χ' of a graph G is the least number of colours that are needed when colouring its edges under the restriction that no two adjacent edges receive the same colour. Vizing's Theorem asserts that a simple graph G with maximum degree Δ either has chromatic index $\chi' = \Delta$ or $\chi' = \Delta + 1$ [12, 25]. A simple graph G is described as *Class 1* (resp. *Class 2*) if its chromatic index is Δ (resp. $\Delta + 1$), although determining which is the case is in general an NP-complete problem [13]. Nevertheless, for some types of graphs the situation is less difficult; for instance, a century-old result of König confirms that all bipartite graphs are Class 1 [17].

If G is a Δ-regular Class 1 graph of even order, then the edges of G can be partitioned into Δ 1-factors (*i.e.*, 1-regular spanning subgraphs); such a partition is called a *1-factorisation* of G. A *Hamilton decomposition* of a Δ-regular graph G consists of a set of Hamilton cycles (plus a 1-factor if Δ is odd) that partition the edges of G. Results pertaining to Hamilton decompositions date back to the nineteenth century when Walecki is described as having found elegant Hamilton decompositions of complete graphs [18]. For a survey paper about Hamilton decompositions of graphs, see [3]. In the event that G is a Hamilton decomposable graph of even order, then it is clear that G admits a 1-factorisation and hence G is Class 1. However, the converse does not hold as there are graphs with 1-factorisations but which do not admit Hamilton decompositions (for an example, refer to Figure 1).

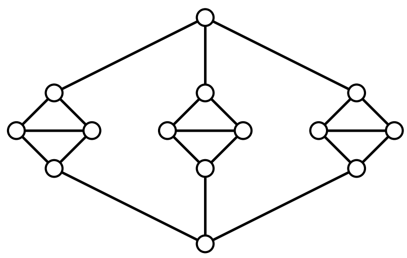

Figure 1: A Class 1 graph that has no Hamilton decomposition

The *line graph* $L(G)$ of a graph G is the graph having the edge set of G as its vertex set, with two vertices of $L(G)$ being adjacent if and only if their corresponding edges in G are adjacent. Line graphs can be viewed as a special case of block intersection graphs, or, alternatively, block intersection graphs can be viewed as a generalisation of line graphs. There are a number of results in the literature concerning 1-factorisations and line graphs. For instance, Alspach proved that any complete graph K_n with an even number of edges has a Class 1 line graph [2]. More generally, Jaeger has established that if G is a Hamilton decomposable graph with an even number of edges then $L(G)$ is Class 1 [15].

One-factorisations are widely used within graph theory as well as in areas such as scheduling. For an in-depth review about the theory and application of 1-factorisations, readers can consult [26]. Among the various problems about 1-factorisations, particularly noteworthy is the 1-Factorisation Conjecture, one of the earliest references to which is [7]:

Conjecture 1.1 *If G is a simple Δ-regular graph on $2k$ vertices and $k \leqslant \Delta$ then G is Class 1.*

A stronger conjecture that implies the 1-Factorisation Conjecture was given by Nash-Williams [20]:

Conjecture 1.2 *If G is a simple Δ-regular graph on n vertices and $n \leqslant 2\Delta$ then G has a Hamilton decomposition.*

Both of these conjectures have recently been solved when the number of vertices is sufficiently large [11]. However, with regard to block intersection graphs of Steiner triple systems, the block intersection graph of a STS(v) has $\frac{v(v-1)}{6}$ vertices and is regular of degree $\frac{3v-9}{2}$. Hence it is only for $v \in \{9, 13\}$ (resp. admissible $v \leqslant 15$) that the block intersection graph of a STS(v) satisfies the hypothesis of the 1-Factorisation Conjecture (resp. Nash-Williams' Conjecture). In [23] it is reported that every STS(v) with order $v \leqslant 15$ has a Hamilton decomposable block intersection graph, but for admissible orders $v \geqslant 19$ the status remains undetermined.

Since the property of being Hamilton decomposable is (for graphs of even order) a stronger property than having a 1-factorisation, it is natural to initially consider the potentially easier question of deciding whether 1-factorisations exist for STS block intersection graphs. Any graph of order n and maximum degree Δ such that the number of edges exceeds $\Delta\lfloor \frac{n}{2} \rfloor$ is called *overfull* and is necessarily Class 2 [8]. Every regular graph of odd order is overfull and hence every STS(v) for which $v \equiv 3$ or 7 (mod 12) must have a Class 2 block intersection graph. We therefore only consider $v \equiv 1$ or 9 (mod 12) for the remainder of this paper, for which we offer this conjecture:

Conjecture 1.3 *Every STS(v) with $v \equiv 1$ or 9 (mod 12) has a Class 1 block intersection graph.*

A *Kirkman triple system* of order v, denoted KTS(v), is a STS(v) for which the blocks can be partitioned into $\frac{v-1}{2}$ sets called *parallel classes*, each of which consists of $\frac{v}{3}$ pairwise disjoint blocks. In Section 2 we show that Conjecture 1.3 is satisfied by every Kirkman triple system of order $v \equiv 9$ (mod 12).

A Steiner triple system of order v is said to be *cyclic* if its automorphism group contains a cyclic subgroup of order v. In Section 3 we consider cyclic Steiner triple systems of order $v \equiv 1$ (mod 12) and we prove that Conjecture 1.3 holds for every such STS with $\frac{\varphi(v)}{v-1} \geqslant \frac{2}{3}$, where φ denotes Euler's totient function. In Section 4 we establish that Conjecture 1.3 is true for all cyclic Steiner triple systems of order $v \equiv 9 \pmod{12}$.

2 Kirkman triple systems

Ray-Chaudhuri and Wilson proved in 1971 that a KTS(v) exists if and only if $v \equiv 3 \pmod 6$ [24]. Here we show that each Kirkman triple system for which $v \equiv 9 \pmod{12}$ gives rise to a block intersection graph that admits a 1-factorisation. Before proving the main result of this section, we state the following theorem which will be used throughout this paper.

Theorem 2.1 (König [17]) *If G is a bipartite graph, then $\chi'(G) = \Delta(G)$.*

We now show that Conjecture 1.3 holds for KTS of order $v \equiv 9 \pmod{12}$.

Theorem 2.2 *A KTS(v) has a Class 1 block intersection graph if and only if $v \equiv 9 \pmod{12}$.*

Proof. It has already been observed that every STS(v) with $v \equiv 3 \pmod{12}$ has a Class 2 block intersection graph, so we only need to now consider Kirkman triple systems with $v \equiv 9 \pmod{12}$. Suppose that $\mathcal{B}$ is the block set of a KTS(v) with $v \equiv 9 \pmod{12}$. Hence $v = 6m + 3$ where m is an odd integer, and the KTS has $\frac{v-1}{2} = 3m + 1$ parallel classes. Let $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_{3m+1}$ be the parallel classes.

There are $\binom{3m+1}{2}$ pairs of parallel classes, and we wish to partition these pairs into $3m$ sets $\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_{3m}$ so that within the pairs of each set $\mathcal{S}_i$ each parallel class occurs exactly once. Such a partition can be easily obtained by making use of a 1-factorisation of K_{3m+1} on the vertex set $\{\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_{3m+1}\}$.

Consider two parallel classes, say $\mathcal{P}_x$ and $\mathcal{P}_y$, that are together as a pair in some set $\mathcal{S}_i$. The blocks of these two parallel classes induce a subgraph $G_{x,y}$ of the block intersection graph G of the KTS such that $G_{x,y}$ is a 3-regular bipartite graph with $2m + 1$ vertices in each of its two parts. Recalling that bipartite graphs are Class 1 by Theorem 2.1, we can therefore edge-colour this cubic bipartite graph with the three colours $3i-2$, $3i-1$ and $3i$. Moreover, we can use these same three colours for each subgraph of G induced by a pair of parallel classes in $\mathcal{S}_i$, since each such pair induces a subgraph of G that is disjoint from the subgraphs induced by the other pairs of $\mathcal{S}_i$.

We have now exhibited a proper colouring of the edges of the block intersection graph that uses only $9m$ colours, and hence we conclude that the KTS has a Class 1 block intersection graph. $\square$

3 Cyclic Steiner triple systems of order $v \equiv 1 \pmod{12}$

In this section we investigate Steiner triple systems of order $v \equiv 1 \pmod{12}$ for which the block set $\mathcal{B}$ can be generated from a set of $\frac{v-1}{6}$ base blocks through the repeated application of a permutation σ of the points of V. More succinctly, we consider Steiner triple systems that are cyclic, which are known to exist if and only if $v \equiv 1$ or 3 (mod 6) and $v \neq 9$ [21]. Without loss of generality we may assume that $V = \mathbb{Z}_v$ and that the permutation σ is $(0, 1, 2, \ldots, v-1)$. We prefer an additive notation for this permutation, so that for any block B, we define $B+i = \{x+i \pmod{v} : x \in B\}$.

Here we will prove that many cyclic Steiner triple systems of order $v \equiv 1$ (mod 12) have Class 1 block intersection graphs. As with the Kirkman triple systems of Section 2 we will find a means of decomposing the block intersection graph into several subgraphs that themselves are Class 1. Many of these subgraphs will be bipartite, but there is another type of subgraph that will also be helpful to us.

Given positive integers n and k, we denote by $P(n,k)$ the *generalised Petersen graph* on $2n$ vertices and $3n$ edges. Specifically, the vertex set of $P(n,k)$ is $\{w_0, w_1, \ldots, w_{n-1}\} \cup \{x_0, x_1, \ldots, x_{n-1}\}$, and for each $i \in \mathbb{Z}_n$ let $\{w_i, w_{i+1}\}$, $\{w_i, x_i\}$ and $\{x_i, x_{i+k}\}$ be edges. The familiar Petersen graph is therefore $P(5,2)$. Although the Petersen graph is Class 2, Castagna and Prins have shown that it is the unique generalised Petersen graph with this property [6].

Given a block B in a cyclic STS(v), we will refer to the set of blocks $\{B, B+1, B+2, \ldots, B+(v-1)\}$ as its orbit. Note that if, in the block intersection graph of a cyclic STS, B is adjacent to $B+i$ then B is also adjacent to $B-i$; the edges $\{B, B+i\}$ and $\{B, B-i\}$ are said to have difference i with respect to the orbit of B. If the six neighbours that a block B has within its orbit are $B \pm i$, $B \pm j$ and $B \pm k$ then we will refer to the three least positive elements of the 6-set $\{\pm i \pmod{v}, \pm j \pmod{v}, \pm k \pmod{v}\}$ as the *orbital differences* for the orbit of B.

Lemma 3.1 *Given two orbits of size v in a cyclic STS(v), if one of them has an orbital difference d that is co-prime to v and e is an orbital difference for the other orbit, then a $P(v, d^{-1}e)$ is formed by the edges of difference d in the first orbit, the edges of difference e in the second orbit and a suitably chosen 1-factor between the two orbits.*

Proof. Let B_0 be a block having an orbit with a difference d that is co-prime to v. For each $i \in \mathbb{Z}_v$ let $B_i = B_0 + i$. Then there exists a v-cycle $(B_0, B_d, B_{2d}, \ldots, B_{(v-1)d})$. B_0 is adjacent to nine vertices in the other orbit of the lemma's hypothesis. Let A_0 be one of these nine neighbours of B_0. We now obtain a set of v edges of the form $\{B_i, A_i\}$, yielding the spokes of a generalised Petersen graph. Since e is an orbital difference for the orbit of A_0, there exists an edge $\{A_{ie}, A_{(i+1)e}\}$ for each $i \in \mathbb{Z}_v$. Relabel $B_0, B_d, B_{2d}, \ldots, B_{(v-1)d}$ (resp. $A_0, A_d, A_{2d}, \ldots, A_{(v-1)d}$) as $w_0, w_1, w_2, \ldots, w_{v-1}$ (resp. $x_0, x_1, x_2, \ldots, x_{v-1}$), respectively. Then we find that x_j is adjacent to x_{j+k} if and only if A_j is adjacent to A_{j+kd}. In the graph that we have constructed, the three neighbours of A_j are A_{j+e}, A_{j-e} and B_j and so we obtain $e \equiv kd \pmod{v}$. Hence

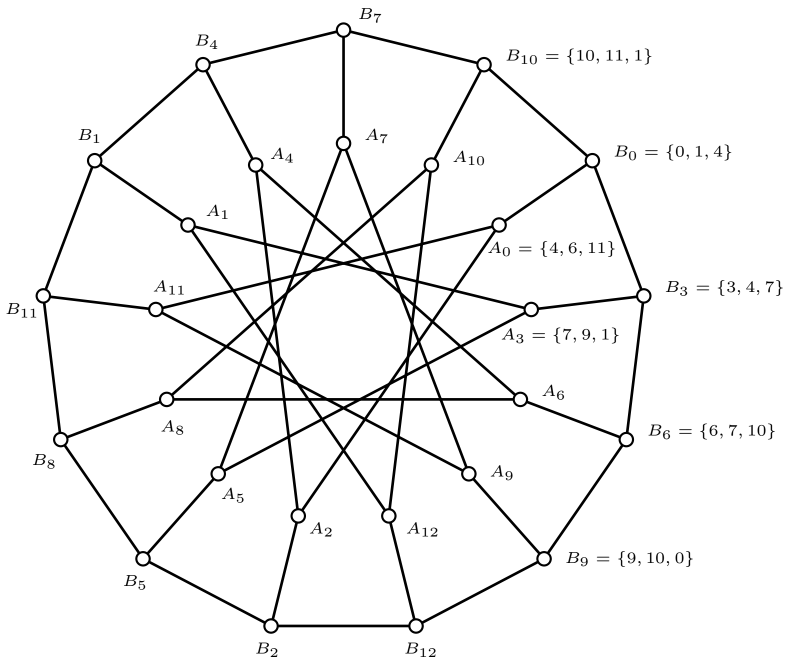

Figure 2: A $P(13, 5)$ subgraph in the block intersection graph of a cyclic STS(13)

$k \equiv d^{-1}e \pmod v$ and thus the edges that we have described constitute a $P(v, d^{-1}e)$ that is a subgraph of the block intersection graph of the cyclic STS(v). □

As an example of such a design, let $v = 13$ and observe that $\mathcal{B} = \big\{\{0, 1, 4\} + i : i \in \mathbb{Z}_{13}\big\} \cup \big\{\{0, 2, 7\} + i : i \in \mathbb{Z}_{13}\big\}$ is the block set of a cyclic STS(13) having $\{0, 1, 4\}$ and $\{0, 2, 7\}$ as base blocks. For each $i \in \mathbb{Z}_{13}$, let A_i be the block $\{4, 6, 11\} + i$ and similarly let $B_i = \{0, 1, 4\} + i$. The orbital differences arising from A_0 are 2, 5 and 6, while those arising from B_0 are 1, 3 and 4. Using orbital difference $d = 3$ (which is co-prime to v and has multiplicative inverse $d^{-1} \equiv 9$) we obtain a 13-cycle $(B_0, B_3, B_6, \dots, B_{10})$. B_0 is adjacent to nine vertices in the orbit of A_0, one of which is A_0 itself. We thus have a set of 13 edges of the form $\{B_i, A_i\}$, yielding the spokes of a generalised Petersen graph. Since $e = 2$ is an orbital difference for the orbit of A_0, we include in our graph the edge $\{A_{2i}, A_{2(i+1)}\}$ for each $i \in \mathbb{Z}_{13}$. The result is the $P(13, 5)$ shown in Figure 2.

To foreshadow results that are yet to come, we can continue with this example and demonstrate how to obtain a 1-factorisation of the block intersection graph of this cyclic STS(13). Since $P(13, 5)$ is a generalised Petersen graph other than the Petersen graph, then we can properly colour its edges with three colours, say 1, 2 and 3. Using orbital differences $d = 1$ and $e = 6$, along with a 1-factor of spokes such as edges of the form $\{A_i, B_{i+3}\}$ we obtain a $P(13, 6)$ which we can edge-colour with colours 4, 5 and 6. The orbital differences $d = 4$ and $e = 5$, along with edges of the form $\{A_i, B_{i+7}\}$ produce another $P(13, 6)$ which we can edge-colour with colours 7, 8 and 9. The remaining uncoloured edges from the block intersection graph of the cyclic STS(13) induce a 6-regular bipartite graph, which can be properly edge-coloured with colours 10 through 15.

Observe that any cyclic STS(v) for which $v \equiv 1 \pmod{12}$ has $\{1, 2, \ldots, \frac{v-1}{2}\}$ as its set of orbital differences. We will want to partition these $\frac{v-1}{2}$ differences into $\frac{v-1}{4}$ pairs of differences, and for each such pair we will want to construct a generalised Petersen graph of order $2v$. The hypothesis of Lemma 3.1 requires that one of the two orbital differences of the pair be co-prime to v. If we let φ denote Euler's totient function then the proportion of the orbital differences that are co-prime to v is $\frac{\varphi(v)}{v-1}$. It is necessary that $\frac{\varphi(v)}{v-1}$ be at least one half in order for our approach to showing that a cyclic STS(v) has a Class 1 block intersection graph to work. We are able to show that $\frac{\varphi(v)}{v-1} \geqslant \frac{2}{3}$ is sufficient.

Theorem 3.1 *Any cyclic STS(v) with $v \equiv 1 \pmod{12}$ and $\frac{\varphi(v)}{v-1} \geqslant \frac{2}{3}$ has a Class 1 block intersection graph.*

Proof. First note that when $v \equiv 1 \pmod{12}$ the blocks of a cyclic STS(v) give rise to $N = \frac{v-1}{6}$ distinct orbits. Observe that N is even and consider the complete graph K_N in which each vertex denotes an orbit of the STS. Since N is even, K_N admits a 1-factorisation in which each 1-factor consists of a set of pairs of orbits.

If, for just one of these 1-factors, say $\mathcal{F}_0$, it is the case that each pair of orbits has at least three (of its six) orbital differences that are co-prime to v then we can use the technique of Lemma 3.1 to construct three edge-disjoint generalised Petersen graphs for each pair of orbits corresponding to an edge of the 1-factor. For the first of the three generalised Petersen graphs constructed from each such pair of orbits, a proper edge-colouring with colours 1, 2 and 3 is possible. For the second (resp. third) generalised Petersen graph arising from each of these pair of orbits, colours 4, 5 and 6 (resp. 7, 8 and 9) can be used in a proper edge-colouring. The remaining edges between these pairs of orbits induce a 6-regular bipartite graph, which is Class 1 by Theorem 2.1 and hence can be properly edge-coloured with colours 10 through 15. Let $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_{N-2}$ denote the remaining 1-factors in the 1-factorisation of K_N. Since all edges that join vertices of the same orbit have already been coloured (with one of the colours in the set $\{1, 2, \ldots, 9\}$), then each edge of these 1-factors corresponds to a pair of orbits for which the uncoloured edges induce a 9-regular bipartite graph. For each $i \in \{1, 2, \ldots, N-2\}$ and each edge of $\mathcal{F}_i$, we use the colours of $\{9i+7, 9i+8, \ldots, 9i+15\}$ to properly colour the edges of these 9-regular bipartite graphs. The result is a Class 1 colouring for the block intersection graph of the STS.

All that now remains is to prove that there is some partition of the N orbits into $\frac{N}{2}$ pairs such that at least three of the orbital differences of each of these pairs are co-prime to v (*i.e.*, we need to prove that there is a way to select the initial 1-factor $\mathcal{F}_0$ of K_N).

For each $j \in \{0, 1, 2, 3\}$ let c_j denote the number of orbits having exactly j orbital differences that are co-prime to v. Clearly $c_0 + c_1 + c_2 + c_3 = N$. Since each orbit with no differences that are co-prime to v must be paired with an orbit having three differences that are co-prime to v, and each orbit with one such difference must be paired with an orbit having at least two, it is readily evident that the following two conditions are necessary for a suitable 1-factor $\mathcal{F}_0$ to exist:

$$c_3 \geqslant c_0 \tag{1}$$

$$c_2 + (c_3 - c_0) \geqslant c_1 \tag{2}$$

Moreover, these two conditions are sufficient since the c_0 orbits having no differences that are co-prime to v can be arbitrarily paired with c_0 of the c_3 orbits having three co-prime differences. The c_1 orbits with one co-prime difference can then be arbitrarily paired with c_1 of the remaining $c_2 + (c_3 - c_0)$ unpaired orbits that have at least two co-prime differences. And lastly, all of the remaining unpaired orbits can be arbitrarily paired together, yielding a suitable 1-factor $\mathcal{F}_0$.

Recall that $\frac{\varphi(v)}{v-1}$ equals the proportion of orbital differences that are co-prime to v, and thus $\frac{c_1+2c_2+3c_3}{3N} = \frac{\varphi(v)}{v-1}$. And since $\frac{\varphi(v)}{v-1} \geqslant \frac{2}{3}$, it follows that

$$c_3 \geqslant 2c_0 + c_1 \tag{3}$$

Clearly $2c_0 + c_1 \geqslant c_0$, and so condition (1) is satisfied. Statement (3) also implies that $c_1 - c_3 \leqslant -2c_0$, and by observing that $-2c_0 \leqslant c_2 - c_0$ holds if and only if $0 \leqslant c_2 + c_0$ (which is always true), we find that $c_1 - c_3 \leqslant c_2 - c_0$ and so condition (2) is satisfied. □

While Theorem 3.1 shows that Conjecture 1.3 holds for any cyclic STS(v) with $v \equiv 1 \pmod{12}$ and $\frac{\varphi(v)}{v-1} \geqslant \frac{2}{3}$, we note that our technique of decomposing the block intersection graph of a cyclic STS into generalised Petersen graphs and bipartite graphs can also often be used when $\frac{1}{2} \leqslant \frac{\varphi(v)}{v-1} < \frac{2}{3}$. The crucial requirement is that the orbital differences be distributed among the orbits of the STS in such a manner that conditions (1) and (2) are satisfied. Cyclic Steiner triple systems for which $\frac{1}{2} \leqslant \frac{\varphi(v)}{v-1} < \frac{2}{3}$ do exist, although they appear to be somewhat sporadic. The smallest order $v \equiv 1 \pmod{12}$ for which $\frac{\varphi(v)}{v-1} < \frac{2}{3}$ is $v = 385$, for which $\frac{\varphi(v)}{v-1} = \frac{5}{8}$.

Cyclic Steiner triple systems for which our technique is certain to fail also exist, namely those for which fewer than half of the orbital differences are co-prime to v. The smallest such order $v \equiv 1 \pmod{12}$ is $v = 37182145$, for which $\frac{\varphi(v)}{v-1} = \frac{95040}{193657}$. If Conjecture 1.3 holds in general, then there must be some means of obtaining a Class 1 edge-colouring of the block intersection graph of a cyclic STS(37182145) other than by the method that we have described.

Interestingly, both 385 and 37182145 are products of consecutive prime numbers. We now investigate some of the conditions on v that pertain to when $\frac{\varphi(v)}{v-1}$ satisfies the hypothesis of Theorem 3.1. Note in particular that if $\frac{\varphi(v)}{v} \geqslant \frac{2}{3}$ then $\frac{\varphi(v)}{v-1} > \frac{2}{3}$. If v has prime factorisation $v = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_t^{\alpha_t}$ then $\frac{\varphi(v)}{v} = (1 - \frac{1}{p_1})(1 - \frac{1}{p_2}) \cdots (1 - \frac{1}{p_t})$, which is independent of the exponents $\alpha_1, \alpha_2, \ldots, \alpha_t$ of the prime factorisation. We can therefore concentrate our attention on the prime factors of v.

Corollary 3.1 *If $v \equiv 1 \pmod{12}$ is a prime power, then any cyclic STS(v) has a Class 1 block intersection graph.*

Proof. As already observed, it suffices to prove that $\frac{\varphi(v)}{v} \geqslant \frac{2}{3}$. Let $v = p^\alpha$ and note that since $v \equiv 1 \pmod{12}$ then $p \neq 2$ and $p \neq 3$. Hence $\frac{\varphi(v)}{v} = 1 - \frac{1}{p} \geqslant 1 - \frac{1}{5} > \frac{2}{3}$. □

Corollary 3.2 *If $v \equiv 1 \pmod{12}$ has only two prime divisors, then any cyclic STS(v) has a Class 1 block intersection graph.*

Proof. Suppose $v = p_1^{\alpha_1} p_2^{\alpha_2}$ where $p_1 < p_2$. Then $p_1 \geqslant 5$ and $p_2 \geqslant 7$. Therefore $\frac{\varphi(v)}{v} \geqslant (1 - \frac{1}{5})(1 - \frac{1}{7}) > \frac{2}{3}$. □

Corollary 3.3 *If $v \equiv 1 \pmod{12}$ has only three prime divisors, and these three divisors are not one of the trios in the set $\mathcal{T} = \big\{\{5, 7, 11\}, \{5, 7, 13\}, \{5, 7, 17\}, \{5, 7, 19\}, \{5, 7, 23\}, \{5, 7, 29\}, \{5, 7, 31\}\big\}$, then any cyclic STS($v$) has a Class 1 block intersection graph.*

Proof. Observe that if v were to equal $p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3}$ where $\{p_1, p_2, p_3\} \in \mathcal{T}$, then $\frac{\varphi(v)}{v}$ would be less than $\frac{2}{3}$, which would not imply the hypothesis of Theorem 3.1. However, with these cases excluded from consideration then $\frac{\varphi(v)}{v} > \frac{2}{3}$. □

4 Cyclic Steiner triple systems of order $v \equiv 9 \pmod{12}$

In this section we will prove that all cyclic Steiner triple systems of order $v \equiv 9 \pmod{12}$ have Class 1 block intersection graphs. Similar to cyclic Steiner triple systems of order $v \equiv 1 \pmod{12}$, we will find a decomposition of the block intersection graph into several subgraphs that themselves are Class 1.

Cyclic Steiner triple systems of order $v \equiv 9 \pmod{12}$ can be generated from a set of base blocks which are a solution of Heffter's second difference problem, through the repeated application of a permutation of the points of V. Recall that Heffter's second difference problem is as follows: given $v = 6n + 3$ is it possible to partition the set $\{1, 2, \ldots, \frac{v-1}{2} = 3n + 1\} \setminus \{2n + 1\}$ into n triples $\{x, y, z\}$ such that $x + y = \pm z \pmod{v}$? For $n \in \mathbb{Z}$ such a partition is always possible [21] . Given a solution to Heffter's second difference problem, one can construct a difference triple $\{0, x, x + y\}$ from each triple $\{x, y, z\}$, and then the set of all difference triples along with $\{0, 2n + 1, 4n + 2\}$ serve as a collection of base blocks. If $\mathcal{B}(v)$ is a collection of base blocks obtained from a solution of Heffter's second difference problem, then a cyclic STS(v) $(\{0, 1, \ldots, v - 1\}, \mathcal{T})$ of order $v \equiv 9 \pmod{12}$ is constructed as follows:

$$\mathcal{T} = \Big\{\{i, x + i, x + y + i\} \mid 0 \leqslant i \leqslant v - 1, \{0, x, x + y\} \in \mathcal{B}(v)\Big\} \\ \bigcup \Big\{\{i, 2n + 1 + i, 4n + 2 + i\} \mid 0 \leqslant i \leqslant 2n\Big\}.$$

For each base block $B \in \mathcal{B}(v)$, we call the set of blocks it generates a full orbit and for the base block $\{0, 2n + 1, 4n + 2\}$, the set of blocks it generates is called a short orbit.

Given a set of blocks $\mathcal{B}$, we denote by $G_{\mathcal{B}}$ the block intersection graph induced by the blocks of $\mathcal{B}$. If $\hat{\mathcal{B}} \subseteq \mathcal{B}$, then $G_{\mathcal{B}}[\hat{\mathcal{B}}]$ denotes the subgraph of $G_{\mathcal{B}}$ induced by $\hat{\mathcal{B}}$.

Lemma 4.1 *Let $(V, \mathcal{B})$ be a cyclic STS(v) of order $v \equiv 9 \pmod{12}$ and d be an orbital difference of the full orbit $\mathcal{O} = \{B_0, B_1, \ldots, B_{v-1}\}$. Then any cycle in $G_{\mathcal{B}}[\mathcal{O}]$ of the form $(B_i, B_{i+d}, \ldots, B_{i+(\ell-1)d})$, where $B_{i+\ell d} = B_i$, is of length $\ell = \frac{v}{\gcd(v,d)}$ and the number of cycles of this form is* $\gcd(v, d)$.

Proof. Without loss of generality we may assume that $i = 0$ and let $B_0, B_d, \ldots, B_{(\ell-1)d}$ be distinct elements and $B_{\ell d}$ be the first element such that $B_{\ell d} = B_{jd}$ for some $0 \leqslant j \leqslant \ell - 1$. Suppose that $B_{\ell d} = B_{jd}$ for some $1 \leqslant j \leqslant \ell - 1$. Then we have $\ell d \equiv jd \pmod v$ and as a result, $(\ell - j)d \equiv 0 \pmod v$ which yields $B_{(\ell-j)d} = B_0$ where $0 \leqslant \ell - j \leqslant \ell - 1$; this is a contradiction. Therefore, $B_{\ell d} = B_0$ and consequently $\ell d \equiv 0 \pmod v$. So ℓ is the least integer such that ℓd is a multiple of v, which is clearly $\frac{v}{\gcd(v,d)}$. Hence the number of cycles of this form is $\frac{v}{\ell} = \gcd(v, d)$. □

Theorem 4.1 *Any cyclic STS(v) with $v \equiv 9 \pmod{12}$ has a Class 1 block intersection graph.*

Proof. First, observe that in a cyclic STS(v) with $v \equiv 9 \pmod{12}$ the number of points and the number of blocks are $v = 6n+3$ and $b = n(6n+3)+(2n+1)$, where n is an odd integer. Hence we have n full orbits of size $v = 6n + 3$ and one short orbit of size $\frac{v}{3} = 2n + 1$, which give rise to $N = n + 1$ distinct orbits. Let $\mathcal{O}_0, \mathcal{O}_1, \ldots, \mathcal{O}_{N-2}$ be the full orbits and $\mathcal{S}$ be the short orbit. We construct a complete graph K_N on N vertices where each vertex represents an orbit of the STS. Since N is even, K_N admits a 1-factorisation and hence we can partition the set of all pairs of orbits into $N - 1$ sets $\mathcal{F}_0, \ldots, \mathcal{F}_{N-2}$ such that each orbit occurs exactly once in each set. For each one of these 1-factors, we have $\frac{N}{2} - 1$ pairs of full orbits and one pair consisting of a full orbit and a short orbit.

Let $\mathcal{F}$ be a 1-factor of K_N. If $(\mathcal{O}_i, \mathcal{O}_j) \in \mathcal{F}$, then we define $G_{\mathcal{F}}[\mathcal{O}_i, \mathcal{O}_j]$ to be the graph on vertex set $\mathcal{O}_i \cup \mathcal{O}_j$ and edge set $E(G_{\mathcal{F}}[\mathcal{O}_i, \mathcal{O}_j]) = \{\{B, \hat{B}\} \mid B \in \mathcal{O}_i, \hat{B} \in \mathcal{O}_j, B \cap \hat{B} \neq \emptyset\}$.

Let $G_{\mathcal{F}} = \Big(\sum_{(\mathcal{O}_i, \mathcal{O}_j) \in \mathcal{F}} G_{\mathcal{F}}[\mathcal{O}_i, \mathcal{O}_j]\Big) + G_{\mathcal{B}}[\mathcal{O}, \mathcal{S}]$, where $(\mathcal{O}, \mathcal{S}) \in \mathcal{F}$. For each $\mathcal{F}$, we show how to colour $G_{\mathcal{F}}$, starting with $G_{\mathcal{F}_0}$.

Without loss of generality we may assume that $\mathcal{F}_0 = \{(\mathcal{O}_0, \mathcal{O}_1), \ldots, (\mathcal{O}_{N-4}, \mathcal{O}_{N-3}), (\mathcal{O}_{N-2}, \mathcal{S})\}$. To colour $G_{\mathcal{F}_0}$, we first properly colour the edges in $G_{\mathcal{F}_0}[\mathcal{O}_0, \mathcal{O}_1], \ldots, G_{\mathcal{F}_0}[\mathcal{O}_{N-4}, \mathcal{O}_{N-3}]$, which induce a 9-regular bipartite graph and hence is 9-edge-colourable by Theorem 2.1, using colours $1, 2, \ldots, 9$. If it is possible to colour the 9-regular subgraph $G_{\mathcal{B}}[\mathcal{O}_{N-2}, \mathcal{S}]$ using the same nine colours, then we can similarly colour each $G_{\mathcal{F}_i}$ properly, where $i \in \{1, 2, \ldots, N-2\}$ using colours $\{9i + 1, 9i + 2, \ldots, 9i + 9\}$. All that remains is to colour $G_{\mathcal{B}}[\mathcal{O}_{N-2}, \mathcal{S}]$ with nine colours.

Now, suppose that $\mathcal{O}_{N-2} = \{B_0, B_1, \ldots, B_{v-1}\}$ with orbital differences d_1, d_2, and d_3 and $\mathcal{S} = \{A_0, A_1, \ldots, A_{\frac{v}{3}-1}\}$. As we showed in Lemma 4.1, for each orbital difference d_k, $k \in \{1, 2, 3\}$, there exist $\gcd(v, d_k)$ cycles of length $\ell_k = \frac{v}{\gcd(v,d_k)}$ of the form $(B_i, B_{i+d_k}, \ldots, B_{i+(\ell_k-1)d_k})$ in $G_{\mathcal{F}_0}[\mathcal{O}_{N-2}]$. In $G_{\mathcal{B}}[\mathcal{O}_{N-2}, \mathcal{S}]$, each B_i, $i \in \{0, 1, \ldots, v-1\}$ is adjacent to three vertices in $\mathcal{S}$. Let A_i be one of these three neighbours of B_i. We then obtain a set of ℓ_k edges $\{B_i, A_i\}, \{B_{i+d_k}, A_{i+d_k}\}, \ldots, \{B_{i+(\ell_k-1)d_k}, A_{i+(\ell_k-1)d_k}\}$, where the indices of the blocks of the full orbit are reduced modulo v and the indices of the blocks of the short orbit are reduced modulo $\frac{v}{3}$. We call each cycle $(B_i, B_{i+d_k}, \ldots, B_{i+(\ell_k-1)d_k})$ along with the set of edges $\{B_i, A_i\}, \{B_{i+d_k}, A_{i+d_k}\}, \ldots, \{B_{i+(\ell_k-1)d_k}, A_{i+(\ell_k-1)d_k}\}$ a configuration. Let G_{d_k} be the subgraph of $G_{\mathcal{B}}[\mathcal{O}_{N-2}, \mathcal{S}]$ induced by the orbital difference d_k, $k \in \{1, 2, 3\}$. Clearly, we have $\gcd(v, d_k)$ configurations for orbital difference d_k which decompose G_{d_k} into edge-disjoint subgraphs. Since the indices of the blocks of the short orbit are reduced modulo $\frac{v}{3}$, each block of the short orbit occurs thrice among the $\gcd(v, d_k)$ configurations. Hence G_{d_k} is a 3-regular graph which has a decomposition into edge-disjoint configurations. It now suffices to properly colour G_{d_1} using colours 1,2,3 and G_{d_2} using colours 4,5,6 and finally G_{d_3} using colours 7,8,9. Note that

$$V_1 = \{B_0, B_1, \ldots, B_{v-1}, A_0, A_1, \ldots, A_{\frac{v}{3}-1}\}$$

is the set of vertices and

$$E_1 = \{\{B_0, A_0\}, \{B_{d_1}, A_{d_1}\}, \ldots, \{B_{(\ell_1-1)d_1}, A_{(\ell_1-1)d_1}\}, \{B_1, A_1\}, \{B_{1+d_1}, A_{1+d_1}\}, \ldots, \{B_{1+(\ell_1-1)d_1}, A_{1+(\ell_1-1)d_1}\}, \ldots, \{B_n, A_n\}, \{B_{n+d_1}, A_{n+d_1}\}, \ldots, \{B_{n+(\ell_1-1)d_1}, A_{n+(\ell_1-1)d_1}\}\} \cup \{(B_0, B_{d_1}, \ldots, B_{(\ell_1-1)d_1}), (B_1, B_{1+d_1}, \ldots, B_{1+(\ell_1-1)d_1}), \ldots, (B_n, B_{n+d_1}, \ldots, B_{n+(\ell_1-1)d_1})\}$$

is the set of edges of G_{d_1}, where

$$(B_0, B_{d_1}, \ldots, B_{(\ell_1-1)d_1}), (B_1, B_{1+d_1}, \ldots, B_{1+(\ell_1-1)d_1}), \ldots, (B_n, B_{n+d_1}, \ldots, B_{n+(\ell_1-1)d_1})$$

are the $\gcd(v, d_1)$ cycles of length $\ell_1 = \frac{v}{\gcd(v,d_1)}$, and $n = \gcd(v, d_1) - 1$.

Suppose $A_0, A_{d_1}, \ldots, A_{(\ell'_1-1)d_1}$ are distinct elements and $A_{\ell'_1 d_1} = A_0$. It is easy to see that $\ell'_1 = \frac{\frac{v}{3}}{\gcd(\frac{v}{3}, d_1)}$. Note that we have two cases: 1) $\ell'_1 = \ell_1$ and 2) $\ell'_1 < \ell_1$.

Case 1 ($\ell'_1 = \ell_1$): If $\ell'_1 = \ell_1 = \frac{v}{3}$, then there will be three configurations such that each A_j, $j \in \{0, 1, \ldots, \frac{v}{3} - 1\}$, is adjacent to exactly one block in each configuration. Since the configuration is a cycle with pendant edges, it is easy to colour the first configuration using three colours 1, 2, and 3. To colour the second configuration, first colour the edges incident to each A_j, $j \in \{0, 1, \ldots, \frac{v}{3} - 1\}$. If the edge incident to A_j is coloured by colour i in the first configuration, then colour the edge incident to A_j in the second configuration by colour $\sigma(i)$, where σ is the permutation $(1, 2, 3)$. The remaining edges in the second configuration can be easily coloured properly using colours 1, 2, and 3. Similarly, to colour the third configuration, we permute the colours in the first configuration by $\sigma^2 = (1, 3, 2)$. Since each vertex A_i

has received a distinct colour in each of the three configurations, then G_{d_1} has been properly 3-edge-coloured.

If $\ell_1' = \ell_1 < \frac{v}{3}$, then there will be more than three configurations. Note that we call the union of configurations in which each block of the short orbit occurs exactly once a layer. Clearly, there are three layers. Since in every layer, each A_j, $j \in \{0, 1, \ldots, \frac{v}{3} - 1\}$, occurs exactly once and a layer is a union of configurations, we can easily colour the layer using three colours 1, 2, and 3. To colour the second and the third layers, again permute the colours in the first layer by $\sigma = (1, 2, 3)$ and $\sigma^2 = (1, 3, 2)$ respectively.

Case 2 ($\ell_1' < \ell_1$): In this case we have $n + 1$ configurations $\mathcal{C}_1, \ldots, \mathcal{C}_{n+1}$ where $n+1 \geqslant 3$ and a partition of the set of the blocks of the short orbit into $n+1$ subsets $\mathcal{A}_1, \ldots, \mathcal{A}_{n+1}$ such that each block in the set $\mathcal{A}_i = \{A_{i_1}, \ldots, A_{i_{\ell_1'}}\}$ occurs three times in exactly one of the configurations, say $\mathcal{C}_i$, $i \in \{1, \ldots, n\}$. In other words, each block in $\mathcal{A}_i$ has degree three in $\mathcal{C}_i$ and degree zero in the other configurations. To colour the configuration $\mathcal{C}_i$, first colour the three edges incident to each block in $\mathcal{A}_i$ by colours 1, 2, and 3. It is then straightforward to colour the remaining edges in the configuration properly using colours 1, 2, and 3.

Similarly, it is possible to colour G_{d_2} using colours 4, 5, 6 and G_{d_3} using colours 7, 8, 9. □

5 Closing Remarks

It has long been known that STS block intersection graphs belong to the family of strongly regular graphs [22]. Specifically, a graph is said to be *strongly regular* if it is regular, each pair of adjacent vertices has a constant number λ of common neighbours and each pair of nonadjacent vertices has a constant number μ of common neighbours. As noted by Cioabă and Li in [9], chromatic indices of strongly regular graphs of even order are not yet well understood. Should Conjecture 1.3 be true, then perhaps a broader conjecture that applies to strongly regular graphs more generally might also hold.

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