AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **66(3)** (2016), Pages 420–424

A note on the locating-total domination number in trees

Nader Jafari Rad   Hadi Rahbani

Department of Mathematics
Shahrood University of Technology
Shahrood
Iran
n.jafarirad@gmail.com

Abstract

A total dominating set of a graph $G = (V, E)$ with no isolated vertex is a set $D \subseteq V(G)$ such that every vertex is adjacent to a vertex in D. A total dominating set D of G is a locating-total dominating set if for every pair of distinct vertices u and v in $V - D$, $N(u) \cap D \neq N(v) \cap D$. Let $\gamma_L^t(G)$ be the minimum cardinality of a locating-total dominating set of G. We show that for a nontrivial tree T of order n, with ℓ leaves and s support vertices, $\gamma_t^L(T) \geq (n + \frac{\ell}{2} - s + 1)/2$, improving some previous bounds presented by Chellali [*Discussiones Math. Graph Theory* 28 (3) (2008), 383–392] and Chen and Young Sohn [*Discrete Appl. Math.* 159 (13-14) (2011), 769–773]. We also characterize the extremal trees achieving the above bound.

1 Introduction

For notation and terminology not given here we refer to [5]. Let $G = (V(G), E(G))$ be a graph. The *open neighborhood* of a vertex $v \in V(G)$ is $N_G(v) = N(v) = \{u \in V(G) | uv \in E(G)\}$, and the *closed neighborhood* of v is $N_G[v] = N[v] = N(v) \cup \{v\}$. The *degree* of v is the size of its open neighborhood. A vertex of degree one in a tree is called a *leaf* and its neighbor is called a *support* vertex. We denote by $L(T)$ (respectively, $S(T)$) the set of leaves (respectively, support vertices) of a tree T. The number of leaves and support vertices of a tree T are $\ell = \ell(T) = |L(T)|$ and $s = s(T) = |S(T)|$, respectively. The subgraph induced in a graph G by a subset of vertices S is denoted by $G[S]$. A subset S is an independent set if $G[S]$ has no edge. A subset D of vertices of G is a *dominating set* if every vertex in $V(G) - D$ is adjacent to a vertex in D. A subset D of vertices of G is a *total dominating set* if every vertex in $V(G)$ is adjacent to a vertex in D. The *total domination number*,

$\gamma_t(G)$ of G, is the minimum cardinality of a total dominating set of G. The literature on the subject of total domination has been surveyed in a recent book [8].

A total dominating set D of a graph G is called a *locating-total dominating* set (LTDS) if for every pair of distinct vertices u and v in $V - D$, $N(u) \cap D \neq N(v) \cap D$. The *locating-total domination number* $\gamma_t^L(G)$ is the minimum cardinality of a locating-total dominating set of G. Locating-total domination was introduced by Haynes et al. [4] and further studies for example in [1, 2, 3, 6, 7].

Let $\mathcal{F}$ be the family of trees that can be obtained from r disjoint copies of P_4 and P_3 by first adding $r - 1$ edges so that they are incident only with support vertices and the resulting graph is connected, and then subdividing each new edge exactly once.

Theorem 1.1 (Chellali [1])**.** *If T is a tree of order $n \geq 2$ with ℓ leaves and s support vertices, then $\gamma_t^L \geq 2(n + \ell - s + 1)/5$, with equality if and only if $T = P_2$ or $T \in \mathcal{F}$.*

Theorem 1.2 (Chellali [1])**.** *If T is a tree of order $n \geq 2$, then $\gamma_t^L \geq (n + 2 - s)/2$.*

Chen and Sohn [3] obtained a new family ζ_2 of trees and gave the following theorem.

Theorem 1.3 (Chen and Sohn [3])**.** *If T is a tree of order $n \geq 3$ with ℓ leaves and s support vertices, then $\gamma_t^L \geq (n + \ell + 1)/2 - s$, with equality if and only if $T \in \zeta_2$.*

In this paper, we show that for any tree T of order $n \geq 2$, with ℓ leaves and s support vertices, $\gamma_t^L \geq \frac{1}{2}(n + \ell/2 - s + 1)$, and characterize trees achieving equality for this bound. We thus improve Theorem 1.1 for trees with $n \geq \frac{3}{2}\ell + s - 1$, Theorem 1.2 for all trees, and Theorem 1.3 for trees with $\ell \leq 2s$. The following is useful.

Lemma 1.4. *For $n \geq 2$, $\gamma_t^L(P_n) = \gamma_t(P_n) = \lfloor n/2 \rfloor + \lceil n/4 \rceil - \lfloor n/4 \rfloor$.*

2 Lower bound on the locating-total domination number of a tree

Let $\zeta = \{P_3\} \cup \{P_{4k} | k \geq 1\}$, S^k be a $\gamma_t^L(P_{4k})$-set for $k \geq 1$, and $S = S(P_3) \cup (\bigcup_{k \geq 1} S^k)$. Let ξ be the family of trees that can be obtained from t disjoint copies of trees in ζ by first adding $t - 1$ edges in such a way that they are incident only with vertices in S and the resulting graph is connected, and then subdividing each new edge exactly once.

Theorem 2.1. *If T is a tree of order $n \geq 2$, with ℓ leaves and s support vertices, then $\gamma_t^L \geq (n + \ell/2 - s + 1)/2$, with equality if and only if $T \in \xi$.*

Proof. Let T be a tree of order $n \geq 2$ with ℓ leaves and s support vertices. Let D be a $\gamma_t^L(T)$-set such that $|L(T) \cap D|$ is minimum. Set $B = \{v \notin D : |N(v) \cap D| = 1\}$ and $C = \{v \notin D | |N(v) \cap D| \geq 2\}$. Then $V(T) = D \cup B \cup C$. Let $Q_1 = D - (L(T) \cup S(T))$

and $Q_2 = B - L(T)$, and ω be the number of components of $T[D]$. Then $D = (L(T) \cap D) \cup S \cup Q_1$. By minimality of $|L(T) \cap D|$, we may assume that $|L(T) \cap D| = \ell - s$ and $|L(T) \cap B| = s$. Let $|[D, B \cup C]|$ be the number of edges with one end-point in D and the other end-point in $B \cup C$. Clearly, $|[D, B \cup C]| \geq |B| + 2|C| = 2n - 2|D| - |B|$. On the other hand, $|[D, B \cup C]| = n - 1 - |E(T[D])| - |E(T[Q_2 \cup C])|$. Thus we obtain $n - 1 - |E(T[D])| - |E(T[Q_2 \cup C])| \geq 2n - 2|D| - |B|$.

Claim 1. $|E(T[Q_2 \cup C])| \geq \frac{|Q_2|}{2}$, and the equality holds if and only if $T[Q_2 \cup C] \cong |C|K_1 + \frac{|Q_2|}{2}K_2$ and C is an independent set in $T[Q_2 \cup C]$.

Proof of Claim 1. Since $\deg(v) \geq 2$ for any $v \in Q_2$, we have $N(v) \cap (C \cup Q_2) \neq \emptyset$. Thus,

$$|E(T[Q_2 \cup C])| = \frac{1}{2} \sum_{v \in Q_2 \cup C} \deg_{T[Q_2 \cup C]}(v) \geq \frac{1}{2} \sum_{v \in Q_2} \deg_{T[Q_2 \cup C]}(v) \geq \frac{|Q_2|}{2}.$$

Assume that equality holds. Then

$$\frac{1}{2} \sum_{v \in Q_2 \cup C} \deg_{T[Q_2 \cup C]}(v) = \frac{1}{2} \sum_{v \in Q_2} \deg_{T[Q_2 \cup C]}(v) = \frac{|Q_2|}{2},$$

and thus $\deg_{T[Q_2 \cup C]}(v) = 0$ for each vertex $v \in C$ and $\deg_{T[Q_2 \cup C]}(v) = 1$ for each vertex $v \in Q_2$. Consequently, $T[Q_2 \cup C] \cong \frac{|Q_2|}{2}K_2 + |C|K_1$ and C is an independent set in $T[Q_2 \cup C]$. The converse is obvious.

Claim 2. $|E(T[D])| \geq \frac{|D|}{2}$, the equality holds if and only if $T[D] \cong \frac{|D|}{2}K_2$.

Proof of Claim 2. Since D is a total dominating set of T, every component of $T[D]$ has at least two vertices. Thus, $\omega \leq \frac{|D|}{2}$ and so $|E(T[D])| = |D| - \omega \geq |D| - \frac{|D|}{2} = \frac{|D|}{2}$. Moreover, the equality $|E(T[D])| = \frac{|D|}{2}$ holds if and only if $T[D] \cong \frac{|D|}{2}K_2$.

By Claims 1 and 2, $2n - 2|D| - |B| \leq n - 1 - \frac{|D|}{2} - \frac{|Q_2|}{2}$. Thus, we have $n + 1 - |B| \leq \frac{3|D|}{2} - \frac{|Q_2|}{2}$. But $|B| = |Q_2| + |B \cap L(T)|$. Thus we obtain that

$$n + 1 - \frac{|Q_2|}{2} - |B \cap L(T)| = n + 1 - \frac{|Q_2|}{2} - s \leq \frac{3|D|}{2}.$$

It is obvious that each vertex of D is adjacent to at most one vertex of Q_2. If $u \in D \cap (L(T) \cup S(T))$ then by the minimality of D, we have $Q_2 \cap N(u) = \emptyset$. We deduce that $|Q_2| \leq |D| - |(L(T) \cap D) \cup S(T)| = |D| - \ell$. We now have $n + 1 + \frac{\ell}{2} - s \leq (\frac{3}{2} + \frac{1}{2})|D|$, and thus, $\gamma_t^L(T) \geq \frac{1}{2}(n + \frac{\ell}{2} - s + 1)$, as desired.

We next prove the equality part. Assume that $\gamma_t^L = \frac{1}{2}(n + \frac{\ell}{2} - s + 1)$. Then $|E(T[Q_2 \cup C])| \cong \frac{|Q_2|}{2}K_2 + |C|K_1$, C is an independent set in $T[Q_2 \cup C]$, $|N(v) \cap D| = 2$ for every vertex $v \in C$, $|Q_2| = |D| - \ell$, and $T[D] \cong \frac{|D|}{2}K_2$. Since $|Q_1| = |D| - \ell$, we obtain $|Q_1| = |Q_2|$. If $|Q_1| = |Q_2| = 0$, then $D \subseteq L(T) \cup S(T)$. For every component C' of $T[D \cup B]$, since $|Q_2| = 0$ and $T[D] = \frac{|D|}{2}K_2$, we have $\gamma_t^L(C') = 2$.

Therefore $C' = P_3$ or P_4. On other hand since every vertex in C' is adjacent to exactly two vertices in D, we get $T \in \xi$. Now we consider the case $|Q_1| = |Q_2| \neq 0$. Let $T_1, T_2, \ldots, T_{\omega_1}$ be the components of $T[D \cup B]$. Clearly, $D \cap V(T_i)$ is a LTDS for T_i for $i = 1, 2, \ldots, \omega_1$. Now

$$\begin{aligned}
\frac{1}{2}(n + \frac{\ell}{2} - s + 1) = |D| &= \sum_{i=1}^{\omega_1} |D \cap V(T_i)| \\
&\geq \sum_{i=1}^{\omega_1} \gamma_t^L(T_i) \\
&\geq \sum_{i=1}^{\omega_1} \frac{1}{2}(|V(T_i)| + \frac{\ell(T_i)}{2} - s(T_i) + 1) \\
&= \frac{1}{2}(n - (\omega_1 - 1) + \frac{\ell}{2} - s + \omega_1)
\end{aligned}$$

Thus $\gamma_t^L(T_i) = \frac{1}{2}(|V(T_i)| + \frac{\ell(T_i)}{2} - s(T_i) + 1)$ for each $i = 1, 2, \ldots, \omega_1$. If $V(T_i) \cap Q_2 = \emptyset$ for some component T_i, then as before we obtain that $T_i \in \{P_3, P_4\}$. Assume that $V(T_i) \cap Q_2 \neq \emptyset$ for some component T_i. We show that T_i is a path of order $4k$ for some integer k. Let $v \in V(T_i)$. Suppose that $\deg_{T_i}(v) \geq 3$. Let $\{x, y, z\} \subseteq N(v)$. If $v \in D$, then we can assume that $\{y, z\} \subseteq B$, since $T[D] \cong \frac{|D|}{2} K_2$. But then $N(y) \cap D = N(z) \cap D = \{v\}$, a contradiction. Thus, $v \notin D$, and so $v \in B$. Since $T[Q_2 \cup C] \cong \frac{|Q_2|}{2} K_2 + |C| K_1$, we can assume that $\{y, z\} \subseteq D$. Then $|N(v) \cap D| \geq 2$, contradicting that $v \in B$. We conclude that $\deg_{T_i}(v) \leq 2$. Consequently, T_i is a path. Since $V(T_i) \cap Q_2 \neq \emptyset$, we have $|V(T_i)| \geq 5$, and thus, $\ell = s = 2$. Then $\gamma_t^L(T_i) = \frac{|V(T_i)|}{2}$. Now the Lemma 1.4 implies that T_i is a path of order $4k$ for some integer k. Thus, for $i = 1, 2, \ldots, \omega_1$, we have $T_i \in \zeta$. Note that every vertex in C is adjacent to exactly two vertices in D, thus $T \in \xi$. The converse is straightforward. □

We note that Theorem 2.1 improves Theorem 1.1 for trees with $n \geq \frac{3}{2}\ell + s - 1$, improves Theorem 1.2 for all trees, and improves Theorem 1.3 for trees with $\ell \leq 2s$. We also note that if $\ell \geq 3$, then by Theorem 2.1, $\gamma_t^L(T) \geq (n + \ell/2 - s + 1)/2 > (n + 2 - s)/2$, and thus a simple calculation leads the following characterization of trees achieving equality of the bound of Theorem 1.2.

Corollary 2.2. *If T is a tree of order $n \geq 2$, then $\gamma_t^L(T) = \frac{1}{2}(n + 2 - s)$ if and only if $T = P_{4k}$ for some integer $k \geq 1$.*

Acknowledgements

The authors would like to thank the referees for their careful review of the paper.

References

[1] M. Chellali, On locating and differentiating-total domination in trees, *Discuss. Math. Graph Theory* 28 (3) (2008), 383–392.

[2] M. Chellali, Locating-total domination critical graphs, *Australas. J. Combin.* 45 (2009), 227–234.

[3] X. Chen and M. Young Sohn, Bounds on the locating-total domination number of a tree, *Discrete Appl. Math.* 159 (13-14) (2011), 769–773.

[4] T. W. Haynes, M. A. Henning, and J. Howard, Locating and total dominating sets in trees, *Discrete Appl. Math.* 154 (2006), 1293–1300.

[5] T. W. Haynes, S. T. Hedetniemi and P. J. Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker, Inc., New York, 1998.

[6] M. A. Henning and N. Jafari Rad, Locating-total domination in graphs, *Discrete Appl. Math.* 160 (13-14) (2012), 1986–1993.

[7] M. A. Henning and C. Löwenstein, Locating-total domination in claw-free cubic graphs, *Discrete Math.* 312 (2012), 3107–3116.

[8] M. A. Henning and A. Yeo, *Total domination in graphs*, (Springer Monographs in Mathematics) 2013.

(Received 23 Feb 2016; revised 21 Sep 2016)