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On k-total edge product cordial graphs

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Abstract

A k-total edge product cordial labeling is a variant of the well-known cordial labeling. In this paper we characterize graphs admitting a 2-total edge product cordial labeling. We also show that dense graphs and regular graphs of degree $2(k-1)$ admit a k-total edge product cordial labeling.

1 Introduction

We consider finite undirected graphs without loops, multiple edges and isolated vertices. If G is a graph, then $V(G)$ and $E(G)$ stand for the vertex set and edge set of G, respectively. Cardinalities of these sets are called the *order* and *size* of G. The sum of the order and size of G is denoted by $\tau(G)$, i.e., $\tau(G) = |V(G)| + |E(G)|$. The subgraph of a graph G induced by $A \subseteq E(G)$ is denoted by $G[A]$. The set of vertices of G adjacent to a vertex $v \in V(G)$ is denoted by $N_G(v)$. For integers p, q we denote by $[p, q]$ the set of all integers z satisfying $p \leq z \leq q$.

Let k be an integer greater than 1. For a graph G, a mapping $\varphi : E(G) \to [0, k-1]$ induces a vertex mapping $\varphi^* : V(G) \to [0, k-1]$ defined by

$$\varphi^*(v) \equiv \prod_{u \in N_G(v)} \varphi(vu) \pmod{k}.$$

Set $\mu_\varphi(i) := |\{v \in V(G) : \varphi^*(v) = i\}| + |\{e \in E(G) : \varphi(e) = i\}|$ for each $i \in [0, k-1]$. A mapping $\varphi : E(G) \to [0, k-1]$ is called a *k-total edge product cordial* (for short k-TEPC) *labeling* of G if

$$|\mu_\varphi(i) - \mu_\varphi(j)| \leq 1 \quad \text{for all} \;\; i, j \in [0, k-1].$$

A graph that admits a k-TEPC labeling is called a *k-total edge product cordial* (k-TEPC) graph.

The following claim is evident.

Observation 1. *A mapping $\varphi : E(G) \to [0, k-1]$ is a k-TEPC labeling of a graph G if and only if*

$$\left\lfloor \frac{\tau(G)}{k} \right\rfloor \leq \mu_\varphi(i) \leq \left\lceil \frac{\tau(G)}{k} \right\rceil \quad \text{for each} \ \ i \in [0, k-1].$$

A k-total edge product cordial labeling is a version of the well-known cordial labeling defined by Cahit [2]. Vaidya and Barasara [5] introduced the concept of a 2-TEPC labeling as the edge analogue of a total product cordial labeling. They called this labeling the total edge product cordial labeling. In [5, 6] they proved that cycles C_n for $n \neq 4$, complete graphs K_n for $n > 2$, wheels, fans, double fans and some cycle related graphs are 2-TEPC. In [7] they proved that any graph can be embedded as an induced subgraph of a 2-TEPC graph. The concept of k-TEPC graphs was defined by Azaizeh et al. in [1]. They proved that paths P_n for $n \geq 4$, cycles C_n for $3 < n \neq 6$, some trees and some unicyclic graphs are 3-TEPC graphs. We refer the reader to [4] for comprehensive references.

In Section 2 we characterize 2-TEPC graphs. In Section 3 we prove that graphs with sufficiently large size and $2(k-1)$-regular graphs are k-TEPC.

2 2-TEPC graphs

For a graph G, denote by $O(G)$ the set of all integers t such that there is a mapping $\varphi : E(G) \to [0, 1]$ satisfying $\mu_\varphi(0) = t$.

As $\mu_\varphi(0) + \mu_\varphi(1) = \tau(G)$, by Observation 1, we immediately have

Observation 2. *A graph G is 2-total edge product cordial if and only if*

$$\left\{ \left\lfloor \frac{\tau(G)}{2} \right\rfloor, \left\lceil \frac{\tau(G)}{2} \right\rceil \right\} \cap O(G) \neq \emptyset.$$

Lemma 1. *An integer t belongs to $O(G)$ if and only if there is a subset A of $E(G)$ such that $\tau(G[A]) = t$.*

Proof. Suppose that $t \in O(G)$. Then there is a mapping $\varphi : E(G) \to [0, 1]$ such that $\mu_\varphi(0) = t$. Set $A = \{e \in E(G) : \varphi(e) = 0\}$. As $\varphi^*(v) = 0$ whenever v is incident with an edge of A, $\mu_\varphi(0) = \tau(G[A])$.

On the other hand, let A be a subset of $E(G)$. Consider the mapping $\psi : E(G) \to [0, 1]$ defined by

$$\psi(e) = \begin{cases} 0 & \text{when} \ \ e \in A, \\ 1 & \text{when} \ \ e \notin A. \end{cases}$$

Clearly, $\mu_\psi(0) = \tau(G[A])$. Therefore, $\tau(G[A]) \in O(G)$. □

Example 1. If $A \subseteq E(K_3)$ then

$$\tau(G[A]) = \begin{cases} 0 & \text{when} \ \ A = \emptyset, \\ 3 & \text{when} \ \ |A| = 1, \\ 5 & \text{when} \ \ |A| = 2, \\ 6 & \text{when} \ \ |A| = 3. \end{cases}$$

Therefore, $O(K_3) = \{0, 3, 5, 6\}$.

Example 2. If $A \subseteq E(K_{1,n})$ then

$$\tau(G[A]) = \begin{cases} 0 & \text{when } A = \emptyset, \\ 2|A| + 1 & \text{when } A \neq \emptyset. \end{cases}$$

Thus, $O(K_{1,n}) = \{0, 3, 5, \dots, 2n + 1\}$.

Lemma 2. *Let G be a connected graph different from K_3 and $K_{1,n}$. Then*

$$O(G) = [0, \tau(G)] - \{1, 2, 4\}.$$

Proof. Evidently, $\tau(H) \notin \{1, 2, 4\}$ for any graph H without isolated vertices. Thus, according to Lemma 1, $O(G) \cap \{1, 2, 4\} = \emptyset$.

On the other hand, denote by p (q) the order (size) of G. As G is different from K_3 and $K_{1,n}$, it contains a path P of length 3. Denote by e_1, e_2, e_3 the edges of P in such a way that e_1 and e_3 are independent edges of P. Clearly, e_1 and e_3 are independent edges of G. Moreover, there is a spanning tree T of G which contains P. Denote by $e_4, \dots, e_{p-1}$ (if $p > 4$) the edges of $E(T) - \{e_1, e_2, e_3\}$ in such a way that the subgraph of G induced by $\{e_1, \dots, e_j\}$ is a connected graph for each $j \in [1, p-1]$. The other edges of G denote by $e_p, \dots, e_q$ (if $q \geq p$).

Suppose that $t \in [0, \tau(G)] - \{1, 2, 4\}$. According to Lemma 1, it is enough to find a set $A \subseteq E(G)$ such that $\tau(G[A]) = t$. Consider the following cases.

A. $t = 0$. Set $A = \emptyset$. Evidently, $\tau(G[A]) = 0 = t$ in this case.

B. $1 \leq t \leq 2p-1$ and $t \equiv 1 \pmod 2$. Then there is a positive integer s such that $t = 2s + 1$ (clearly, $s \leq p - 1$). Set $A = \{e_1, e_2, \dots, e_s\}$. The graph $G[A]$ is connected and it is a subgraph of T. Thus, it is a tree and so $|E(G[A])| = s$, $|V(G[A])| = s + 1$, i.e., $\tau(G[A]) = t$.

C. $1 < t < 2p-1$ and $t \equiv 0 \pmod 2$. Then there is a positive integer s such that $t = 2s + 2$ (clearly, $2 \leq s < p - 1$ in this case). Set $A = \{e_1, e_3, e_4, \dots, e_{s+1}\}$. The graph which we obtain from $G[A]$ by adding the edge e_2 is a tree. Therefore, $G[A]$ is a forest with two connected components and so $|E(G[A])| = s$, $|V(G[A])| = s + 2$, i.e., $\tau(G[A]) = t$.

D. $t \geq 2p$. Then there is a positive integer s such that $t = s + p$ ($s \geq p$ in this case). Set $A = \{e_1, e_2, \dots, e_s\}$. The graph $G[A]$ is a connected spanning subgraph of G. Thus, $|E(G[A])| = s$, $|V(G[A])| = p$, i.e., $\tau(G[A]) = t$. □

The union of two disjoint graphs G and H is denoted by $G \cup H$ and the union of $m \geq 1$ disjoint copies of a graph G is denoted by mG.

If A is a subset of $E(G \cup H)$ then $A = \big(A \cap E(G)\big) \cup \big(A \cap E(H)\big)$. Thus, according to Lemma 1, we have

Observation 3. *If G and H are disjoint graphs then*

$$O(G \cup H) = \{t + l : t \in O(G), l \in O(H)\}.$$

Lemma 3. *Let G_1 and G_2 be disjoint 2-TEPC graphs. If $\tau(G_1)$ is even then $G_1 \cup G_2$ is also a 2-TEPC graph.*

Proof. The graphs G_1 and G_2 are both 2-TEPC. Then there are sets $A_1 \subset E(G_1)$, $A_2 \subset E(G_2)$ such that $\tau(G_1[A_1]) = \tau(G_1)/2$ ($\tau(G_1)$ is even) and $\tau(G_2[A_2]) \in \{\lfloor \tau(G_2)/2 \rfloor, \lceil \tau(G_2)/2 \rceil\}$. For $A = A_1 \cup A_2$ we have

$$\begin{aligned}\tau(G[A]) &= \tau(G[A_1]) + \tau(G[A_2]) = \tau(G_1[A_1]) + \tau(G_2[A_2]) \\ &\in \{\tau(G_1)/2 + \lfloor \tau(G_2)/2 \rfloor, \tau(G_1)/2 + \lceil \tau(G_2)/2 \rceil\} \\ &= \{\lfloor \tau(G)/2 \rfloor, \lceil \tau(G)/2 \rceil\},\end{aligned}$$

i.e., G is a 2-TEPC graph. □

Lemma 4. *Let G be a graph and let $t \in [0, \tau(G)]$. Then there is a set $A \subseteq E(G)$ such that $|t - \tau(G[A])| \leq 1$.*

Proof. If G is a connected graph then the assertion follows from Example 1, Example 2 and Lemma 2.

Suppose that $G = G_1 \cup \cdots \cup G_c$, where G_i, $i \in [1, c]$, is a connected component of G. For every $j \in [0, c]$ define the set A_j and the integer r_j by $A_0 = \emptyset$, $r_0 = 0$, $A_j = A_{j-1} \cup E(G_j)$ and $r_j = r_{j-1} + \tau(G_j)$. As $t \leq \tau(G) = r_c$, there is an integer $i \in [1, c]$ such that $t \in [r_{i-1}, r_i]$. Set $t^* = t - r_{i-1}$. Clearly, $t^* \in [0, \tau(G_i)]$. The graph G_i is connected and so there is a set $A^* \subseteq E(G_i)$ such that $|t^* - \tau(G[A^*])| \leq 1$. Then, the set $A = A_{i-1} \cup A^*$ satisfies

$$|t - \tau(G[A])| = |(r_{i-1} + t^*) - (\tau(G[A_{i-1}]) + \tau(G[A^*]))| = |t^* - \tau(G[A^*])| \leq 1,$$

because $\tau(G[A_{i-1}]) = r_{i-1}$. □

Lemma 5. *Let G be a graph whose each component is a star. If G is neither nK_2 nor $K_{1,2} \cup nK_2$, for an odd integer n, then it is a 2-TEPC graph.*

Proof. Suppose that G is a counterexample with a minimum number c of connected components. Then $G = K_{1,t_1} \cup \cdots \cup K_{1,t_c}$, where $t_1 \geq \cdots \geq t_c \geq 1$.

If $c = 1$ then $G = K_{1,t_1}$ for $t_1 \geq 2$, because G is not $1K_2 = K_{1,1}$. Clearly, $\tau(G) = 2t_1 + 1$. Let A be a subset of $E(G)$ such that $|A| = \lfloor t_1/2 \rfloor$. Thereout $\tau(G[A]) = t_1 + 1 = \lceil \tau(G)/2 \rceil$, for t_1 even, and $\tau(G[A]) = t_1 = \lfloor \tau(G)/2 \rfloor$, for t_1 odd. According to Lemma 1 and Observation 2, G is a 2-TEPC graph, a contradiction.

If $c = 2$ then $\tau(G) = 2(t_1 + t_2 + 1) \neq 8$, because G is different from $K_{1,2} \cup 1K_2 = K_{1,2} \cup K_{1,1}$. If $t_1 + t_2$ is even then choose $A \subset E(K_{1,t_1})$ such that $|A| = (t_1 + t_2)/2$. Thereout $\tau(G[A]) = t_1 + t_2 + 1 = \tau(G)/2$ and by Lemma 1 and Observation 2, G is a 2-TEPC graph, a contradiction. If $t_1 + t_2$ is odd then $t_1 + t_2 \geq 5$ and there are sets $A_1 \subset E(K_{1,t_1})$, $A_2 \subset E(K_{1,t_2})$ such that $|A_1| \geq |A_2| \geq 1$ and $|A_1| + |A_2| = (t_1 + t_2 - 1)/2$. For $A = A_1 \cup A_2$ we have $\tau(G[A]) = \tau(G[A_1]) + \tau(G[A_2]) = t_1 + t_2 + 1 = \tau(G)/2$. Therefore, G is a 2-TEPC graph, a contradiction.

Thus, $c \geq 3$. If $t_{c-1} = 1$ then set $G_1 = K_{1,t_{c-1}} \cup K_{1,t_c} = 2K_2$ and $G_2 = K_{1,t_1} \cup \cdots \cup K_{1,t_{c-2}}$. As G_1 and G_2 have both less components than c, they are 2-TEPC. Moreover, $\tau(G_1) = 6$ is even. By Lemma 3, G is a 2-TEPC graph, a contradiction.

Therefore, $t_{c-1} \geq 2$. If $t_1 \geq 3$ then set $G_1 = K_{1,t_1} \cup K_{1,t_c}$ and $G_2 = K_{1,t_2} \cup \cdots \cup K_{1,t_{c-1}}$. They have both less components than c and $\tau(G_1)$ is even. According to Lemma 3, G is a 2-TEPC graph, a contradiction.

So, $t_1 = \cdots = t_{c-1} = 2$. Set $G_1 = K_{1,t_1} \cup K_{1,t_2} = 2K_{1,2}$ and $G_2 = K_{1,t_3} \cup \cdots \cup K_{1,t_c}$. The graph G_1 is 2-TEPC and $\tau(G_1)$ is even. If G_2 is a 2-TEPC graph then, by Lemma 3, G is also 2-TEPC, a contradiction. Therefore, G_2 is either K_2 or $K_{1,2} \cup K_2$. Consequently, G is either $2K_{1,2} \cup K_2$ or $3K_{1,2} \cup K_2$. It is easy to see that both of them are 2-TEPC graphs. This means that there is no counterexample. □

Theorem 1. *A simple graph with no isolated vertex is* 2*-TEPC if and only if it is neither of the following graphs:*

(i) *an unicyclic graph of order* 4*,*

(ii) $K_3 \cup K_{1,2} \cup K_2$*,*

(iii) nK_2*, for an odd integer* n*,*

(iv) $K_{1,2} \cup nK_2$*, for an odd integer* n*.*

Proof. Let G be a graph. Consider the following cases.

A. $\tau(G) = 3$. Then $G = K_2 = K_{1,1}$, i.e., a graph of type (iii). According to Example 2, $\{1, 2\} \cap O(G) = \emptyset$. By Observation 2, G is not 2-TEPC.

B. $5 \leq \tau(G) \leq 7$. Let A be a subset of $E(G)$ such that $|A| = 1$. As $\tau(G[A]) = 3 \in \{\lfloor \tau(G)/2 \rfloor, \lceil \tau(G)/2 \rceil\}$, by Observation 2, G is 2-TEPC.

C. $\tau(G) = 8$. As $\tau(G)/2 = 4$ and $\tau(H) \neq 4$ for any graph H without isolated vertices, the graph G is not 2-TEPC. However, G is either an unicyclic graph (type (i)) or $K_{1,2} \cup K_2$ (type (iv)) in this case.

D. $\tau(G) \geq 9$ and every component of G is either a star or K_3. Consider the following subcases.

D1. Every component of G is a star. By Lemma 5, G is a 2-TEPC graph except for G is either nK_2 or $K_{1,2} \cup nK_2$, for an odd integer n.

If $G = nK_2$, for odd n, then $\tau(G) = 3n$. As n is odd, $\lfloor \tau(G)/2 \rfloor \equiv 1 \pmod 3$ and $\lceil \tau(G)/2 \rceil \equiv 2 \pmod 3$. However, $\tau(G[A]) = 3|A| \equiv 0 \pmod 3$ for any set $A \subset E(G)$. Therefore, G is not a 2-TEPC graph in this case.

If $G = K_{1,2} \cup nK_2$, for an odd integer n, then $\tau(G) = 3n + 5$ and $\tau(G)/2 = 3(n + 1)/2 + 1 \equiv 1 \pmod 3$. However, $\tau(G[A]) = 3|A| \equiv 0 \pmod 3$ for any set $A \subset E(G)$ containing at most one edge of $K_{1,2}$ and $\tau(G[A]) = 3|A| - 1 \equiv 2 \pmod 3$ for any set $A \subset E(G)$ containing both edges of $K_{1,2}$. Therefore, G is not a 2-TEPC graph.

D2. Every component of G is K_3. Thus, $G = rK_3$. Let A be a subset of $E(G)$ such that A contains exactly one edge of each its component. Clearly, $|A| = r$. As $\tau(G[A]) = 3r = \tau(G)/2$, by Observation 2, G is 2-TEPC.

D3. $G = rK_3 \cup S$, where $r \geq 1$ and every component of S is a star.

If S is a 2-TEPC graph then, by Lemma 3, G is also a 2-TEPC graph.

If $S = nK_2$, for odd n, then $\tau(G) = 6r + 3n$. As n is an odd integer, $\lceil \tau(G)/2 \rceil = 3r + 2 + 3(n - 1)/2$. Let A_1 be a subset of $E(rK_3)$ such that A_1 contains at least one edge of each component of rK_3 and $|A_1| = 1 + r$. Similarly, let A_2 be a subset of $E(S)$ such that $|A_2| = (n-1)/2$. For $A = A_1 \cup A_2$ we have $\tau(G[A]) = \tau(rK_3[A_1]) + \tau(S[A_2]) = 3r + 2 + 3(n-1)/2$. Therefore, G is a 2-TEPC graph.

If $S = K_{1,2} \cup nK_2$, for an odd integer n, then $\tau(G) = 6r + 5 + 3n$. As n is odd, $\tau(G)/2 = 3r+4+3(n-1)/2$. Set $G^* = G - K_{1,2}$ (i.e., $G^* = rK_3 \cup nK_2$). In the same way as above we choose a set $A^* \subset E(G^*)$ such that $\tau(G^*[A^*]) = 3r+2+3(n-1)/2$. If $G[A^*]$ contains an isolated edge $e \in A^*$ then for $A = (A^* - e) \cup E(K_{1,2})$ we have

$$\tau(G[A]) = \tau(G^*[A^*]) - 3 + 5 = \tau(G)/2,$$

therefore, G is a 2-TEPC graph. $G[A^*]$ contains no isolated edge when $r = 1$ and $n = 1$. In this case $G = K_3 \cup K_{1,2} \cup K_2$, i.e., a graph of type (ii). It is easy to see that $7 \notin O(G)$. According to Observation 2, G is not 2-TEPC.

E. $\tau(G) \geq 9$ and G contains a component C different from K_3 and a star. Note that $\tau(C) \geq 7$.

If G is connected (i.e., $G = C$) then $\lceil \tau(G)/2 \rceil \geq 5$. By Lemma 2, $\lceil \tau(G)/2 \rceil \in O(G)$. Thus, G is a 2-TEPC graph.

If $C \neq G$ then set $H = G - C$ (i.e., $G = C \cup H$). If $\tau(C) \geq \tau(H)$ then $5 \leq \lceil \tau(G)/2 \rceil = \lceil (\tau(C) + \tau(H))/2 \rceil \leq \tau(C)$. According to Lemma 2, $\lceil \tau(G)/2 \rceil \in O(C) \subset O(G)$, i.e., G is a 2-TEPC graph. If $\tau(C) < \tau(H)$ then set $t = \lceil \tau(G)/2 \rceil - 6$. As $\tau(H) > \tau(C) \geq 7$, $t \in [0, \tau(H)]$. By Lemma 4, there is a set $A_H \subset E(H)$ such that $|t - \tau(H[A_H])| \leq 1$. Similarly, by Lemma 2, there is a set $A_C \in E(C)$ such that

$$\tau(C[A_C]) = \begin{cases} 5 & \text{when } \ t - \tau(H[A_H]) = -1, \\ 6 & \text{when } \ t - \tau(H[A_H]) = 0, \\ 7 & \text{when } \ t - \tau(H[A_H]) = 1. \end{cases}$$

For $A = A_C \cup A_H$ we have $\tau(G[A]) = \tau(C[A_C]) + \tau(H[A_H]) = \lceil \tau(G)/2 \rceil$. Therefore, G is a 2-TEPC graph □

3 Dense graphs

A *matching* in a graph is a set of pairwise nonadjacent edges. A matching is *perfect* if every vertex of the graph is incident with exactly one edge of the matching. A *maximum matching* is a matching that contains the largest possible number of edges. The number of edges in a maximum matching of a graph G is denoted by $\alpha(G)$. An *edge cover* of a graph G is a subset A of $E(G)$ such that every vertex of G is incident with an edge in A. The smallest number of edges in any edge cover of G is denoted by $\rho(G)$. Note that only graphs with no isolated vertices have an edge cover. For such graphs Gallai [3] proved that $\alpha(G) + \rho(G) = |V(G)|$.

Theorem 2. *Let k be an integer greater than* 1 *and let G be a simple graph with no isolated vertex. If* $|E(G)| > (2k-1)|V(G)| - k\big(\alpha(G) + 1\big)$ *then G is a k-total edge product cordial graph.*

Proof. For G we have

$$\tau(G) = |V(G)| + |E(G)| > k\big(2|V(G)| - \alpha(G) - 1\big) = k\big(|V(G)| + \rho(G) - 1\big).$$

Therefore, $\lceil \tau(G)/k \rceil \geq |V(G)| + \rho(G)$. Thus, there exists an edge cover $A_0 \subset E(G)$ such that $|A_0| = \lceil \tau(G)/k \rceil - |V(G)|$. Then

$$|E(G) - A_0| = \tau(G) - \big(|V(G)| + |A_0|\big) = \tau(G) - \lceil \tau(G)/k \rceil$$

and there is a partition $A_1, \ldots, A_{k-1}$ of $E(G) - A_0$ such that

$$\lceil \tau(G)/k \rceil \geq |A_1| \geq \cdots \geq |A_{k-1}| \geq \lfloor \tau(G)/k \rfloor.$$

Now consider the mapping $\varphi : E(G) \to [0, k-1]$ given by

$$\varphi(e) = i \quad \text{when} \quad e \in A_i.$$

As every vertex of G is incident with an edge in A_0, $\varphi^*(v) = 0$ for each $v \in V(G)$. So, $\mu_\varphi(0) = |V(G)| + |A_0| = \lceil \tau(G)/k \rceil$. Similarly, for $i \in [1, k-1]$, $\mu_\varphi(i) = |A_i| \in \big\{\lfloor \tau(G)/k \rfloor, \lceil \tau(G)/k \rceil\big\}$. According to Observation 1, the mapping φ is a k-TEPC labeling of G. □

We consider only graphs without isolated vertices. So $\alpha(G) \geq 1$ and we have immediately:

Corollary 1. *Let G be a graph of size at least $(2k-1)\big(|V(G)| - 1\big)$. Then G is a k-TEPC graph.*

For a composite number k we are able to prove a stronger result. First we prove the following auxiliary assertion.

Lemma 6. *Let G be a connected graph with minimum degree $\delta(G) \geq 2$ which is different from an odd cycle. Then G contains two disjoint edge covers where each of them has size at most $|V(G)| - 1$.*

Proof. If A is an edge cover of G and $G[A]$ contains a cycle then the set which we get from A by deleting an edge of the cycle is also an edge cover. Thus, any edge cover contains a subset A' which is also an edge cover and $G[A']$ is acyclic, i.e., $|A'| \leq |V(G)| - 1$. Therefore, it is sufficient to find two disjoint edge covers.

If G is a regular graph of degree 2 then it is an even cycle. Two perfect matchings of G are desired edge covers in this case.

If the maximum degree of G is at least 3 then choose a vertex $v \in V(G)$ of maximum degree. Suppose that G has $2s$ vertices of odd degree. Let G^* be a graph which we get from G by adding s new pairwise nonadjacent edges joining vertices of odd degree ($G^* = G$ when $s = 0$). Clearly, G^* is an Eulerian graph. Therefore, there is an ordering $e_1, e_2, \ldots e_q$ of $E(G^*)$ which forms an Eulerian trail of G^* starting (and finishing) at v. Moreover, we can assume that e_1 is an adding (new) edge when v is of odd degree in G. For $p \in \{0, 1\}$, set $A_p := \{e_i \in E(G) : i \equiv p \pmod 2\}$. Evidently, $A_0 \cap A_1 = \emptyset$. Also, any vertex of G is incident with two consecutive edges (belonging to $E(G)$) of the Eulerian trail. One of the consecutive edges belongs to A_0, the other to A_1. Thus, A_0 and A_1 are desired edge covers. □

Theorem 3. *Let k be a composite number greater than 4. Let G be a graph of minimum degree $\delta(G) \geq 2$. If $|E(G)| \geq (k-1)\big(|V(G)|-1\big)$ then G is a k-total edge product cordial graph.*

Proof. As $k > 4$ is a composite number, there are integers p, q such that $k > p > q > 1$ and $pq \equiv 0 \pmod{k}$.

Let $G_1, \dots, G_c$ be connected components of G. For all $i \in [0, c]$ and $s \in [1, 3]$, define the set B_i^s recursively in the following way.

Set $B_0^s = \emptyset$, for all $s \in [1, 3]$.

If G_i is an odd cycle then we choose its edge e_i. The set $E(G_i) - \{e_i\}$ can be partitioned into disjoint matchings M_i^2 and M_i^3 of G_i, where $|M_i^2| = |M_i^3| = \big(|V(G_i)|-1\big)/2$. Set $B_i^1 = B_{i-1}^1 \cup \{e_i\}$, $B_i^2 = B_{i-1}^2 \cup M_i^2$, and $B_i^3 = B_{i-1}^3 \cup M_i^3$.

If G_i is not an odd cycle then, by Lemma 6, there are disjoint edge covers C_i^2 and C_i^3 of G_i, where $|C_i^2| \leq |C_i^3| \leq |V(G_i)| - 1$. Set $B_i^1 = B_{i-1}^1$, $B_i^2 = B_{i-1}^2 \cup C_i^2$, and $B_i^3 = B_{i-1}^3 \cup C_i^3$.

Clearly, the sets B_c^1, B_c^2, B_c^3 are disjoint subsets of $E(G)$, $|B_c^1| = r$ where r denote the number of components of G isomorphic to an odd cycle, $|B_c^2| \leq |V(G)| - c - r$ and similarly $|B_c^3| \leq |V(G)| - c - r$.

Let $t_0, \dots, t_{k-1}$ be integers such that

$$\lceil \tau(G)/k \rceil \geq t_0 \geq \cdots \geq t_{k-1} \geq \lfloor \tau(G)/k \rfloor \quad \text{and} \quad t_0 + \cdots + t_{k-1} = \tau(G).$$

Evidently, $t_j \in \{\lfloor \tau(G)/k \rfloor, \lceil \tau(G)/k \rceil\}$ for each $j \in [0, k-1]$.

For G we have

$$\begin{aligned}\tau(G) = |V(G)| + |E(G)| &\geq |V(G)| + (k-1)\big(|V(G)|-1\big) \\ &= k|V(G)| - k + 1 > k\big(|V(G)|-1\big).\end{aligned}$$

Therefore, $\lceil \tau(G)/k \rceil \geq |V(G)|$. Thus, there is a partition $A_0, \dots, A_{k-1}$ of $E(G)$ satisfying

(i) $|A_0| = t_0 - |V(G)| + 2r$,

(ii) $B_c^1 \subset A_1$ and $|A_1| = t_1$,

(iii) $B_c^2 \subset A_p$ and $|A_p| = t_p - r$,

(iv) $B_c^3 \subset A_q$ and $|A_q| = t_q - r$,

(v) $|A_j| = t_j$ for $j \in [2, k-1] - \{p, q\}$.

Now consider the mapping $\varphi : E(G) \to [0, k-1]$ given by

$$\varphi(e) = j \quad \text{when} \quad e \in A_j.$$

If a vertex v is incident with a chosen edge e_i then $\deg(v) = 2$ and it is also incident with the second edge e' where $\varphi(e_i) = 1$ and $\varphi(e')$ is equal to either p or q. Therefore, $\varphi^*(v) \in \{p, q\}$. Moreover, if v and v' are incident with e_i then $\{\varphi^*(v), \varphi^*(v')\} = \{p, q\}$. If $u \in V(G)$ is not incident with any chosen edge then it is incident with an edge belonging to B_c^2 and another belonging to B_c^3. As the values of these edges are p and q (and $pq \equiv 0 \pmod{k}$), $\varphi^*(u) = 0$. Since there are precisely r chosen edges, we have: $\mu_\varphi(0) = |A_0|+|V(G)|-2r = t_0$, $\mu_\varphi(p) = |A_p|+r = t_p$, $\mu_\varphi(q) = |A_q|+r = t_q$ and similarly $\mu_\varphi(j) = |A_j| = t_j$ for all $j \in [1, k-1] - \{p, q\}$. This means that φ is a k-TEPC labeling of G. □

We are able to prove a similar result also for $k = 4$.

Theorem 4. *Let G be a graph with a 2-factor. If $|E(G)| > 3\big(|V(G)| - 1\big)$ then G is a 4-total edge product cordial graph.*

Proof. A 2-factor of a graph G denote by F. For G we have

$$\tau(G) = |V(G)| + |E(G)| \geq |V(G)| + 3\big(|V(G)| - 1\big) + 1 = 4\big(|V(G)| - 1\big) + 2.$$

Therefore, $\lceil \tau(G)/4 \rceil \geq |V(G)|$ and there is a partition $A_0, \ldots, A_3$ of $E(G)$ satisfying

(i) $|A_0| = \lceil \tau(G)/4 \rceil - |V(G)|$,

(ii) $E(F) \subset A_2$ and $|A_2| \geq |V(G)|$,

(iii) $\lceil \tau(G)/4 \rceil \geq |A_2| \geq |A_1| \geq |A_3| \geq \lfloor \tau(G)/4 \rfloor$.

Now consider the mapping $\psi : E(G) \to [0, 3]$ given by

$$\psi(e) = j \quad \text{when} \quad e \in A_j.$$

As every vertex $v \in V(G)$ is incident with two edges belonging to $F \subset A_2$, we have: $\psi^*(v) = 0$ and consequently $\mu_\psi(0) = |V(G)| + |A_0| = \lceil \tau(G)/4 \rceil$. Similarly, for each $i \in [1, 3]$ we have: $\mu_\psi(i) = |A_i| \in \{\lfloor \tau(G)/4 \rfloor, \lceil \tau(G)/4 \rceil\}$. Thus, ψ is a 4-TEPC labeling. □

We conclude this paper with the following result.

Theorem 5. *Let k be an integer greater than 2. Then any regular graph of degree $2(k-1)$ is a k-total edge product cordial graph.*

Proof. As G is a regular graph of degree $2(k-1)$, it is decomposable into $k-1$ edge-disjoint 2-factors $F_1, \ldots, F_{k-1}$. Moreover, $|V(G)| \geq 2k - 1$ and $|E(G)| = (k-1)|V(G)|$. According to Theorems 3 and 4, the assertion is true for any composite number k.

Suppose that $k \geq 3$ is a prime number. F_1 is a 2-regular graph of order at least 5 and, by Theorem 1, it is a 2-TEPC graph. Therefore, there is a 2-TEPC labeling $\eta : E(F_1) \to [0, 1]$. Clearly, $\mu_\eta(0) = \mu_\eta(1) = |E(F_1)| = |V(G)|$. Now consider the mapping $\varphi : E(G) \to [0, k-1]$ given by

$$\varphi(e) = \begin{cases} \eta(e) & \text{when} \quad e \in E(F_1), \\ j & \text{when} \quad e \in E(F_j),\ j \in [2, k-1]. \end{cases}$$

As each vertex $v \in V(G)$ is incident with precisely two edges belonging to F_j, $j \in [1, k-1]$, we have:

$$\varphi^*(v) \equiv \eta^*(v) \cdot \left(\prod_{j=2}^{k-1} j^2\right) \equiv \eta^*(v) \pmod{k}.$$

Therefore, $\mu_\varphi(0) = \mu_\eta(0) = |V(G)| = \mu_\eta(1) = \mu_\varphi(1)$ and similarly for all $j \in [2, k-1]$ we have: $\mu_\varphi(j) = |E(F_j)| = |V(G)|$. So, φ is a k-total edge product cordial labeling of G. □

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