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Group divisible designs of three groups and block size five with configuration $(1, 2, 2)$

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Abstract

The subject matter for this paper is GDDs with three groups and block size five in which each block has configuration $(1, 2, 2)$; that is, each block has exactly one point from one of the three groups and two points from each of the other two groups. We provide necessary and sufficient conditions of the existence of a GDD $(n, 3, 5; \lambda_1, \lambda_2)$ with configuration $(1, 2, 2)$. A highlight of this paper is a technique which uses two and then three idempotent MOLS consecutively to construct a required family of GDDs.

1 Introduction

Group divisible designs (GDDs) have been studied for their usefulness in statistics and for their universal application to constructions of new designs [12, 16, 17]. Certain difficulties are present especially when the number of groups is smaller than the block size. In [3, 4], the question of existence of GDDs for block size three was settled. There is a more technical proof given in the book "Triple Systems" [2]. Similar

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results were established for GDDs with block size four in [5, 7, 9, 14, 18]. In [6, 8], results about GDDs with two groups and block size five with fixed block configuration were presented. In [13], results about GDDs with fours groups and block size five with fixed block configuration were established. In [15], generalizations of designs for block size five from Clatworthy's Table are given. In [10], results about GDDs with block size six with fixed block configuration were studied.

A group divisible design, GDD$(n, m, k; \lambda_1, \lambda_2)$, is a collection of k-element subsets of a v-set V called *blocks* which satisfies the following properties: the $v = nm$ elements of V are partitioned into m subsets (called *groups*) of size n each; each point of V appears in $r = \frac{\lambda_1(n-1)+\lambda_2 n(m-1)}{k-1}$ (called the *replication number*) of the $b = \frac{nmr}{k}$ blocks; points within the same group are called *first associates* of each other and appear together in λ_1 blocks; any two points not in the same group are called *second associates* of each other and appear together in λ_2 blocks.

In [5, 18], the necessary conditions are proved to be sufficient for the existence of a $GDD(n, 3, 4; \lambda_1, \lambda_2)$ with configuration $(1, 1, 2)$, that is, each block has exactly one point from each of the two groups and two points from the third group. The purpose of this paper is to establish results for GDDs with block size five and three groups (i.e. GDD$(n, 3, 5; \lambda_1, \lambda_2)$) in which each block has configuration $(1, 2, 2)$, that is, each block has exactly one point from one of the three groups and two points from each of the other two groups. Unless otherwise stated, GDDs addressed in this paper all have the configuration $(1, 2, 2)$. First we find the relationship between λ_2 and λ_1.

Theorem 1.1 *Necessary conditions for the existence of a GDD* $(n, 3, 5;\ \lambda_1, \lambda_2)$ *are* $n \geq 2$ *and* $\lambda_2 = \frac{2(n-1)\lambda_1}{n}$.

Proof: Suppose a GDD$(n, 3, 5; \lambda_1, \lambda_2)$ exists, then the replication number r for an arbitrary point is $\frac{\lambda_1(n-1)+\lambda_2(2n)}{4}$. Also, since $vr = bk$, we have $b = \frac{3n\times[\lambda_1(n-1)+\lambda_2(2n)]}{20}$. On the other hand, since every block must contain exactly two first associate pairs (with configuration $(1, 2, 2)$), the group size n should be greater than or equal to 2. Also, the number of blocks b must equal the number of the first associates pairs $\frac{3n(n-1)\lambda_1}{2}$ divided by 2, i.e., $b = \frac{3n(n-1)\lambda_1}{4}$. We have $b = \frac{3n(n-1)\lambda_1}{4} = \frac{3n\times[\lambda_1(n-1)+\lambda_2(2n)]}{20}$, that is, $\lambda_2 = \frac{2(n-1)\lambda_1}{n}$. □

Since $\lambda_2 = \frac{2(n-1)\lambda_1}{n}$, if we let $\lambda_1 = \frac{nt}{2}$ for any positive integer t such that $nt \equiv 0 \pmod 2$, then $\lambda_2 = (n-1)t$. Also, since the replication number $r = \frac{\lambda_1(n-1)+\lambda_2(2n)}{4} = \frac{5n(n-1)t}{8}$, $n(n-1)t \equiv 0 \pmod 8$. Combine $nt \equiv 0 \pmod 2$ and $n(n-1)t \equiv 0 \pmod 8$, and we have the following corollary.

Corollary 1.1 *Suppose* t *is a positive integer and let* $\lambda_1 = \frac{nt}{2}$*. The necessary conditions for* t *are as follows.*

1) If $n \equiv 0 \pmod 8$*, then there is no condition for* t*.*

2) If $n \equiv 1, 4, 5 \pmod 8$*, then* t *must be even.*

3) If $n \equiv 2, 3, 6, 7 \pmod 8$*, then* $t \equiv 0 \pmod 4$*.*

Remark 1.1 Notice that if a GDD$(n, 3, 5; \lambda_1, \lambda_2)$ exists, then for any integer $c \geq 1$, a GDD$(n, 3, 5; c\lambda_1, c\lambda_2)$ exists by taking c copies of GDD$(n, 3, 5;\ \lambda_1, \lambda_2)$. Therefore, we can reduce the problem to find a GDD$(n, 3, 5;\ \lambda_1, \lambda_2)$ for the minimum value of λ_1.

2 GDD$(n, 3, 5; \lambda_1, \lambda_2)$ for $n \equiv 2, 3, 6, 7 \pmod 8$

Theorem 2.1 *Necessary conditions are sufficient for a GDD*$(2, 3, 5; \lambda_1,\ \lambda_2)$ *and a GDD*$(3, 3, 5; \lambda_1, \lambda_2)$*, respectively.*

Proof: By Theorem 1.1 and Corollary 1.1, the necessary condition for the existence of a GDD$(2, 3, 5; \lambda_1, \lambda_2)$ is $\lambda_2 = \lambda_1 = t$ where $t \equiv 0 \pmod 4$. Thus, the minimum values of λ_1 and λ_2 are both 4. A GDD$(2, 3, 5; 4, 4)$ on the three groups $\{1, 2\}$, $\{3, 4\}$, and $\{5, 6\}$ is as follows: $\{1, 2, 3, 4, 5\}$, $\{1, 2, 3, 4, 6\}$, $\{1, 2, 3, 5, 6\}$, $\{1, 2, 4, 5, 6\}$, $\{1, 3, 4, 5, 6\}$, and $\{2, 3, 4, 5, 6\}$. By Remark 1.1, we have a GDD$(2, 3, 5; \lambda_1, \lambda_2)$.

By Theorem 1.1 and Corollary 1.1, the necessary condition for the existence of a GDD$(3, 3, 5; \lambda_1, \lambda_2)$ is $\lambda_2 = \frac{4\lambda_1}{3}$ and $t \equiv 0 \pmod 4$. The minimum values of t, λ_1 and λ_2 are 4, 6 and 8 respectively. A GDD$(3, 3, 5; 6, 8)$ on the three groups $\{1, 2, 3\}$, $\{4, 5, 6\}$ and $\{7, 8, 9\}$ is given in Figure 1. Note that each column represents a block. By Remark 1.1, we have a GDD$(3, 3, 5; \lambda_1, \lambda_2)$. □

1	1	1	1	1	1	1	1	1	1	1	1	2	2	2	2	2	2	4	4	4	4	4	4	5	5	5
2	2	2	2	2	2	3	3	3	3	3	3	3	3	3	3	3	3	5	5	5	6	6	6	6	6	6
4	4	5	7	7	8	4	4	5	7	7	8	4	4	5	7	7	8	7	7	8	7	7	8	7	7	8
5	6	6	8	9	9	5	6	6	8	9	9	5	6	6	8	9	9	8	9	9	8	9	9	8	9	9
7	8	9	6	4	5	8	9	7	6	4	5	7	8	9	5	6	4	1	2	3	3	1	2	2	3	1

Figure 1: A GDD$(3, 3, 5; 6, 8)$

Definition 2.1 A 1-*factor* of a graph G is a set of pairwise disjoint edges which partition the vertex set. A 1-*factorization* of a graph G is a set of 1-factors which partition the edge set of the graph.

A 1-factorization of K_{2n} contains $2n - 1$ 1-factors.

Definition 2.2 A 2-*factor* of a graph G is a spanning subgraph of G which is regular of degree 2. A 2-*factorization* of a graph G is an edge disjoint decomposition of G into 2-factors.

It is known that a K_{2n+1} $(n \geq 1)$ has n 2-factors. We will use the same notation, say G to denote a set or a group and a complete graph on $|G|$ points labeled with elements from G. Similarly, a pair of elements (a, b) and an edge (a, b) will be used interchangeably. The context should make the intention clear.

Lemma 2.1 *A GDD*$(n, 3, 5; 2n, 4(n-1))$ *exists for any* $n \geq 2$.

Proof: Suppose that $G_1 = \{v_1, v_2, \ldots, v_n\}$, $G_2 = \{u_1, u_2, \ldots, u_n\}$ and $G_3 = \{w_1, w_2, \ldots, w_n\}$ are the three groups. If n is odd, then there are $\frac{n-1}{2}$ 2-factors for each group. Note that each 2-factor has n edges. Let $A_i = \{a_{i1}, a_{i2}, \ldots, a_{in}\}$ be the ith 2-factor for G_1, where $i = 1, 2, \ldots, \frac{n-1}{2}$, and $B_i = \{b_{i1}, b_{i2}, \ldots, b_{in}\}$ be the ith 2-factor for G_2, and $C_i = \{c_{i1}, c_{i2}, \ldots, c_{in}\}$ be the ith 2-factor for G_3. For $i = 1, 2, \ldots, \frac{n-1}{2}$ and $s = 1, 2, \ldots, n$, we construct n blocks $\{a_{is}\} \cup \{b_{i1}\} \cup \{w_s\}$, $\{a_{is}\} \cup \{b_{i2}\} \cup \{w_{s+1}\}$, $\ldots$, $\{a_{is}\} \cup \{b_{in}\} \cup \{w_{s-1}\}$, where subscripts of w's are modulo n from $\{1, 2, \ldots, n\}$. As a result, we have $\frac{n^2(n-1)}{2}$ blocks. Do the same to get $\frac{n^2(n-1)}{2}$ blocks in the form of $\{c_{is}\} \cup \{a_{i1}\} \cup \{u_s\}$, $\{c_{is}\} \cup \{a_{i2}\} \cup \{u_{s+1}\}$, $\ldots$, $\{c_{is}\} \cup \{a_{in}\} \cup \{u_{s-1}\}$, and $\frac{n^2(n-1)}{2}$ blocks in the form of $\{b_{is}\} \cup \{c_{i1}\} \cup \{v_s\}$, $\{b_{is}\} \cup \{c_{i2}\} \cup \{v_{s+1}\}$, $\ldots$, $\{b_{is}\} \cup \{c_{in}\} \cup \{v_{s-1}\}$. Thus, we have a total of $\frac{3n^2(n-1)}{2}$ blocks as required. For $n = 5$, $A_1 = \{(1, 2), (2, 3), (3, 4), (4, 5), (5, 1)\}$, $B_1 = \{(6, 7), (7, 8), (8, 9), (9, 10), (10, 6)\}$, and $G_3 = \{11, 12, 13, 14, 15\}$, see the 25 blocks constructed using A_1, B_1 and elements from G_3 in Figure 2 for an example.

1	1	1	1	1	2	2	2	2	2	3	3	3	3	3	4	4	4	4	4	5	5	5	5	5
2	2	2	2	2	3	3	3	3	3	4	4	4	4	4	5	5	5	5	5	1	1	1	1	1
6	7	8	9	10	6	7	8	9	10	6	7	8	9	10	6	7	8	9	10	6	7	8	9	10
7	8	9	10	6	7	8	9	10	6	7	8	9	10	6	7	8	9	10	6	7	8	9	10	6
11	12	13	14	15	12	13	14	15	11	13	14	15	11	12	14	15	11	12	13	15	11	12	13	14

Figure 2: The first 25 blocks of a GDD$(5, 3, 5; 10, 16)$

Notice that in the above construction, when we used the 2-factors A_i's and B_i's, every edge of A_i is used n times. Again, when we used the 2-factors C_i's and A_i's, every edge of A_i is used n times. That is, every edge of A_i is used a total of $2n$ times in the construction. Similarly, every edge of B_i and every edge of C_i are used a total of $2n$ times, respectively. This implies that $\lambda_1 = 2n$. Furthermore, when we used 2-factors of G_1 and G_2 and elements from G_3 to construct blocks, edges (a, c) and (b, c) occur twice, and edge (a, b) occur 4 times, where $a \in G_1$, $b \in G_2$ and $c \in G_3$. Since there are $\frac{n-1}{2}$ 2-factors of each G_i's, edges (a, c) and (b, c) appear $2 \times (\frac{n-1}{2})$ times, respectively, and edge (a, b) appears $4 \times (\frac{n-1}{2})$ times. Similarly, the blocks constructed using 2-factors of G_2 and G_3 have $2 \times (\frac{n-1}{2})$ (a, b) and (a, c) edges, respectively, and $4 \times (\frac{n-1}{2})$ (b, c) edge. The blocks constructed using 2-factors of G_3 and G_1 have $2 \times (\frac{n-1}{2})$ (a, b) and (b, c) edges, respectively, and $4 \times (\frac{n-1}{2})$ (a, c) edge. Thus, each edge corresponding to the second associate pair occurs $4 \times (\frac{n-1}{2}) + 2 \times (\frac{n-1}{2}) + 2 \times (\frac{n-1}{2}) = 4(n-1)$ times, that is, $\lambda_2 = 4(n-1)$ as required.

If n is even, we construct the blocks in a manner similar to the construction of the blocks when n is odd, except that we use 1-factors for n even. Notice that there are $n-1$ 1-factors of each G_i's, and there are $\frac{n}{2}$ edges in each 1-factor. When we use 1-factors A_i from G_1 and B_i from G_2 and elements from G_3, where $i = 1, 2, \ldots, n-1$, we construct $\frac{n^2}{4}$ blocks $\{a_{is}\} \cup \{b_{i1}\} \cup \{w_s\}$, $\{a_{is}\} \cup \{b_{i2}\} \cup \{w_{s+1}\}$, $\ldots$, $\{a_{is}\} \cup \{b_{im}\} \cup \{w_{s-1}\}$ for $s = 1, 2, \ldots, \frac{n}{2}$, where $m = \frac{n}{2}$ and subscripts of w's take modulo of m from $\{1, 2, \ldots, m\}$. Notice that we only used half of the elements from

G_3. Duplicating these blocks, and then in each block of the duplicated $\frac{n^2}{4}$ blocks, replace only the element w_j from G_3 to w_{j+m}. The resulting $\frac{n^2}{2}$ blocks used all elements from G_3 $\frac{n}{2}$ times. Since there are $n-1$ 1-factors of G_1 and G_2, respectively, we have a total of $\frac{n^2(n-1)}{2}$ blocks. Similarly, construct $\frac{n^2(n-1)}{2}$ blocks using 1-factors from G_3 and G_1 and $\frac{n^2(n-1)}{2}$ blocks using 1-factors from G_2 and G_3, respectively. We have a total of $\frac{3n^2(n-1)}{2}$ blocks as required. For $n = 10$, $A_1 = \{(1,2),(3,4),(5,6),(7,8),(9,10)\}$, $B_1 = \{(11,12),(13,14),(15,16),(17,18),(19,20)\}$, and $G_3 = \{21,\ldots,25,26,\ldots,30\}$, see the 50 blocks constructed using A_1, B_1 and elements from G_3 in Figure 3 for an example.

1	1	1	1	1	3	3	3	3	3	5	5	5	5	5	7	7	7	7	7	9	9	9	9	9
2	2	2	2	2	4	4	4	4	4	6	6	6	6	6	8	8	8	8	8	10	10	10	10	10
11	13	15	17	19	11	13	15	17	19	11	13	15	17	19	11	13	15	17	19	11	13	15	17	19
12	14	16	18	20	12	14	16	18	20	12	14	16	18	20	12	14	16	18	20	12	14	16	18	20
21	22	23	24	25	22	23	24	25	21	23	24	25	21	22	24	25	21	22	23	25	21	22	23	24

1	1	1	1	1	3	3	3	3	3	5	5	5	5	5	7	7	7	7	7	9	9	9	9	9
2	2	2	2	2	4	4	4	4	4	6	6	6	6	6	8	8	8	8	8	10	10	10	10	10
11	13	15	17	19	11	13	15	17	19	11	13	15	17	19	11	13	15	17	19	11	13	15	17	19
12	14	16	18	20	12	14	16	18	20	12	14	16	18	20	12	14	16	18	20	12	14	16	18	20
26	27	28	29	30	27	28	29	30	26	28	29	30	26	27	29	30	26	27	28	30	26	27	28	29

Figure 3: The first 50 blocks of a GDD(10, 3, 5; 20, 36)

Notice that in the above construction for n even, when we used the 1-factors A_i's and B_i's, every edge of A_i is used n times. Again, when we used the 1-factors C_i's and A_i's, every edge of A_i is used n times. That is, every edge of A_i is used a total of $2n$ times in the construction. Similarly, every edge of B_i and every edge of C_i are used a total of $2n$ times, respectively. This implies that $\lambda_1 = 2n$. Furthermore, when we used 1-factors of G_1 and G_2 and elements from G_3 to construct blocks, edges (a, c) and (b, c) occur once, and edge (a, b) occur twice, where $a \in G_1$, $b \in G_2$ and $c \in G_3$. Since there are $n-1$ 1-factors of each G_i's, edges (a, c) and (b, c) appear $n-1$ times, respectively, and edge (a, b) appears $2(n-1)$ times. Similarly, the blocks constructed using 1-factors of G_2 and G_3 have $n-1$ (a, b) and (a, c) edges, respectively, and $2(n-1)$ (b, c) edge. The blocks constructed using 1-factors of G_3 and G_1 have $n-1$ (a, b) and (b, c) edges, respectively, and $2(n-1)$ (a, c) edge. Thus, each second associate edge occurs $2(n-1)+(n-1)+(n-1) = 4(n-1)$ times, that is, $\lambda_2 = 4(n-1)$ as required. We conclude that a GDD$(n, 3, 5; 2n, 4(n-1))$ exists for any $n \geq 2$. □

Theorem 2.2 *Necessary conditions are sufficient for a GDD$(n, 3, 5; \lambda_1, \lambda_2)$ for $n \equiv 2, 3, 6, 7 \pmod{8}$.*

Proof: By Corollary 1.1, if $n \equiv 2, 3, 6, 7 \pmod{8}$, then $t \equiv 0 \pmod{4}$. The minimum value of t is 4, thus the minimum values of λ_1 and λ_2 are $2n$ and $4(n-1)$, respectively. Since a GDD$(n, 3, 5; 2n; 4(n-1))$ exists by Lemma 2.1, the necessary conditions are sufficient by Remark 1.1. □

3 GDD$(n, 3, 5; \lambda_1, \lambda_2)$ for $n \equiv 0 \pmod 8$

If $n \equiv 0 \pmod 8$, then there is no condition on t by Corollary 1.1. The minimum value of t is 1, thus the minimum values of λ_1 and λ_2 are $\frac{nt}{2} = \frac{n}{2}$ and $n-1$, respectively. The following definitions and results which can be found in [1, 11] will be used in our proofs of GDDs.

Definition 3.1 A *Latin square* L of side (or order) n is an $n \times n$ array in which each cell contains a single symbol from an n-set S, such that each symbol occurs exactly once in each row and exactly once in each column. Two Latin squares L_1 and L_2 of the same order are *orthogonal* if $L_1(a,b) = L_1(c,d)$ and $L_2(a,b) = L_2(c,d)$, implies $a = c$ and $b = d$. A set of Latin squares $L_1, \ldots, L_m$ is *mutually orthogonal*, or a set of MOLS, if for every $1 \le i < j \le m$, L_i and L_j are orthogonal.

Definition 3.2 A Latin square L of order n on $\{1, 2, \ldots, n\}$ is called an *idempotent Latin square* if $L(i,i) = i$ for $i = 1, 2, \ldots, n$ and is called a *symmetric Latin square* if $L(i,j) = L(j,i)$ for all i and j in $\{1, 2, \ldots, n\}$. A Latin square where rows and columns are labeled by $\{1, 2, \ldots, n\}$ is referred to as a *quasigroup* $(L, \circ)$ where $i \circ j = L(i,j)$.

Theorem 3.1 *If there are m MOLS of order n, then there are $m-1$ idempotent MOLS of order n.* (*See page 129 in [11]*).

Theorem 3.2 *There are three idempotent MOLS for all positive integers n except for $n \in \{2, 3, 4, 6, 10\}$.* (*See Table III.3.88 on page 186 in [1]*).

Definition 3.3 A *partially balanced incomplete block design* with m associate classes (PBIBD(m)) is a block design based on a v-set X with b blocks each of size k such that if the elements x and y are ith associates, then they are together in precisely λ_i blocks, $1 \le i \le m$.

Theorem 3.3 *If two idempotent MOLS of order n exist, then a PBIBD(3) with block size 5, $\lambda_1 = 4$, $\lambda_2 = 7$, $\lambda_3 = n-1$ and $v = 3n$ exists.*

Proof: Let $V = \{a_1, \ldots, a_n, b_1, \ldots, b_n, c_1, \ldots, c_n\}$ be a set of $3n$ elements. Suppose there exist two idempotent Latin squares of order n on symbols $\{1, 2, \ldots, n\}$, namely (L_1, o_1), and (L_2, o_2).

For $1 \le i, j \le n$ and $i \ne j$, we construct the following blocks $\{a_i, a_j, b_{io_1j}, b_{io_2j}, c_{io_2j}\}$, $\{b_i, b_j, c_{io_1j}, c_{io_2j}, a_{io_2j}\}$, and $\{c_i, c_j, a_{io_1j}, a_{io_2j}, b_{io_2j}\}$. By the definitions of the Latin square and MOLS, we observe the following for the blocks $\{a_i, a_j, b_{io_1j}, b_{io_2j}, c_{io_2j}\}$ where $1 \le i, j \le n$ and $i \ne j$: pairs (a_i, a_j) $(i \ne j)$ appear twice and pairs (a_s, b_u) $(s \ne u)$ appear four times as in a Latin square each entry appear exactly once in each row and column. Similarly, for $s \ne u$, pairs (a_s, c_u) appear twice, pairs (b_s, b_u) appear twice, and pairs (b_s, c_u) appear once. Also, pairs (b_s, c_s) appear $n-1$

times because these pairs appear from Latin squares L_2 and hence have the same index, and as the Latin square is idempotent, every index appear $n-1$ times as the non-diagonal entry. Similarly, we can count the occurrences of pairs in the remaining blocks. Hence, altogether, pairs (a_i, b_j), (a_i, c_j), (b_i, c_j) $(i \neq j)$ appear seven times each, but (a_i, b_i), (b_i, c_i) and (c_i, a_i) appear $n-1$ times and pairs (a_i, a_j), (b_i, b_j), (c_i, c_j) appear four times. Therefore, we have a PBIBD(3) with $\lambda_1 = 4$, $\lambda_2 = 7$ and $\lambda_3 = n-1$. □

Since for $n = 8$, there exist three idempotent MOLS by Theorem 3.2, we have the following result using the construction in Theorem 3.3.

Corollary 3.1 *For $n = 8$, we have a GDD$(8, 3, 5;4, 7)$ with groups $G_1 = \{a_1, \ldots, a_n\}$, $G_2 = \{b_1, \ldots, b_n\}$ and $G_3 = \{c_1, \ldots, c_n\}$.*

Theorem 3.4 *Necessary conditions are sufficient for a GDD$(n, 3, 5; \lambda_1, \lambda_2)$ for $n \equiv 0 \pmod 8$, i.e., a GDD$(8k, 3, 5; 4k, 8k-1)$ exists for $k \geq 1$.*

Proof: The construction is similar to the construction in Theorem 3.3. We use three idempotent MOLS here instead of two as in Theorem 3.3. Let $n = 8k (k \geq 1)$ and $V = \{a_1, \ldots, a_n, b_1, \ldots, b_n, c_1, \ldots, c_n\}$ be a set of $3n$ elements. By Theorem 3.2, there exist three idempotent MOLS of order n on symbols $\{1, 2, \ldots, n\}$, namely (L_1, o_1), (L_2, o_2) and (L_3, o_3).

For $1 \leq i, j \leq n$ and $i \neq j$, we construct the following blocks $\{a_i, a_j, b_{io_1j}, b_{io_2j}, c_{io_2j}\}$, $\{b_i, b_j, c_{io_1j}, c_{io_2j}, a_{io_2j}\}$, $\{c_i, c_j, a_{io_1j}, a_{io_2j}, b_{io_2j}\}$, and $k-1$ copies of each of the following blocks: $\{a_i, a_j, b_{io_1j}, b_{io_2j}, c_{io_3j}\}$, $\{b_i, b_j, c_{io_1j}, c_{io_2j}, a_{io_3j}\}$, and $\{c_i, c_j, a_{io_1j}, a_{io_2j}, b_{io_3j}\}$. We can count the occurrences of the pairs as in Theorem 3.3: pairs (a_i, a_j), (b_i, b_j) and (c_i, c_j) $(i \neq j)$ appear $4 + 4(k-1) = 4k$ times, and pairs (a_i, b_j), (a_i, c_j) and (b_i, c_j) $(i \neq j)$ appear $7 + 8(k-1) = 8k-1$ times, and pairs (a_i, b_i), (b_i, c_j) and (a_i, c_j) appear $\frac{n(n-1)}{n} = 8k-1$ times. That is, $\lambda_1 = 4k$ and $\lambda_2 = 8k-1$ as required. Thus, a $GDD(8k, 3, 5; 4k, 8k-1)$ exists for $k \geq 1$. □

4 GDD$(n, 3, 5; \lambda_1, \lambda_2)$ for $n \equiv 1, 4, 5 \pmod 8$

If $n \equiv 1, 4, 5 \pmod 8$, then t must be even by Corollary 1.1. The minimum value of t is 2, thus the minimum values of λ_1 and λ_2 are $\frac{nt}{2} = n$ and $2(n-1)$, respectively.

Example 4.1 A GDD$(4, 3, 5; 4, 6)$ is given in Figure 4.

Lemma 4.1 *A GDD$(n = 8s+4, 3, 5; n = 8s+4, 2(n-1) = 2(8s+3))$ exists for any $s \geq 0$.*

Proof: For $n = 4$, we have a GDD$(4, 3, 5; 4, 6)$ in Example 4.1. For $n = 8s+4 (s \geq 1)$, we construct blocks in a manner similar to the constructions in Theorems 3.3 and 3.4.

9	9	9	10	10	10	11	11	11	12	12	12
1	3	2	2	3	4	1	2	4	4	1	3
2	1	3	3	4	2	2	4	1	1	3	4
5	5	5	6	6	6	7	7	7	8	8	8
6	7	8	7	5	8	8	5	6	5	6	7
5	5	5	6	6	6	7	7	7	8	8	8
1	1	1	3	3	3	4	4	4	2	2	2
2	3	4	2	1	4	2	3	1	1	3	4
10	10	12	11	9	9	9	10	9	9	9	10
12	11	11	12	12	11	12	12	10	11	10	11
1	1	1	2	2	2	3	3	3	4	4	4
6	7	6	5	6	5	7	7	8	8	5	8
7	8	8	6	7	7	5	8	5	6	6	5
10	10	10	12	12	12	11	11	11	9	9	9
9	12	11	10	11	9	10	9	12	12	11	10

Figure 4: A GDD$(4, 3, 5; 4, 6)$

By Theorem 3.2, there exist three idempotent MOLS of order $n = 8s + 4$ ($s \geq 1$) on symbols $\{1, 2, \dots, n\}$, namely (L_1, o_1), (L_2, o_2) and (L_3, o_3).

For $1 \leq i, j \leq n$ and $i \neq j$, we construct the following blocks: two copies of each of the blocks $\{a_i,\ a_j,\ b_{io_1j},\ b_{io_2j},\ c_{io_2j}\}$, $\{b_i,\ b_j,\ c_{io_1j},\ c_{io_2j},\ a_{io_2j}\}$, $\{c_i,\ c_j,\ a_{io_1j},\ a_{io_2j},\ b_{io_2j}\}$, and $2s - 1$ copies of each of the blocks $\{a_i, a_j, b_{io_1j}, b_{io_2j}, c_{io_3j}\}$, $\{b_i, b_j, c_{io_1j}, c_{io_2j}, a_{io_3j}\}$, and $\{c_i, c_j, a_{io_1j},\ a_{io_2j},\ b_{io_3j}\}$. We can count the occurrences of the pairs as in Theorem 3.3: pairs (a_i, a_j), (b_i, b_j) and (c_i, c_j) $(i \neq j)$ appear $2 \times 4 + 4(2s - 1) = 8s + 4$ times, and pairs (a_i, b_j), (a_i, c_j) and (b_i, c_j) $(i \neq j)$ appear $2 \times 7 + 8(2s - 1) = 16s + 6 = 2(8s + 3)$ times. That is, $\lambda_1 = 8s + 4$ and $\lambda_2 = 2(8s + 3)$ as required. Thus, a $GDD(8s + 4, 3, 5; 8s + 4, 2(8s + 3))$ exists for $s \geq 0$. □

Example 4.2 A GDD$(5, 3, 5; 5, 8)$ is given in Figure 5.

1	2	3	4	5	1	2	3	4	5	1	2	3	4	5	1	2	3	4	5	1	2	3	4	5
2	3	4	5	1	2	3	4	5	1	2	3	4	5	1	2	3	4	5	1	2	3	4	5	1
6	7	8	9	10	7	8	9	10	6	8	9	10	6	7	9	10	6	7	8	10	6	7	8	9
7	8	9	10	6	8	9	10	6	7	9	10	6	7	8	10	6	7	8	9	6	7	8	9	10
11	11	11	11	11	12	12	12	12	12	13	13	13	13	13	14	14	14	14	14	15	15	15	15	15
1	3	5	2	4	1	3	5	2	4	1	3	5	2	4	1	3	5	2	4	1	3	5	2	4
3	5	2	4	1	3	5	2	4	1	3	5	2	4	1	3	5	2	4	1	3	5	2	4	1
11	13	15	12	14	13	15	12	14	11	15	12	14	11	13	12	14	11	13	15	14	11	13	15	12
13	15	12	14	11	15	12	14	11	13	12	14	11	13	15	14	11	13	15	12	11	13	15	12	14
6	6	6	6	6	7	7	7	7	7	8	8	8	8	8	9	9	9	9	9	10	10	10	10	10
6	8	10	7	9	6	8	10	7	9	6	8	10	7	9	6	8	10	7	9	6	8	10	7	9
8	10	7	9	8	8	10	7	9	8	8	10	7	9	8	8	10	7	9	8	8	10	7	9	8
11	12	13	14	15	12	13	14	15	11	13	14	15	11	12	14	15	11	12	13	15	11	12	13	14
12	13	14	15	11	13	14	15	11	12	14	15	11	12	13	15	11	12	13	14	11	12	13	14	15
1	1	1	1	1	2	2	2	2	2	3	3	3	3	3	4	4	4	4	4	5	5	5	5	5

Figure 5: A GDD$(5, 3, 5; 5, 8)$

Lemma 4.2 *A* $GDD(n = 4s + 1, 3, 5; n = 4s + 1, 2(n - 1) = 8s)$ *exists for any* $s \geq 1$.

Proof: Example 4.2 in Figure 5 illustrates the construction for $s = 1$ where $G_1 = \{1, \dots, 5\}$, $G_2 = \{6, \dots, 10\}$ and $G_3 = \{11, \dots, 15\}$ and each G_i $(i = 1, 2, 3)$ has two 2-factors. Similarly, we construct the GDDs for $s > 1$ as follows.

Let $G_1 = \{a_1, \dots, a_{4s+1}\}$, $G_2 = \{b_1, \dots, b_{4s+1}\}$ and $G_3 = \{c_1, \dots, c_{4s+1}\}$. Since a K_{4s+1} has $2s$ 2-factors, each group has $2s$ 2-factors. Suppose G_1 has 2-factors $\{T_1, \dots, T_{2s}\}$, G_2 has 2-factors $\{S_1, \dots, S_{2s}\}$, and G_3 has 2-factors $\{R_1, \dots, R_{2s}\}$. Note that each 2-factor has $n = 4s + 1$ edges. Let $T_i = \{x_1, x_2, \dots, x_{4s+1}\}$ and $S_i = \{y_1, y_2, \dots, y_{4s+1}\}$. For each $i = 1, 2, \dots, s$, we construct $(4s + 1)^2$ blocks as follows. The first $4s+1$ blocks are $\{x_j, y_j, c_1\}$, i.e., $\{x_j\} \cup \{y_j\} \cup \{c_1\}$, for $j = 1, 2, \dots, 4s+1$. In general, the k^{th} set of $4s + 1$ blocks is $\{\{x_j, y_{j+(k-1)}, c_k\} : j = 1, \dots, 4s + 1\}$ for $k = 1, 2, \dots, 4s + 1$. As a result, we have $s(4s + 1)^2$ blocks. Similarly, we construct the next $s(4s + 1)^2$ blocks using 2-factor T_i where $i = s + 1, \dots, 2s$ and 2-factor R_i ($4s + 1$ edges need to be cycled for every $4s + 1$ blocks) and point b_j from G_2 where $j = 1, \dots, 4s + 1$. Similarly, we construct the last $s(4s + 1)^2$ blocks using 2-factor S_i where $i = s + 1, \dots, 2s$ and 2-factor R_{i-s} ($4s + 1$ edges need to be cycled for every $4s + 1$ blocks) and point a_j from G_1 where $j = 1, \dots, 4s + 1$. We have a total of $3s(4s + 1)^2$ blocks as required.

From the above construction, it is clear that $\lambda_1 = 4s + 1$ since each 2-factor T_i, S_i and R_i $(i = 1, \dots, 2s)$ appear $4s + 1$ times.

Now we will check the count for λ_2. First, we consider the blocks obtained using 2-factors of G_1 and G_2 and points from G_3. As to construct these blocks, with each 2-factor T_i, we cycle S_i, $i = 1, 2, \cdots, s$, λ_2 count between the elements of G_1 and G_2 increases by 4 and between the elements of $G_1 \cup G_2$ and G_3 by 2. Therefore, as we use s 2-factors of G_1 with s 2-factors of group G_2, it contributes $4s$ towards λ_2 count between the elements of G_1 and G_2, and $2s$ towards λ_2 count between the elements of $G_1 \cup G_2$ and G_3. Similarly, counting the contribution towards λ_2 from the blocks obtained by using s 2-factors of G_1 and G_3, and then from the blocks obtained by using s 2-factors of G_2 and G_3, we obtain $\lambda_2 = 8s$ as required. We conclude that a GDD$(n = 4s + 1, 3, 5; n = 4s + 1, 2(n - 1) = 8s)$ exists for any $s \geq 1$. □

Combining Lemmas 4.1 and 4.2 (notice that Lemma 4.2 applies for both $n \equiv 1$ and 5 (mod 8)), we have the following theorem.

Theorem 4.1 *Necessary conditions are sufficient for a GDD$(n, 3, 5; \lambda_1, \lambda_2)$ for $n \equiv 1, 4, 5 \pmod 8$.*

Combining Theorems 2.2, 3.4 and 4.1, we have the following.

Theorem 4.2 *Necessary conditions are sufficient for a GDD$(n, 3, 5; \lambda_1, \lambda_2)$.*

5 Summary

In this paper we have obtained complete existence results for GDD$(n, 3, 5; \lambda_1, \lambda_2)$ with configuration $(1, 2, 2)$. It is worth noting that although Latin squares have been used to construct BIBDs and GDDs in the past, to the best of our knowledge, an application like that which we have used in Theorem 3.4 and Lemma 4.1, using two and then three idempotent MOLS consecutively, has not been done to construct other designs.

References

[1] C. J. Colbourn and D. H. Dinitz. (Eds), *Handbook of Combinatorial Designs (2nd ed.)*, Chapman and Hall, CRC Press, Boca Raton, 2007.

[2] C. J. Colbourn and A. Rosa, *Triple Systems*, Oxford Science Publications, Clarendon Press, Oxford, 1999.

[3] H. L. Fu and C. A. Rodger, Group divisible designs with two associate classes: $n = 2$ or $m = 2$, *J. Combin. Theory Ser. A* 83 (1998), 94–117.

[4] H. L. Fu, C. A. Rodger and D. G. Sarvate, The existence of group divisible designs with first and second associates having block size three, *Ars Combin.* 54 (2000), 33–50.

[5] D. Henson, S. P. Hurd and D. G. Sarvate, Group divisible designs with three groups and block size four, *Discrete Math.* 307 (2007), 1693–1706.

[6] S. P. Hurd, N. Mishra and D. G. Sarvate, Group divisible designs with two groups and block size five with fixed block configuration, *J. Combin. Math. Combin. Comput.* 70 (2009), 15–31.

[7] S. P. Hurd and D. G. Sarvate, Odd and even group divisible designs with two groups and block size four, *Discrete Math.* 284 (2004), 189–196.

[8] S. P. Hurd and D. G. Sarvate, Group divisible designs with two groups and block configuration $(1, 4)$, *J. Combin., Inf. System Sciences* 32 (2007), 1–4.

[9] S. P. Hurd and D. G. Sarvate, Group divisible designs with block size four and two groups, *Discrete Math.* 308 (2008), 2663–2673.

[10] M. Keranen and M. Laffin, Fixed block configuration group divisible designs with block size six, *Discrete Math.* 308 (2008), 2663–2673.

[11] C. C. Lindner and C. A. Rodger, *Design Theory (2nd Ed.)*, CRC Press, Boca Raton, 2009.

[12] R. C. Mullin and H. D. O. Gronau, PBDs and GDDs: The Basics, pp. 185–213 in *The CRC Handbook of Combinatorial Designs*, (Eds. C.J. Colbourn and D.H. Dinitz), CRC Press, Boca Raton, FL, 1996.

[13] R. Mwesigwa, D. G. Sarvate and L. Zhang, Group divisible designs of four groups and block size five with configuration $(1, 1, 1, 2)$, *J. Algebra Comb. Discrete Struct. Appl.* 3 (3) (2016), 187–194.

[14] N. Punnim and D. G. Sarvate, A construction for group divisible designs with two groups, *Congr. Numer.* 185 (2007), 57–60.

[15] D. G. Sarvate, N. Mishra and K. Namyalo, Group divisible designs with block size five from Clatworthy's table, *Communications in Statistics—Theory and Methods* (to appear).

[16] A. P. Street and D. J. Street, *Combinatorics of Experimental Design*, Clarendon Press, Oxford, 1987.

[17] A. P. Street and D. J. Street, Partially Balanced Incomplete Block Designs, pp. 419–423 in *The CRC Handbook of Combinatorial Designs*, (Eds. C.J. Colbourn and D.H. Dinitz), CRC Press, Boca Raton, FL, 1996.

[18] M. Zhu and G. Ge, Mixed group divisible designs with three groups and block size four, *Discrete Math.* 310 (2010), 2323–2326.

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