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Total coloring of the corona product of two graphs

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Abstract

A total coloring of a graph is an assignment of colors to all the elements (vertices and edges) of the graph such that no two adjacent or incident elements receive the same color. In this paper, we prove the tight bound of the Behzad and Vizing conjecture on total coloring for the corona product of two graphs G and H, when H is a cycle, a complete graph or a bipartite graph.

1 Introduction

All graphs considered here are finite, simple and undirected. Let $G = (V(G), E(G))$ be a graph with vertex set $V(G)$ and edge set $E(G)$. A *total coloring* of G is a mapping $f : V(G) \cup E(G) \to C$, where C is a set of colors, satisfying the following three conditions (a)–(c):

(a) $f(u) \neq f(v)$ for any two adjacent vertices $u, v \in V(G)$;

(b) $f(e) \neq f(e')$ for any two adjacent edges $e, e' \in E(G)$; and

(c) $f(v) \neq f(e)$ for any vertex $v \in V(G)$ and any edge $e \in E(G)$ incident to v.

The *total chromatic number* of a graph G, denoted by $\chi''(G)$, is the minimum number of colors that suffice in a total coloring. It is clear that $\chi''(G) \geq \Delta(G) + 1$, where $\Delta(G)$ is the maximum degree of G. Behzad [1] and Vizing [14] conjectured (Total Coloring Conjecture (TCC)) that for every graph G, $\Delta(G) + 1 \leq \chi''(G) \leq \Delta(G) + 2$. If a graph G is total colorable with $\Delta(G) + 1$ colors then the graph is called type-I, and if it is total colorable with $\Delta(G) + 2$ colors, then it is type-II. This conjecture was verified by Rosenfeld [12] and Vijayaditya [13] for $\Delta(G) = 3$ and by Kostochka [9, 10, 11] for $\Delta(G) \leq 5$. For planar graphs, the conjecture was verified by Borodin [2] for $\Delta(G) \geq 9$. In 1992, Yap and Chew [15] proved that any graph G has a total coloring with at most $\Delta(G) + 2$ colors if $\Delta(G) \geq |V(G)| - 5$. In 1993, Hilton and Hind [6] proved that any graph G has a total coloring with at most $\Delta(G) + 2$ colors if $\Delta(G) \geq \frac{3}{4}|V(G)|$. Zmazek and Žerovnik [16] proved that if G and H are total colorable graphs then their Cartesian product $G \square H$ is also total colorable. It is known that the total coloring problem, which asks to find a total coloring of a given graph G with the minimum number of colors, is NP-hard [3]. In particular, McDiarmid and Arroyo [3] proved that the problem of determining the total coloring of μ-regular bipartite graph is NP-hard, $\mu \geq 3$.

The *corona product* of G and H is the graph $G \circ H$ obtained by taking one copy of G, called the center graph, $|V(G)|$ copies of H, called the outer graph, and making the i^{th} vertex of G adjacent to every vertex of the i^{th} copy of H, where $1 \leq i \leq |V(G)|$. This graph product was introduced by Frucht and Harary [4] in 1970. The following theorems are due to Yap [15].

Theorem 1.1. *Let K_n be the complete graph. Then* $\chi''(K_n) = \begin{cases} n, & \text{if } n \text{ is odd} \\ n+1, & \text{if } n \text{ is even.} \end{cases}$

Theorem 1.2. *Let C_n be the cycle graph. Then* $\chi''(C_n) = \begin{cases} 3, & \text{if } n \equiv 0 \mod 3 \\ 4, & \text{otherwise.} \end{cases}$

In this paper, we prove the tight bound for the total coloring of the corona product of G and H where H is a cycle, a complete graph, or a bipartite graph. Here, we prove that $G \circ H$ is a type-I graph where H is a cycle, a complete graph, or a bipartite graph, and this is independent of G and H being a type-I or type-II graph.

2 Corona Product

Let G and H be two graphs. The corona product of G and H, denoted by $G \circ H$, was defined in the previous section. Several authors have developed diverse theoretical works on the corona product. Equitable colorings of the corona multiproducts of graphs was found by Furmańczyk [5]. The concept of the corona product has some applications in chemistry for representing chemical compounds [7]. Other applications of this concept include navigation of robots in networks [8]; or every time we have to divide a system with binary conflict relations into equal or almost equal conflict-free subsystems.

The corona product is not commutative. For example, $K_3 \circ P_2 \not\simeq P_2 \circ K_3$ since the number of vertices differs. Also the corona product is not associative. Figure 1 shows a total coloring of $K_4 \circ P_3$. It is easy to prove that $\chi''(G \circ H) \leq \Delta(G \circ H) + 2$ for all G and H for the corona product. We are interested in proving the tight bound of the Behzad–Vizing conjecture for the corona product of certain classes of graphs.

Theorem 2.1. *For any total colorable graph G and a cycle C_n, $n \geq 3$,*

$$\chi''(G \circ C_n) = \Delta(G \circ C_n) + 1.$$

Proof. Let G be a total colorable graph with m vertices and let C_n be a cycle with n vertices. The maximum degree of $G \circ C_n$ is $\Delta(G) + n$. We give a total coloring of $G \circ C_n$ by distinguishing two cases.

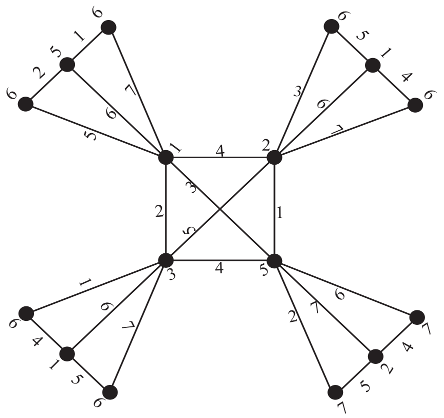

Fig. 1: Total coloring of $K_4 \circ P_3$.

Let $C = \{1, 2, 3, \ldots, \Delta(G) + 1, \Delta(G) + 2, \ldots, \Delta(G) + n + 1\}$ be a set of colors with $\Delta(G) + n + 1$ colors.

Case (i): G is Type-I.

Color all the elements of G using first $\Delta(G) + 1$ colors. Assign $\Delta(G) + 2, \Delta(G) + 3, \ldots, \Delta(G) + n + 1$ colors to the edges between the vertices of C_n and a vertex of G. Take three different colors c_1, c_2, c_3 from the total coloring of G, with c_1 being a vertex color in G. Now assign a coloring to the vertices and edges of C_n with $c_2, c_3, \Delta(G) + 2$ cyclically starting from the vertex v_0. For the n^{th} vertex of C_n, assign the color $\Delta(G) + 3$, and assign the color c_1 to the edge between first and n^{th} vertex in C_n.

Case (ii): G is Type-II.

Color all the elements of G using first $\Delta(G)+2$ colors. Since the maximum degree of G is $\Delta(G)$, at each vertex in G there will be at least one missing color. Let c_1 be one of the missing colors at a vertex. Now assign $c_1, \Delta(G) + 3, \ldots, \Delta(G) + n + 1$ colors to the edges between the vertices of C_n and a vertex of G. Take three different colors c_1, c_2, c_3 from G, with c_2 being a vertex color in G. Now assign a coloring to the elements of C_n with i $c_1, c_3, \Delta(G)$ cyclically starting from the vertex v_0. For the n^{th} vertex, assign $\Delta(G) + 4$, and use the color c_2 between the first and n^{th} vertices of C_n.

Therefore, $\chi''(G \circ C_n) = \Delta(G \circ C_n) + 1$. □

Theorem 2.2. *For any total colorable graph G and any complete graph K_n,*

$$\chi''(G \circ K_n) = \Delta(G \circ K_n) + 1.$$

Proof. Let G be a total colorable graph with m vertices and let K_n be a complete graph with n vertices. The maximum degree of $G \circ K_n$ is $\Delta(G) + n$. We give a total coloring of $G \circ K_n$ by taking a total coloring of G in to two cases.

Let $C = \{1, 2, 3, \ldots, \Delta(G) + 1, \Delta(G) + 2, \ldots, \Delta(G) + n + 1\}$ be a set of colors, with $\Delta(G) + n + 1$ colors.

Case (i): G is Type-I.

Sub-case (i): n is odd.

Color all the elements of G using first $\Delta(G)+1$ colors. Let us consider the remaining n colors $\Delta(G)+2, \Delta(G)+3, \ldots, \Delta(G)+n+1$. From the assignment of colors of the elements of G, choose two colors c_1 and c_2. We need exactly n colors to color K_n. Consider the set of colors

$$C_1 = \{c_1, c_2, c_3 = \Delta(G)+2, c_4 = \Delta(G)+3, \ldots, c_{n+2} = \Delta(G)+n+1\}.$$

Now we take these $n+2$ colors to give colors to the elements of K_n in the following way.

Pilśniak and Woźniak [10] introduced a *proper total colorings distinguishing adjacent vertices by sums*. Here, it is easy to see that these $n+2$ colors are equidistant on a circle of K_n. Let $v_1, v_2, \ldots, v_n$ be the vertices of K_n; we denote by $c(v_i)$ the color of the vertex v_i, and we denote by $c(v_iv_j)$ the color of the edge v_iv_j. In the first step, we color all edges incident with v_1 such that the color of $c(v_1v_i) = c_i$, for $i = 2, 3, \ldots, n$ and the vertex v_1, $c(v_1) = c_1$. Next we consider the vertex v_2: one edge is already colored with c_2, so we put $c(v_2) = c_3$ and $c(v_2v_i) = c_{i+1}$, for $i = 3, 4, \ldots n$. In general, $c(v_j) = c_j$, and $c(v_jv_i) = c_{(j+i-1) \bmod (n+2)}$, for $i = j+1, \ldots n$. This gives a proper total coloring of K_n using $n+2$ colors. Now at each vertex in K_n we have exactly two missing colors. From these missing colors we give distinct colors to the edges between a vertex in G and K_n. If the color c_1 or c_2 is a vertex color in G then we make a shift with $i \to (i+1) \mod (n+2)$ in colors to color K_n.

Sub-case (ii): n is even.

Color all the elements of G using first $\Delta(G)+1$ colors from the color class. Let us consider the remaining n colors $\Delta(G)+2, \Delta(G)+3, \ldots, \Delta(G)+n+1$. From the assignment of colors of the elements of G, choose one color c_1. We need exactly $n+1$ colors to color K_n. Consider the set of colors

$$C_2 = \{c_1, c_2 = \Delta(G)+2, c_3 = \Delta(G)+3, \ldots, c_{n+1} = \Delta(G)+n+1\};$$

$n+1$ colors are equidistant on a circle of K_n. Now we take these $n+1$ colors to give colors to the elements of K_n, in the following way:

Let $v_1, v_2, \ldots, v_n$ be the vertices of K_n; we denote by $c(v_i)$ the color of the vertex v_i and by $c(v_iv_j)$ the color of the edge v_iv_j. In the first step, we color all edges incident with v_1 so that the color $c(v_1v_i) = c_i$, for $i = 2, 3, \ldots, n$, and a vertex v_1 with color c_1. Next we consider v_2: one edge is already colored with c_2, so we put $c(v_2) = c_3$ and $c(v_2v_i) = c_{i+1}$, for $i = 3, 4, \ldots n$. In general, $c(v_j) = c_j$, and $c(v_jv_i) = c_{j+i+1 \bmod (n+1)}$, for $i = j+1, \ldots n$. This gives a total coloring of K_n using $n+1$ colors. Now at each vertex in K_n we have exactly one missing color. Using the missing color, we give the colors to the edges between the vertices of G and K_n. If the color c_1 is on a vertex, then we make a shift with $i \to (i+1) \mod (n+1)$ in colors to color K_n.

Case (ii): G is Type-II.

Sub-case (i): n is odd.

Color all the elements of G using first $\Delta(G) + 2$ colors. Let us consider the remaining $n-1$ colors $\Delta(G)+3, \Delta(G)+4, \ldots, \Delta(G)+n+1$. Choose three different colors c_1, c_2, c_3 from the total coloring of G. We need exactly n colors to color K_n. We color the elements of K_n using the set of colors

$$C_3 = \{c_1, c_2, c_3, c_4 = \Delta(G) + 3, \ldots, c_{n+2} = \Delta(G) + n + 1\}.$$

Now, we color the elements of K_n and edges between a vertex and K_n as given in sub-case(i) of case (i).

Sub-case (ii): n is even.

Color all the elements of G using first $\Delta(G) + 2$ colors. Consider the colors $\Delta(G) + 3, \Delta(G) + 4, \ldots, \Delta(G) + n + 1$. Choose two colors c_1 and c_2 from the total coloring of G. We need $n + 1$ colors to color K_n. We color the elements of K_n using the color class $C_4 = \{c_1, c_2, c_3 = \Delta(G) + 3, \ldots, c_{n+1}\Delta(G) + n + 1\}$. We give the color assignment of elements of K_n and color assignment of edges between K_n and a vertex as given in sub-case (ii) of case (i).

Therefore $\chi''(G \circ K_n) = \Delta(G \circ K_n) + 1$. □

Theorem 2.3. *For any total colorable graph G and a complete bipartite graph $K_{m,n}$,*

$$\chi''(G \circ K_{m,n}) = \Delta(G \circ K_{m,n}) + 1.$$

Proof. Let G be a total colorable graph with p vertices and let $K_{m,n}$ be a complete bipartite graph with bipartition $X = \{u_1, u_2, \ldots, u_m\}$ and $Y = \{v_1, v_2, \ldots, v_n\}$. The maximum degree of $G \circ K_{m,n}$ is $\Delta(G) + (m+n)$. We give a total coloring of $G \circ K_{m,n}$ by taking a total coloring of G in two cases.

Let $C = \{1, 2, 3, \ldots, \Delta(G) + 1, \Delta(G) + 2, \ldots, \Delta(G) + m + n + 1\}$ be a set of colors with $\Delta(G) + m + n + 1$ colors.

Case (i): G is Type-I.

Color all the elements of G with $1, 2, 3, \ldots, \Delta(G)+1$ colors. Choose three different colors c_1, c_2, c_3, from the total coloring of G.

Let

$$C_1 = \{c_1, c_2, c_3, c_4 = \Delta(G) + 2, c_5 = \Delta(G) + 3, \ldots, c_{m+n+3} = \Delta(G) + m + n + 1\}.$$

We consider a permutation $\pi(i) = (c_i, c_{i+1}, \ldots, c_{i+n-1})$ on the colors from C_1. Now we color the edges of $K_{m,n}$ which are incident with the vertex u_i with colors in $\pi(i)$. It is easy to observe that the colors assigned to the edges incident with vertices in X are distinct. Finally, there are four colors from the color set C which are not assigned to any of the edges of $K_{m,n}$. From these four colors, assign two colors to vertices in X and Y. Now in each vertex in $K_{m,n}$ there will be $m + 2$ and $n + 2$ (including the remaining two colors) missing colors at each of the vertices of X and Y respectively. Using these missing colors, we color the edges between a vertex in G and $K_{m,n}$.

Case (ii): G is Type-II.

Color all the elements of G with $1, 2, 3, \ldots, \Delta(G)+1$ colors. We choose three different colors c_1, c_2, c_3, from the total coloring of G.

Let

$$C_1 = \{c_1, c_2, c_3, c_4 = \Delta(G)+3, c_5 = \Delta(G)+4 \ldots, c_{m+n+2} = \Delta(G)+(m+n+1)\}.$$

We consider a permutation $\pi(i) = (c_i, c_{i+1}, \ldots, c_{i+n-1})$ on the colors from C_1. Now we color the edges of $K_{m,n}$ which are incident with the vertex u_i with colors in $\pi(i)$. It is easy to observe that the edges incident with vertices in X are distinct. Finally, there are three colors from the color set C which are not assigned to any of the edges of $K_{m,n}$. From these three colors, assign two colors to vertices in X and Y. Now in each vertex in $K_{m,n}$ there will be $m+2$ and $n+2$ missing colors (including the remaining two colors) in X and Y respectively. There are $m+n$ edges incident between a vertex in G and the vertices in $K_{m,n}$. In the above process, we give only $m+n-1$ distinct colors to the edges incident between a vertex in G and the vertices in $K_{m,n}$. For one edge we give the color, which is a missing color either from the vertex of G or vertex of $K_{m,n}$.

Therefore, $\chi''(G \circ K_{m,n}) = \Delta(G \circ K_{m,n}) + 1$. □

Figure 2 shows a total coloring of $K_3 \circ K_{2,3}$.

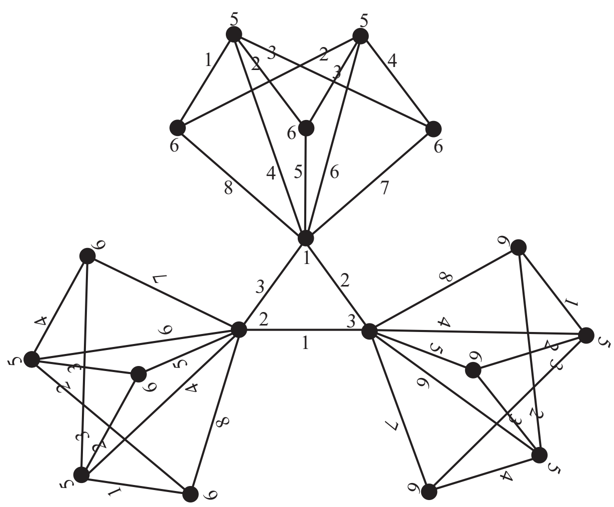

Fig. 2: Total coloring of $K_3 \circ K_{2,3}$.

Corollary 2.1. *For any total colorable graph G and any bipartite graph H,*

$$\chi''(G \circ H) = \Delta(G \circ H) + 1.$$

Proof. Let $X = \{u_1, u_2, \ldots, u_m\}$ and $Y = \{v_1, v_2, \ldots, v_n\}$ be the two partition sets of the vertices of H. Consider the graph $G \circ K_{m,n}$, $G \circ K_{m,n} = \Delta(G) \circ H$. By Theorem 2.3, we can color the elements of $G \circ K_{m,n}$. Now delete the edges from $K_{m,n}$ in $G \circ K_{m,n}$, such that we get $G \circ H$.

Therefore $\chi''(G \circ H) = \Delta(G \circ H) + 1$. □

Corollary 2.2. *For any total colorable graph G and a path P_n,*

$$\chi''(G \circ P_n) = \Delta(G \circ P_n) + 1.$$

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