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Trees with the same global domination number as their square

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Abstract

A set $S \subseteq V$ is a global dominating set of a graph $G = (V, E)$ if S is a dominating set of G and $\overline{G}$, where $\overline{G}$ is the complement graph of G. The global domination number $\gamma_g(G)$ equals the minimum cardinality of a global dominating set of G. The square graph G^2 of a graph G is the graph with vertex set V and two vertices are adjacent in G^2 if they are joined in G by a path of length one or two. In this paper we provide a characterization of all trees T whose global domination number equals the global domination number of the square of T.

1 Introduction and preliminary results

For terminology and notation on graph theory not given here, the reader is referred to [5, 9]. Let $G = (V, E)$ be a graph with vertex set V and edge set E. The *square*

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graph G^2 of G is the graph with vertex set V and two vertices u and v are adjacent in G^2 whenever $d_G(u,v) \leq 2$. The *complement* $\overline{G}$ of G is the graph with vertex set V and with exactly the edges that do not belong to G. The *open neighborhood* of a vertex $v \in V$ is the set $N(v) = \{u \in V \mid uv \in E\}$, and its *closed neighborhood* is the set $N[v] = N(v) \cup \{v\}$. The *degree* of v, denoted by $\deg_G(v)$, is the cardinality of its open neighborhood. A set $S \subseteq V$ is a *dominating set* of G if every vertex of $V - S$ is adjacent to at least one vertex of S. The minimum cardinality of a dominating set of G, denoted by $\gamma(G)$, is called the *domination number* of G. A dominating set of cardinality $\gamma(G)$ is called a γ-set of G. A set $S \subseteq V$ is a *global dominating set* of G if S is a dominating set of G and $\overline{G}$. The minimum cardinality of a global dominating set of G, denoted by $\gamma_g(G)$, is called the *global domination number* of G. A global dominating set of cardinality $\gamma_g(G)$ is called a γ_g-set of G. Global domination is studied for example in [1, 3, 8], and elsewhere.

One of many applications of global domination as given in chapter 11 of [4], relates to a communication network modeled by a graph G, where subnetworks are defined by some matching M_i of cardinality k. The necessity of these subnetworks could be due for reasons of security, redundancy or limitations of recipients for different classes of messages. For this practical case, the global domination number represents the minimum number of master stations needed such that a message issued simultaneously from all masters reaches all desired recipients after traveling over only one communication link. We note that Carrington [2] gave two other applications of global dominating sets for graph partitioning commonly used in the implementation of parallel algorithms.

A set D of vertices in a graph G is a *packing* if the vertices in D are pairwise at distance at least 3 apart in G, or equivalently, for every vertex $v \in V$, $|N[v] \cap D| \leq 1$. A set $S \subseteq V$ is a *distance* 2*-dominating set* of G if $d_G(u,S) \leq 2$ for every vertex $u \in V - S$. The minimum cardinality of a distance 2-dominating set of G, denoted by $\gamma^2(G)$, is called the *distance* 2*-domination number* of G, for more see [4, 5]. We note that every graph G satisfies $\gamma^2(G) = \gamma(G^2)$, since every distance 2-dominating set of G is a dominating set of G^2, and every dominating set of G^2 is a distance 2-dominating set of G. We also mention that a distance 2-dominating set of G and $\overline{G}$ (that is, global distance 2-dominating set of G, $\gamma_g^2(G)$ is not necessarily a global dominating set of G^2, and vice versa. For example, if $G = C_5$, then $\gamma_g^2(C_5) = 1$ and $\gamma_g(C_5^2) = 5$.

A vertex that is adjacent to a leaf is called a *support vertex*. We denote by $L(G)$ and $S(G)$ the set of leaves and support vertices of a graph G, respectively. The *eccentricity* of a vertex v is $ecc(v) = \max\{d_G(v,w) : w \in V\}$. The *radius* of G is $\mathrm{rad}(G) = \min\{ecc(v) : v \in V\}$ and the diameter of G is $\mathrm{diam}(G) = \max\{ecc(v) : v \in V\}$. It is well known that for every graph G, $\mathrm{diam}(G^2) = \lceil \frac{\mathrm{diam}(G)}{2} \rceil$ and $\mathrm{rad}(G) \geq \frac{\mathrm{diam}(G)}{2}$. In particular, if T is a tree, then $\mathrm{rad}(T) = \lceil \frac{\mathrm{diam}(T)}{2} \rceil$. The *center* $C(G)$ of a connected graph G is the set of vertices of minimum eccentricity. If T is a tree and $u_0, u_1, \ldots, u_k$ is a longest path in T, then $C(T) = \{u_{\frac{k}{2}}\}$, when k is even and $C(T) = \{u_{\frac{k-1}{2}}, u_{\frac{k+1}{2}}\}$, when k is odd.

For some families of graphs, the global domination number is known or at least restricted to within a fairly limited range (see [4, 5]). For instance:

- If G or $\overline{G}$ is disconnected, then $\gamma_g(G) = max\{\gamma(G), \gamma(\overline{G})\}$.
- If G is a triangle-free graph, then $\gamma(G) \leq \gamma_g(G) \leq \gamma(G) + 1$ [4].
- Max $\{\gamma(G), \gamma(\overline{G})\} \leq \gamma_g(G) = \gamma_g(\overline{G}) \leq \gamma(G) + \gamma(\overline{G})$.

Lemma 1.1 *For any graph G, if $rad(G) \geq 3$, then every dominating set of G is a dominating set of $\overline{G}$.*

Proof. Let S be a dominating set for G and not be a dominating set for $\overline{G}$. Therefore there exists a vertex u in $V(G)$ such that u is adjacent to every vertex of S in G and so $\mathrm{rad}(G) \leq 2$, a contradiction. □

Since for every graph G, $\mathrm{rad}(G) \geq \frac{\mathrm{diam}(G)}{2}$, we have the following corollary.

Corollary 1.2 *If G is a graph with $\mathrm{diam}(G) \geq 5$, then $\gamma_g(G) = \gamma(G)$.*

In [6], Raczek gave a characterization of all trees and all unicyclic graphs with equal domination and distance 2-domination numbers. Raczek defined the family τ of trees to consist of those trees T that can be obtained from sequence $T_1, T_2, \ldots, T_j$ $(j \geq 1)$ of trees such that T_1 is the path P_2 and $T = T_j$, and, if $j > 1$, then T_{i+1} can be obtained recursively from T_i by the operation τ_1, τ_2 or τ_3:

- **Operation** τ_1. The tree T_{i+1} is obtained from T_i by adding a vertex x_1 and the edge x_1y, where $y \in V(T_i)$ is a support vertex of T_i.
- **Operation** τ_2. The tree T_{i+1} is obtained from T_i by adding a path x_1, x_2, x_3 and the edge x_1y, where $y \in V(T_i)$ is neither a leaf nor a support vertex in T_i.
- **Operation** τ_3. The tree T_{i+1} is obtained from T_i by adding a path x_1, x_2, x_3, x_4 and the edge x_1y, where $y \in V(T_i)$ is a support vertex in T_i.

Theorem 1.3 (Raczek [6]) *If T is a tree, then $\gamma(T) = \gamma(T^2)$ if and only if T belongs to the family τ.*

In [6], Raczek also showed that the set of support vertices of every tree $T \in \tau$ is both a packing and a γ-set of T.

In this paper, we characterize the trees T satisfying $\gamma_g(T) = \gamma_g(T^2)$. In Section 2, we consider graph parameters when restricted to pruned subgraphs. Using these results, in Sections 3, 4, 5, 6 and 7, we discuss trees having a fixed diameter.

2 The pruned subgraphs

Let $G = (V, E)$ be a graph. For every $u \in V$, delete all the leaves from $N(u)$ except one. The remaining graph is called the *pruned subgraph* (or *pruned subtree*, if G is a tree) of G and is denoted by G_p.

Proposition 2.1 *If T is a tree, then* $\operatorname{diam}(T_p) = \operatorname{diam}(T)$ *if and only if $T \neq K_{1,t}$ with $t \geq 2$.*

Proof. Let T be a tree different from a star $K_{1,t}$ with $t \geq 2$. Clearly, $\operatorname{diam}(T) \neq 2$. If $\operatorname{diam}(T) = 0$ or $\operatorname{diam}(T) = 1$, then obviously $\operatorname{diam}(T_p) = \operatorname{diam}(T)$. Hence assume that $\operatorname{diam}(T) \geq 3$, and let a and b be two leaves of T such that $d_T(a, b) = \operatorname{diam}(T)$. Let $P = a, u_1, u_2, \ldots, u_k, b$ be a diametral path in T. Since $\operatorname{diam}(T) \geq 3$, a and b have distinct support vertices. Hence we can assume, without loss of generality, that $a, b \in V(T_p)$. Since $\deg_T(u_i) \geq 2$, each $u_i \in V(T_p)$ and therefore P remains a path linking a and b in T_p. It follows that $\operatorname{diam}(T) \leq \operatorname{diam}(T_p)$, and the equality is obtained from the fact that $\operatorname{diam}(T) \geq \operatorname{diam}(T_p)$.

Conversely, let $T = K_{1,t}$ with $t \geq 2$. Then $T_p = P_2$ and clearly $\operatorname{diam}(T) = 2 > \operatorname{diam}(T_p) = 1$. $\square$

Proposition 2.2 *If T is a tree, then* $\operatorname{rad}(T_p) = \operatorname{rad}(T)$.

Proof. If T is a star, then $T_p = P_2$, and so $\operatorname{rad}(T_p) = \operatorname{rad}(T) = 1$. If T is not a star, then by Proposition 2.1, $\operatorname{diam}(T_p) = \operatorname{diam}(T)$. Since $\operatorname{rad}(T) = \lceil \frac{\operatorname{diam}(T)}{2} \rceil$ and $\operatorname{rad}(T_p) = \lceil \frac{\operatorname{diam}(T_p)}{2} \rceil$ we obtain the desired result. $\square$

Corollary 2.3 *If G is a graph, then $\gamma(G_p) = \gamma(G)$.*

Proof. The result is valid if G has order $n = 1$ or 2. Let $n \geq 3$, and A be a γ-set of G. It is clear that $B = (A - L(G)) \cup S(T)$ is a γ-set of G, too. Since B does not include any leaves of G, B is a γ-set of G_p. $\square$

Let $\mathfrak{F}$ denote the class of trees T with $n \geq 2$ vertices and either radius one (that is, stars) or radius two having a vertex u with $\deg_T(u) \geq 2$ and $\deg_T(v) \geq 3$ for all $v \in N(u)$ [1]. Let $\mathfrak{F}'$ denote the class of trees T with radius two having a vertex u with $\deg_T(u) \geq 2$ and $\deg_T(v) \geq 3$ for all $v \in N(u)$. Additionally, letting $\mathcal{S}$ denote the class of stars on $n \geq 2$ vertices, let $\mathfrak{F} = \mathfrak{F}' \cup \mathcal{S}$.

Theorem 2.4 *If T is a tree, then $\gamma_g(T_p) = \gamma_g(T)$ if and only if $T \notin \mathfrak{F}'$.*

Proof. If $\operatorname{diam}(T) \in \{0, 1, 2\}$, then it is clear that $\gamma_g(T_p) = \gamma_g(T)$. Hence we may assume that $\operatorname{diam}(T) \geq 3$, and let S be the set of support vertices of T. If $\operatorname{diam}(T) = 3$, then $C(T)$ is a γ-set of T and T_p, and so $\gamma_g(T_p) = \gamma_g(T) = 2$. Now suppose that $\operatorname{diam}(T) = 4$, and let u_0, u_1, u_2, u_3, u_4 be a path in T and also in T_p.

Suppose that $T \in \mathfrak{F}'$. Then $S \cup \{u_2\}$ is a γ_g-set of T, while $(S - \{u_3\}) \cup \{u_4\}$ is a γ_g-set of T_p. Hence $\gamma_g(T_p) = \gamma_g(T) - 1$. Suppose now that $T \notin \mathfrak{F}'$. Then T has either a support vertex, say u_3, of degree two or u_2 is a support vertex. If the first situation occurs, then $(S - \{u_3\}) \cup \{u_4\}$ is a γ_g-set of T and T_p, and if the second one occurs, then the set of support vertices is a γ_g-set of T and T_p. Finally, if $\mathrm{diam}(T) \geq 5$, then by Proposition 2.1, and Corollaries 1.2 and 2.3 we have $\gamma_g(T_p) = \gamma_g(T)$. □

3 Trees T with diam$(T) \leq 4$ or diam$(T) \geq 9$

We begin by considering trees T with diameter at most four. The following result has been obtained independently by Brigham and Dutton [1] and Rall [7].

Theorem 3.1 (Brigham and Dutton [1], Rall [7]) *If T is a tree, then either $T \in \mathfrak{F}$ and $\gamma_g(T) = \gamma(T) + 1$, or $\gamma_g(T) = \gamma(T)$.*

It is clear that if $\mathrm{diam}(T) = 0$ or $\mathrm{diam}(T) = 1$, then $\gamma_g(T^2) = \gamma_g(T)$.

Theorem 3.2 *If* $\mathrm{diam}(T) = 2$ *or* 3, *then* $\gamma_g(T^2) \neq \gamma_g(T)$.

Proof. If $\mathrm{diam}(T) = 2$, then T is a star $K_{1,p}$ for $p \geq 2$, so $\gamma_g(T) = 2$ and $\gamma_g(T^2) = p + 1 \geq 3$.

If $\mathrm{diam}(T) = 3$ with $P = u_0, u_1, u_2, u_3$ as a diametral path in T, then clearly $S = \{u_1, u_2\}$ is a γ_g-set of T, so $\gamma_g(T) = 2$. Suppose that $\gamma_g(T^2) = 2$ and let $S = \{u, v\}$ be a γ_g-set of T^2. If u and v are adjacent in T, then $d_T(u, a) = 2$ for every $a \in N(v) - \{u\}$ and likewise $d_T(v, b) = 2$ for every $b \in N(u) - \{v\}$. But then S is not a dominating set of the complement of T^2. Hence u and v are not adjacent in T^2, that is $d_T(u, v) = 2$ or 3. Then for every vertex x on the path between u and v, we have $d_T(x, u) = 1$ or 2 and $d_T(x, v) = 1$ or 2. Thus x is an isolated vertex in the complement of T^2 and cannot be dominated by $\{u, v\}$. Therefore $\gamma_g(T^2) > 2$. □

Lemma 3.3 *Let S be the set of support vertices of a tree T. If* $\mathrm{diam}(T) \in \{2, 3, 4, 5\}$, *then $\gamma(T) = |S|$.*

Proof. Clearly, $\gamma(T) \geq |S|$. Since $\mathrm{diam}(T) \leq 5$, every vertex of T is either a support vertex or adjacent to a support vertex. Hence S dominates all vertices of T, implying that $\gamma(T) \leq |S|$ and the equality follows. □

Lemma 3.4 *If T is a tree of diameter* 4, *then* $\gamma_g(T^2) = 3$.

Proof. Let $P = u_0, u_1, u_2, u_3, u_4$ be a path of length 4 in T. It is easy to see that the set $A = \{u_0, u_2, u_4\}$ is a global dominating set of T^2. We shall show that A is a γ_g-set of T^2. Suppose to the contrary that $\gamma_g(T^2) = 2$. Since $\mathrm{diam}(T) = 4$, $d_T(x, u_2) \leq 2$ for every $x \in V(T)$, so u_2 belongs to every γ-set of the complement of T^2 and hence to every γ_g-set of T^2. Therefore, let $A_1 = \{u_2, x\}$ be a global dominating set of T^2. If $xu_2 \in E(T)$, then each of u_1 and u_3 is at distance at most two from u_2 and x. But then A_1 does not dominate at least one of u_1 or u_3 in the complement of T^2. Thus $xu_2 \notin E(T)$. Let w be any vertex of T adjacent to both u_2 and x. Then A_1 does not dominate w in the complement of T^2, a contradiction. We deduce that $\gamma_g(T^2) \neq 2$, and so $\gamma_g(T^2) = 3$. □

Theorem 3.5 *If T is a tree of diameter 4, then $\gamma_g(T^2) = \gamma_g(T)$ if and only if*

a) $T \notin \mathfrak{F}$ and T has 3 support vertices or

b) $T \in \mathfrak{F}$ and T has 2 support vertices.

Proof. By Lemma 3.3 we have $\gamma(T) = |S(T)|$ and by Lemma 3.4 $\gamma_g(T^2) = 3$. Now by Theorem 3.1 and Lemma 3.4 the result holds. □

We turn our attention to trees with diameter at least nine.

Proposition 3.6 *If T is a tree of diameter at least 9, then $\gamma_g(T^2) = \gamma_g(T)$ if and only if $T \in \tau$.*

Proof. Since $\mathrm{diam}(T) \geq 9$, we have $\mathrm{diam}(T^2) \geq 5$. The result follows by applying Corollary 1.2 to both T and T^2, and by using Theorem 1.3. □

4 Trees with diameter five

In this section we characterize the trees T with diameter 5 such that $\gamma_g(T) = \gamma_g(T^2)$. Thoughout this section, we let $L(u)$ denote the set of leaves attached at a support vertex u.

Lemma 4.1 *Let T be a tree with $\mathrm{diam}(T) \geq 5$. If T' is a tree obtained from T by adding a new vertex attached at a support vertex of T, then $\gamma_g(T'^2) \leq \gamma_g(T^2)$.*

Proof. Let u be a support vertex of T and a be the new vertex attached at u. Let M be a γ_g-set of T^2. We will show that T'^2 has a global dominating set of cardinality $|M|$. Assume first that $M - N_T[u] \neq \emptyset$. Then vertex a is dominated in T'^2 by M as well as any vertex of $L(u)$ in T^2. Also, since $M - N_T[u] \neq \emptyset$, vertex a is at distance at least three from some vertices of $M - N_T[u]$, and so vertex a remains dominated by $M - N_T[u]$ in $\overline{T'^2}$. Therefore for that case, M is a global dominating set of T'^2. Now assume that $M - N_T[u] = \emptyset$. It is clear that $M = N_T[u]$. Since $\mathrm{diam}(T) \geq 5$, we have $\mathrm{rad}(T) \geq 3$. Thus there is a vertex $t \in V(T)$ such that $d_T(u,t) = 3$. It follows that $M_1 = (M - \{u\}) \cup \{t\}$ is a γ_g-set of T^2, too. Now since $M_1 - N_T[u] \neq \emptyset$, we deduce, as previously seen, that M_1 is a global dominating set of T'^2. Therefore $\gamma_g(T'^2) \leq \gamma_g(T^2)$. □

Lemma 4.2 *Let T be a tree and M a γ_g-set of T^2. If $\operatorname{diam}(T) \geq 4$, then $|L(u) \cap M| \leq 1$ for every $u \in S(T)$.*

Proof. To the contrary, suppose there is a support vertex $u \in S(T)$ such that $|L(u) \cap M| \geq 2$. Let $a, b \in M \cap L(u)$. If $M - N_T[u] \neq \emptyset$, then it is clear that $M - \{a\}$ is a global dominating set of T^2, a contradiction. Thus $M \subseteq N[u]$. Observe that since $d_T(z, M) \leq 2$ for every $z \in N_T[u]$, we have $M = N_T[u]$. It follows that $\operatorname{diam}(T) \leq 6$. At first let $\operatorname{diam}(T) = 4$ and $P = u_0, u_1, u_2, u_3, u_4$ be a longest path in T. The set $\{u_0, u_2, u_4\}$ is a global dominating set of T^2, hence $\gamma_g(T^2) \leq 3$, but $|N[u]| \geq 4$, that is a contradiction. Now suppose that $\operatorname{diam}(T) = 5$ and let $P = u_0, u_1, u_2, u_3, u_4, u_5$ be a longest path in T. Note that since $\{u_0, u_2, u_3, u_5\}$ is a global dominating set of T^2, we have $\gamma_g(T^2) \leq 4$. Clearly, if $u \notin C(T)$, then either $d_T(u, u_0) = 4$ or $d_T(u, u_5) = 4$. Hence $d_T(u_0, M) = 3$ or $d_T(u_5, M) = 3$, a contradiction. Thus $u \in C(T)$. But, then $|N[u]| \geq 5$, contradicting the fact that $|M| = |N_T[u]| \leq 4$. Hence we may assume that $\operatorname{diam}(T) = 6$. Let $P = u_0, u_1, u_2, u_3, u_4, u_5, u_6$ be a longest path in T. If $u \in C(T)$, then $(N_T[u] - L(u)) \cup \{u_0\}$ is a global dominating set of T^2 smaller than M, a contradiction. Hence $u \notin C(T)$. But then either $d(u_0, N_T[u]) \geq 3$ or $d(u_6, N_T[u]) \geq 3$, a contradiction. □

Theorem 4.3 *If T is a tree, then $\gamma_g(T_p^2) \neq \gamma_g(T^2)$ if and only if*

a) T is a star or

b) $\operatorname{diam}(T) = 3$ and $\deg_T(u) \geq 3$ for every $u \in S(T)$.

Proof. Let T be a tree of order n. If $\operatorname{diam}(T) \in \{0, 1\}$, then $T = T_p$ and so $\gamma_g(T_p^2) = \gamma_g(T^2)$. If $\operatorname{diam}(T) = 2$, then T is a star, T_p is K_2 and so $\gamma_g(T_p^2) = 2$ while $\gamma_g(T^2) = n > 2$. If $\operatorname{diam}(T) = 3$, then T_p is P_4. Without loss of generality let $P = u_0, u_1, u_2, u_3$ be a longest path in T and in T_p. Then $\{u_0, u_1, u_2\}$ is a γ_g-set of T_p^2, and so $\gamma_g(T_p^2) = 3$. If $\deg_T(u) \geq 3$ for every $u \in S(T)$, then $\{u_0, u_1, u_2, u_3\}$ is a γ_g-set of T^2 and so $\gamma_g(T^2) = 4$. If $\deg_T(u) = 2$ for some $u \in S(T)$, for example u_1, then the set $\{u_0, u_1, u_2\}$ is a γ_g-set of T^2 and so $\gamma_g(T^2) = 3$. Now if $\operatorname{diam}(T) = 4$, then by Proposition 2.1 and Lemma 3.4, $\gamma_g(T_p^2) = \gamma_g(T^2) = 3$. Now let $\operatorname{diam}(T) \geq 5$. Since T can be obtained from T_p by adding a new vertex at each time attached at a support vertex of T_P, Lemma 4.1 inductively implies that $\gamma_g(T^2) \leq \gamma_g(T_p^2)$. Now if M is a γ_g-set of T^2, then by Lemma 4.2 we have $|L(u) \cap M| \leq 1$ for every $u \in S(T)$. Thus, without loss of generality, we can assume that vertices of M belong to $V(T_p)$. Therefore M is a global dominating set of T_p^2. Hence $\gamma_g(T_p^2) \leq |M| = \gamma_g(T^2)$, and the desired equality follows. □

Theorem 4.4 *If T is a tree of diameter 5, then $\gamma_g(T^2) = \gamma_g(T)$ if and only if T_p is one of the trees in Figure 1.*

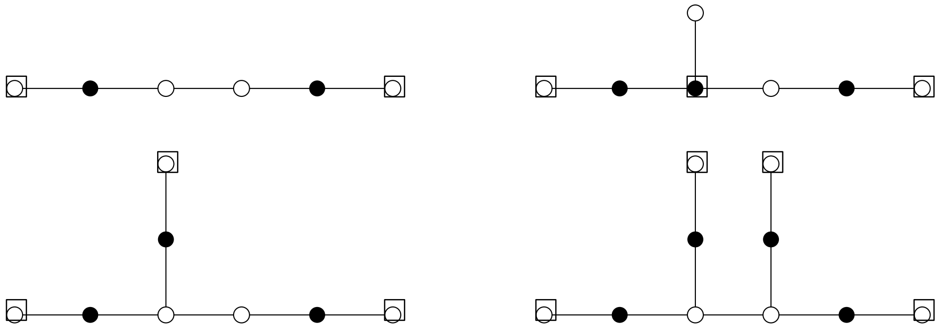


Figure **1**

Proof. In each of the figures given in Figure 2, the set of black vertices represent a γ_g-set of T while the squared vertices represent a γ_g-set of T^2. Let $P = u_0, u_1, u_2, u_3, u_4, u_5$ be a longest path in T. Since $\{u_0, u_2, u_3, u_5\}$ is a global dominating set of T^2, we have $\gamma_g(T^2) \leq 4$ and by Theorem 4.3 we have $\gamma_g(T_p^2) \leq 4$, too. By Proposition 2.1 we have $diam(T_p) = 5$ and by Lemma 3.3 and Corollary 1.2, if T_p has more than four support vertices, then $\gamma_g(T_p^2) \neq \gamma_g(T_p)$. Now the only pruned subtrees of diameter 5 with at most four support vertices are given in Figures **1** and **2.** However, for every tree in Figure **1** we have $\gamma_g(T_p^2) = \gamma_g(T_p)$, and for every tree in Figure **2**, $\gamma_g(T_p^2) \neq \gamma_g(T_p)$. Now our result follows easily from Theorems 2.4 and 4.3.

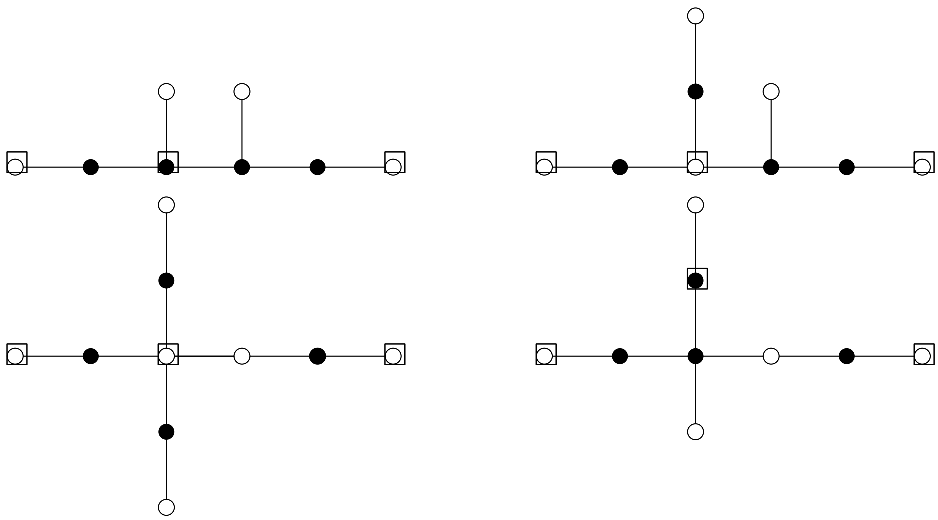


Figure **2**

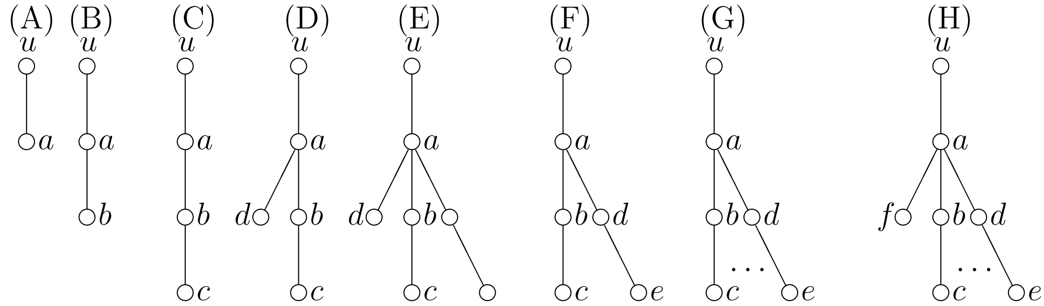
□

5 Trees with diameter six

In this section we characterize all trees T with diameter 6 such that $\gamma_g(T) = \gamma_g(T^2)$.

Lemma 5.1 *Let T be a tree with diameter 6 and center $C(T) = \{u\}$. If $\gamma_g(T^2) = \gamma_g(T)$, then $\deg_{T_p}(x) < 3$ for every $x \in N_{T_p}(u)$.*

Proof. Let T be a tree with diameter 6 such that $\gamma_g(T^2) = \gamma_g(T)$. By Theorems 2.4 and 4.3, $\gamma_g(T_p^2) = \gamma_g(T_p)$. Clearly, $N_{T_p}[u]$ is a global dominating set of T_p^2, and so $\gamma_g(T_p^2) \leq \left|N_{T_p}[u]\right| = d_{T_p}(u) + 1$. On the other hand, since $\mathrm{diam}(T) \geq 5$, we have $\gamma_g(T) = \gamma(T) \geq |S(T)| = |S(T_p)|$. For every $w \in N_{T_p}(u)$, the component of $T_p - (N_{T_p}(u) - \{w\})$ which includes w is named a *branch of* T_p *containing edge* wu. One can easily see that each branch attached at u in T_p is one of the trees in Figure 3.



Family of figures $Q = \{(A), (B), (C), (D), (E), (F), (G), (H)\}$

Figure **3**

Note that since each support vertex in T_p has exactly one leaf, there is at most one branch (A) attached at u in T_P. Now let us define $n(X)$ as the number of branches (X) attached at u in T_p, where $X \in \{A, B, C, D, E, F, G, H\}$. Hence $\deg_{T_p}(u) = \sum_{X\in\{A,B,C,D,E,F,G,H\}} n(X)$. Observe that each of the branches (A), (B) and (C) attached at u in T_p contains one vertex of $N_{T_p}(u)$ and one vertex of $S(T_p)$. Also, each of the branches (D), (E), (F), (G) and (H) attached at u in T_p contains one vertex of $N_{T_p}(u)$ and at least two vertices of $S(T_p)$. We will show that the only branches attached at u in T_p are among branches (A), (B) and (C). Now let us consider the following cases.

Case 1. T_p has at least one branch among branches (E), (G) and (H) attached at u. Each of the branches (E), (G) and (H) has at least three support vertices, and so

$$\begin{aligned}
\gamma_g(T_p) &= \gamma(T_p) \geq |S(T_p)| \\
&\geq n(A) + n(B) + n(C) + n(D) + n(F) + 3(n(E) + n(G) + n(H)) \\
&= n(A) + n(B) + n(C) + n(D) + n(E) + n(F) + n(G) + n(H) + 2(n(E) \\
&\qquad + n(G) + n(H)) \\
&\geq \deg_{T_p}(u) + 2 \geq \gamma_g(T_p^2) + 1.
\end{aligned}$$

Hence $\gamma_g(T_p^2) < \gamma_g(T_p)$, a contradiction. From now on we may assume that T_p has no branches (E), (G) and (H) attached at u.

Case 2. T_p has at least two branches among branches (D) and (F) attached at u. Each of the branches (D) and (F) has two support vertices. It follows that

$$\begin{aligned}\gamma_g(T_p) = \gamma(T_p) &\geq |S(T_p)| = n(A) + n(B) + n(C) + 2(n(D) + n(F)) \\ &= n(A) + n(B) + n(C) + n(D) + n(F) + (n(D) + n(F)) \\ &\geq \deg_{T_p}(u) + 2 \geq \gamma_g(T_p^2) + 1,\end{aligned}$$

and so $\gamma_g(T_p^2) < \gamma_g(T_p)$, a contradiction. Thus T_p has at most one branch among branches (D) and (F) attached at u.

In cases 3 and 4, we will show that T_p has no branches (D) and (F) attached at u.

Case 3. T_p has one branch (D) attached at u. Clearly since $\text{diam}(T_p) = 6$, there is at least one branch (C) attached at u. Let M be the set containing all vertices c of branches (C) plus vertex b of (D). The following situations can occur.

a) If there is no branch (A) or (B) attached at u, then M is a γ_g-set of T_p^2 and $S(T_p)$ is a γ_g-set of T_p. Therefore $\gamma_g(T_p^2) = \gamma_g(T_p) - 1$, a contradiction.

b) If there exist branches (A) or (B) attached at u, then $M \cup \{u\}$ is a global dominating set of T_p^2 and $S(T_p)$ is γ_g-set of T_p. Therefore $\gamma_g(T_p^2) \leq \gamma_g(T_p) - 1$, a contradiction.

Case 4. T_p has one branch (F) attached at u. Since $\text{diam}(T_p) = 6$, T_p has at least one branch (C) attached at u. We observe the following situations.

a) T_p has exactly one branch (C) attached at u. If there are no branches (A) and (B) attached at u, then $S(T_p) \cup \{u\}$ is a γ_g-set of T_p and vertices a of branches (F) and (C) plus u form a γ_g-set of T_p^2. Therefore $\gamma_g(T_p^2) = \gamma_g(T_p) - 1$, a contradiction.
Now if there is one branch (A) and no branch (B) attached at u, then $S(T_p)$ is a γ_g-set of T_p and vertices a and c of branch (C) and vertex a of branch (F) form a γ_g-set of T_p^2. Therefore $\gamma_g(T_p^2) = \gamma_g(T_p) - 1$, a contradiction.
If there is one branch (B) and no branch (A) attached at u, then $S(T_p)$ is a γ_g-set of T_p and vertices c of branch (C) and a of branch (F) plus b of branch (B) form a γ_g-set of T_p^2. Therefore $\gamma_g(T_p^2) = \gamma_g(T_p) - 1$, a contradiction.
If there are at least two branches (B) and no branch (A) attached at u, then $S(T_p)$ is a γ_g-set of T_p and vertices c of branch (C) and a of branch (F) and b of one of the branches (B) plus vertex u form a γ_g-set of T_p^2. Therefore $\gamma_g(T_p^2) \leq \gamma_g(T_p) - 1$, a contradiction.
If there is one branch (A) and at least one branch (B) attached at u, then $S(T_p)$ is a γ_g-set of T_p and vertices c of branch (C) and a of branch (F) plus vertex u form a γ_g-set of T_p^2. Therefore $\gamma_g(T_p^2) \leq \gamma_g(T_p) - 2$, a contradiction.

b) T_p has at least two branches (C) attached at u. If there is no branch (B) attached at u, then $S(T_p) \cup \{u\}$ is a γ_g-set of T_p and vertices c of branches (C) plus vertex a of branch (F) form a γ_g-set of T_p^2. Therefore $\gamma_g(T_p^2) = \gamma_g(T_p) - 2$, a contradiction. Now if T_p has at least one branch (B) attached at u, then $S(T_p)$ is a γ_g-set of T_p, and vertices c of branches (C) plus u and vertex a of branch (F) form a γ_g-set of T_p^2. Therefore $\gamma_g(T_p^2) \leq \gamma_g(T_p) - 1$.

So the only branches that are attached to u are (A), (B) and (C) and the vertex a in each of them has degree one or two. □

Theorem 5.2 *If T is a tree with diameter 6 and $C(T) = \{u\}$, then $\gamma_g(T^2) = \gamma_g(T)$ if and only if one of the following conditions holds:*

a) T_p has one branch (A), two branches (C) and no other branches are attached at u,

b) T_p has one branch (B), at least two branches (C) and no other branches are attached at u.

Proof. If T satisfies the conditions (a) and (b), then for T_p, the vertices c of branches (C) plus vertex u form a γ_g-set of T_p^2 and the set $S(T_p)$ forms a γ_g-set of T_p , so $\gamma_g(T_p^2) = \gamma_g(T_p)$ and by Theorems 2.4 and 4.3, we obtain $\gamma_g(T^2) = \gamma_g(T)$.

Now let $\gamma_g(T^2) = \gamma_g(T)$ and suppose that T does not satisfy conditions (a) or (b). By Theorems 2.4 and 4.3, $\gamma_g(T_p^2) = \gamma_g(T_p)$. Also by Lemma 5.1, $\deg_{T_p}(x) < 3$ for every $x \in N_{T_p}(u)$. Hence the only branches attached at u in T_p are among branches (A), (B) and (C). Moreover, since $\mathrm{diam}(T) = 6$, T_p contains at least two branches (C) attached at u. We now consider the following cases.

Case 1. One branch (A), at least one branch (B) and at least two branches (C) are attached at u in T_p. In this case, $S(T_p)$ is a γ_g-set of T_p. Also vertices c of branches (C) plus vertex u form a γ_g-set of T_p^2. Therefore $\gamma_g(T_p^2) \leq \gamma_g(T_p)-1$, a contradiction.

Case 2. One branch (A), at least three branches (C) and no branch (B) are attached at u in T_p. In this case, $S(T_p)$ is a γ_g-set of T_p. Also vertex a of one branch (C) plus vertices c of the remaining branches (C) form a γ_g-set of T_p^2. Therefore $\gamma_g(T_p^2) = \gamma_g(T_p) - 1$, a contradiction.

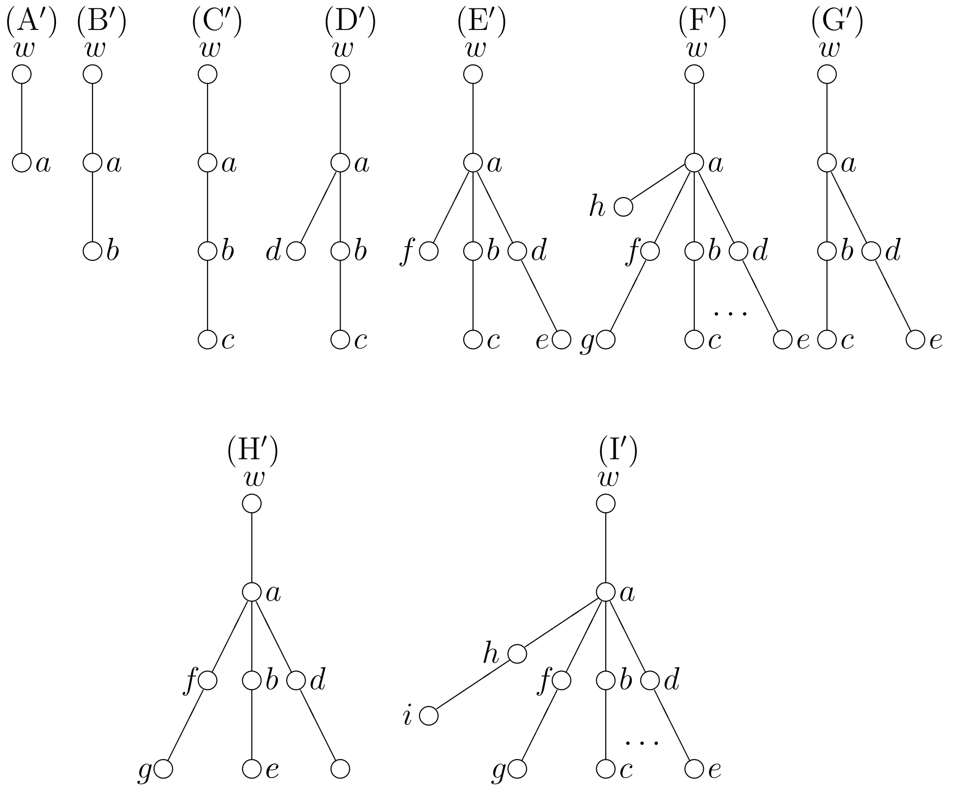
Case 3. At least two branches (B) and at least two branches (C) and no branch (A) are attached at u in T_p. In this case $S(T_p)$ is a γ_g-set of T_p. Also vertices c of branches (C) plus vertex u form a γ_g-set of T_p^2. Therefore $\gamma_g(T_p^2) \leq \gamma_g(T_p) - 1$, a contradiction. □

6 Trees with diameter seven

In this section we characterize all trees T of diameter 7 such that $\gamma_g(T) = \gamma_g(T^2)$.

Lemma 6.1 *Let T be a tree with diameter 7 and center $C(T) = \{u, v\}$. If $\gamma_g(T^2) = \gamma_g(T)$, then $\deg_{T_p}(x) < 3$ for every $x \in N_{T_p}(u) \cup N_{T_p}(v) - \{u, v\}$.*

Proof. Let $\gamma_g(T^2) = \gamma_g(T)$. By Theorems 2.4 and 4.3 we have $\gamma_g(T_p^2) = \gamma_g(T_p)$. Clearly, $N_{T_p}(u) \cap N_{T_p}(v)$ is a global dominating set of T_p^2 and so $\gamma_g(T_p^2) \leq \deg_{T_p}(u) + \deg_{T_p}(v)$. Let Q' be the family of all possible branches of T_p attached at u or v. We note that vertex w in different branches is u or v.



Family of figures $Q' = \{(A'), (B'), (C'), (D'), (E'), (F'), (G'), (H'), (I')\}$

Figure **4**

Note that since each support vertex in T_p has exactly one leaf, there is at most one branch (A') attached at u and at most one branch (A') attached at v in T_P. Let $n(X)$ be the number of branches (X) that are attached at u or v in T_p, where $X \in \{A', B', C', D', E', F', G', H', I'\}$. Hence

$$d_{T_p}(u) + d_{T_p}(v) = \sum_{X \in \{A',B',C',D',E',F',G',H',I'\}} n(X) + 2.$$

Observe that each branch (X) attached at u or v in T_p contains one vertex of $N_{T_p}(u) \cup N_{T_p}(v) - \{u, v\}$ and at least one vertex of $S(T_p)$. More precisely, if $X \in \{A', B', C'\}$, then branch (X) contains one vertex of $S(T_p)$. If $X \in \{D', G'\}$, then branch (X) contains two vertices of $S(T_p)$. If $X \in \{E', H'\}$, then branch (X) contains three vertices of $S(T_p)$. If $X \in \{F', I'\}$, then branch (X) contains at least four vertices of $S(T_p)$. It is sufficient to show that the only branches that are attached at u or v in T_p are among branches (A′), (B′) and (C′). Now if there exists a vertex $x \in N_{T_p}(u) \cup N_{T_p}(v) - \{u, v\}$ such that $\deg_{T_p}(x) \geq 3$, then we are in one of the following cases.

Case 1. T_p has at least one branch among branches (F′) and (I′) attached at u or v. Since each of the branches (F′) and (I′) has at least four support vertices, we obtain

$$\begin{aligned}\gamma_g(T_p) &= \gamma(T_p) \geq |S(T_p)| \\ &\geq n(A') + n(B') + n(C') + n(D') + n(E') + n(G') + n(H') + 4(n(F')+n(I')) \\ &= n(A') + n(B') + n(C') + n(D') + n(E') + n(F') + n(G') + n(H') + n(I') \\ &\qquad + 3(n(F') + n(I')) \\ &\geq \deg_{T_p}(u) + d_{T_p}(v) - 2 + 3 = \deg_{T_p}(u) + d_{T_p}(v) + 1 \geq \gamma_g(T_p^2) + 1.\end{aligned}$$

Hence $\gamma_g(T_p^2) < \gamma_g(T_p)$, a contradiction. Thus, for the next cases T_p has no branch (F′) nor (I′).

Case 2. T_p has at least two branches among branches (E′) and (H′) attached at u or v. Since each of the branches (E′) and (H′) has three support vertices, we obtain that

$$\begin{aligned}\gamma_g(T_p) &= \gamma(T_p) \geq |S(T_p)| \\ &\geq n(A') + n(B') + n(C') + n(D') + n(G') + 3(n(E') + n(H')) \\ &= n(A')+ n(B')+ n(C')+ n(D') + n(E') + n(G') + n(H')+2(n(E')+n(H')) \\ &\geq \deg_{T_p}(u) + d_{T_p}(v) - 2 + 4 = \deg_{T_p}(u) + d_{T_p}(v) + 2 \geq \gamma_g(T_p^2) + 2.\end{aligned}$$

Hence $\gamma_g(T_p^2) < \gamma_g(T_p)$, a contradiction.

Case 3. T_p has at least one branch (E′) and at least one branch among branches (D'), (G') and(H') attached at u or v. In this case, we have

$$\begin{aligned}\gamma_g(T_p) &= \gamma(T_p) \geq |S(T_p)| \\ &\geq n(A') + n(B') + n(C') + 2(n(D') + n(G') + n(H')) + 3n(E') \\ &= n(A') + n(B') + n(C') + n(D') + n(E') + n(G') \\ &\qquad + n(H') + (n(D') + n(G') + n(H')) + 2n(E') \\ &\geq \deg_{T_p}(u) + d_{T_p}(v) - 2 + 1 + 2 = \deg_{T_p}(u) + d_{T_p}(v) + 1 \geq \gamma_g(T_p^2) + 1.\end{aligned}$$

Hence $\gamma_g(T_p^2) < \gamma_g(T_p)$, a contradiction.

Case 4. T_p has at least one branch (H′) and at least one branch among branches (D′), (E′) and (G′) attached at u or v. In this case, we have

$$\begin{aligned}\gamma_g(T_p) &= \gamma(T_p) \geq |S(T_p)| \\ &\geq n(A') + n(B') + n(C') + 2(n(D') + n(E') + n(G')) + 3n(H') \\ &= n(A') + n(B') + n(C') + n(D') + n(E') \\ &\qquad + n(G') + n(H') + (n(D') + n(E') + n(G')) + 2n(H') \\ &\geq \deg_{T_p}(u) + d_{T_p}(v) - 2 + 1 + 2 = \deg_{T_p}(u) + d_{T_p}(v) + 1 \geq \gamma_g(T_p^2) + 1.\end{aligned}$$

Hence $\gamma_g(T_p^2) < \gamma_g(T_p)$, a contradiction.

Case 5. T_p has one branch (E′) and no branch among branches (D′), (G′) and (H′) attached at u or v. Without loss of generality, we assume that (E′) is attached at v. Since $\mathrm{diam}(G) = 7$, T_p has at least one branch (C′) attached at u. We consider the following two situations.

a) No branch (A′) or (B′) is attached at u or v in T_p. Then $S(T_p) \cup \{u\}$ is γ_g-set of T_p, and vertices c of branches (C′) plus vertex a of (E′) form a γ_g-set of T_p^2 . Therefore $\gamma_g(T_p^2) = \gamma_g(T_p) - 3$, that is a contradiction.

b) T_p contains some branches (A′) or (B′) attached at u or v. Then the set M formed by vertices a of branches (C′) and (E′) plus u, v is a global dominating set of T_p^2, implying that $|M| \leq |S(T_p)| - 1$. Hence $\gamma_g(T_p^2) \leq |M| < |S(T_p)| \leq \gamma(T_p) = \gamma_g(T_p)$, a contradiction.

Case 6. T_p has one branch (H′) and no branch among branches (D′), (E′) and (G′) attached at u or v. Without loss of generality we assume that (H′) is attached at v. Since $\mathrm{diam}(G) = 7$, T_p has at least one branch (C′) attached at u. We consider the following two situations.

a) No branch (A′) or (B′) is attached at u or v in T_p. Then $S(T_p) \cup \{u\}$ is γ_g-set of T_p, and vertices c of (C′) plus vertex a of (H′) form a γ_g-set of T_p^2. Therefore $\gamma_g(T_p^2) = \gamma_g(T_p) - 3$, a contradiction.

b) T_p contains some branches (A′) or (B′) attached at u or v. Then the set M formed by vertices a of branches (C′) and (H′) plus u, v is a global dominating set of T_p^2, implying that $|M| \leq |S(T_p)| - 1$. Therefore $\gamma_g(T_p^2) \leq |M| < |S(T_p)| \leq \gamma(T_p) = \gamma_g(T_p)$, a contradiction.

Hence for the remaining cases we consider T_p has no branch (E′) nor (H′).

Case 7. T_p has at least three branches among branches (D′) and (G′) attached at u or v. Since each of the branches (D′) and (G′) has two support vertices, we obtain that

$$\begin{aligned}\gamma_g(T_p) = \gamma(T_p) &\geq |S(T_p)| \geq n(A') + n(B') + n(C') + 2(n(D') + n(G')) \\ &= n(A') + n(B') + n(C') + n(D') + n(G') + (n(D') + n(D')) \\ &\geq \deg_{T_p}(u) + d_{T_p}(v) - 2 + 3 = \deg_{T_p}(u) + d_{T_p}(v) + 1 \geq \gamma_g(T_p^2) + 1.\end{aligned}$$

Hence $\gamma_g(T_p^2) < \gamma_g(T_p)$, a contradiction.

Case 8. T_p has one or two branches among branches (D′) and (G′) attached at u or v. Note that in this case, T_p may also contain some branches among branches (A′), (B′) and (C′) attached at u or v. It is clear that for every tree T with diameter 7 and center $\{u, v\}$, by adding every branch (C′) to u or v, the amounts $\gamma_g(T_p)$ and $\gamma_g(T_p^2)$ increase exactly by 1. We consider the trees in Figure 5.

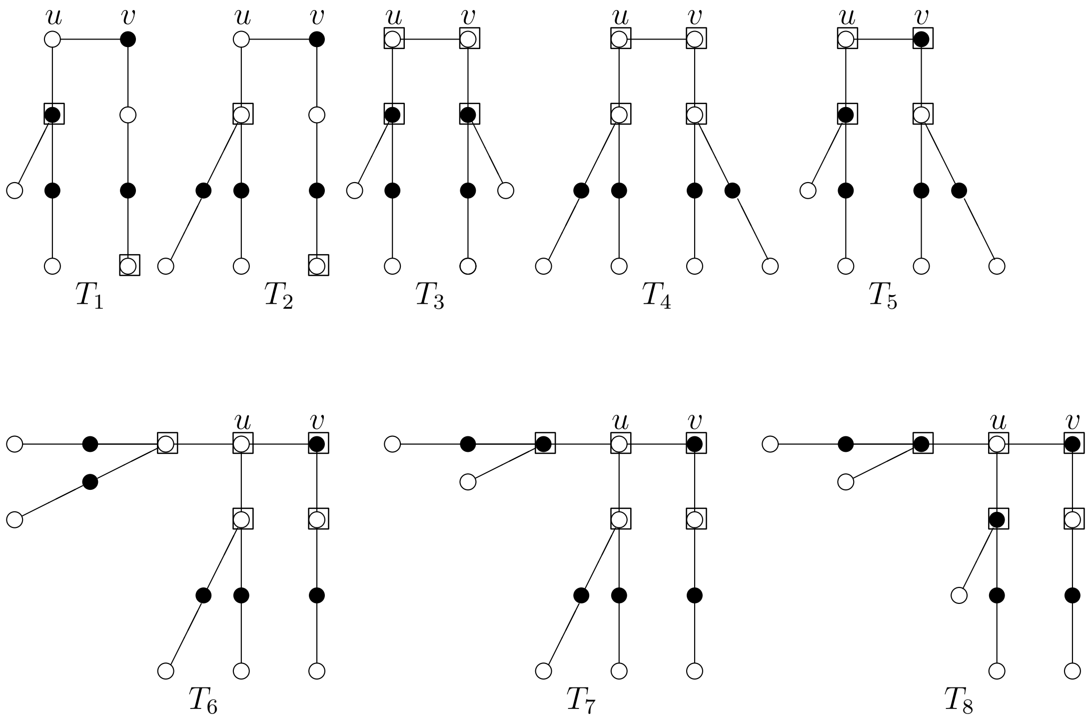


Figure **5**

The tree T_p can be obtained by adding some branches among branches (A'), (B'),(C') to vertices u or v of one of the trees T_i, $i = 1, 2, \ldots, 8$. Note that T_p has at most one branch (A') attached at u or at v. In this case, for each tree T_i in Figures **6**,**7**,**8** a γ_g-set, named M_i, by black vertices and a global dominating set of T_i^2, named N_i, by square shapes is determined. So for trees T_i, $i = 1, 2, 5, 6, 7, 8$ we have $\gamma_g(T_i^2) \leq |N_i| < |M_i| = \gamma_g(T_i)$ and for T_i, $i = 3, 4$, we have $\gamma_g(T_i^2) \leq |N_i| = |M_i| = \gamma_g(T_i)$. By adding some branches among branches (A') and (B') to u or v in T_i, $i = 3, 4$, the amount $|M_i|$ increases at least by 1 but $|N_i|$ does not change. By adding some branches among branches (A') and (B') to u or v in T_i, $i = 5, 6, 7, 8$, the amount $|M_i|$ does not change or increases by at least one, but $|N_i|$ does not change. Hence, if T_p is made by adding some branches among branches (A′), (B′) and (C′) to T_i, $i = 3, 4, \ldots, 8$, then we have $\gamma_g(T_p^2) < \gamma_g(T_p)$, that is a contradiction.

Now we consider the trees in Figure **6** that are obtained from the trees T_1 or T_2 by attaching a branch (A′) to one of the vertices u and v.

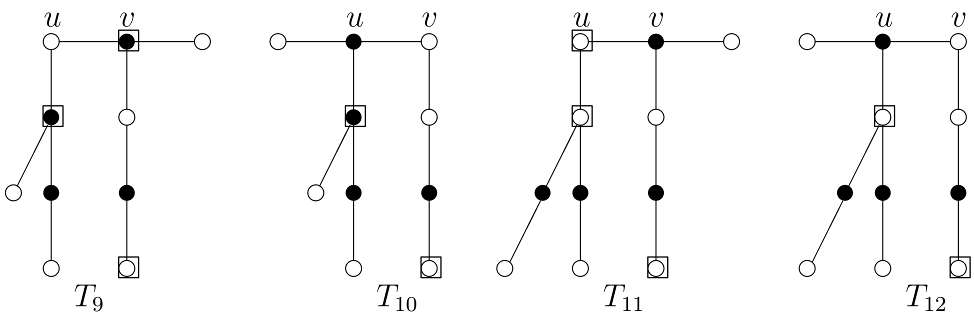


Figure **6**

For trees T_i, $i = 9, 10, 11, 12$ we have $\gamma_g(T_i^2) \leq |N_i| < |M_i| = \gamma_g(T_i)$. Now consider the trees in Figure **7** which are the trees T_i, $i = 1, 2, 9, 10, 11, 12$, with new a global dominating set of T_i^2.

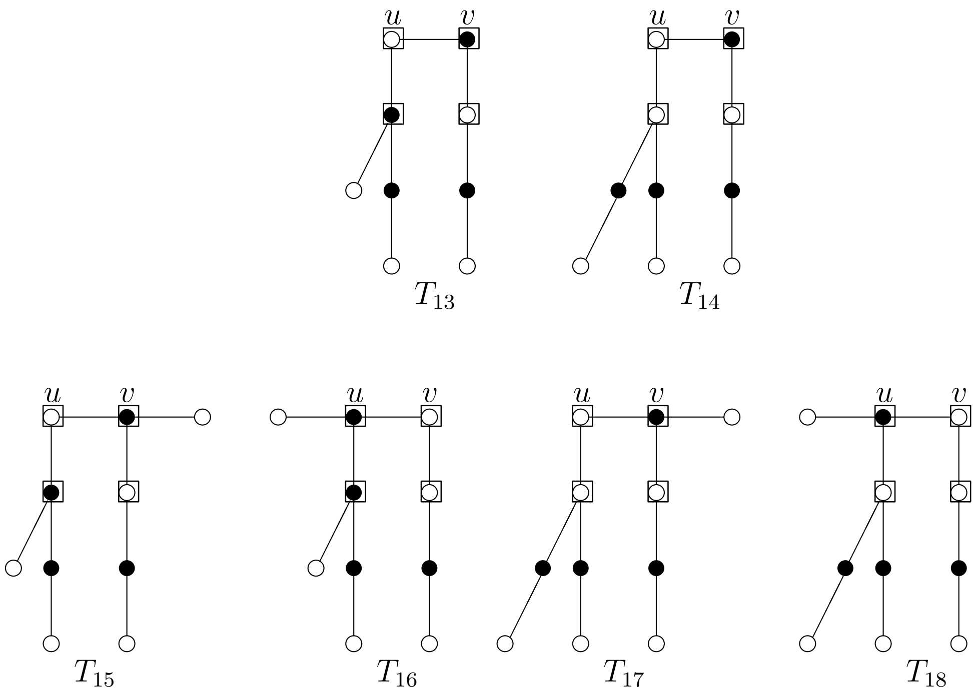


Figure **7**

By adding every branch among branches (A') and (B') to u or v in T_i, $i = 15, 16, 17, 18$ the amount $|M_i|$ increases at least by 1, but $|N_i|$ doesn't change, and by adding every branch (B') to u or v in T_i, $i = 13, 14$, then $|M_i|$ increases by one but $|N_i|$ does not change.
Consequently if T_p is obtained from T_1 or T_2 by attaching some branches among branches (A′), (B′) and (C′) at u or v we have $\gamma_g(T^2) < \gamma_g(T)$, a contradiction. Hence T_p has no branches among branches (D′) and (G′) and the only branches attached at u and v in T_p are among branches (A′), (B′) and (C′). □

Theorem 6.2 *If T is a tree with diameter 7 and center $C(T) = \{u, v\}$, then $\gamma_g(T^2) = \gamma_g(T)$ if and only if at least one branch (C') and just one branch (B') are attached at u, also at v in T_p and T_p has no other branches attached at u and v.*

Proof. Let $\gamma_g(T^2) = \gamma_g(T)$. Then by Lemma 6.1, $\deg_{T_p}(x) < 3$ for every $x \in N_{T_p}(u) \cup N_{T_p}(v) - \{u, v\}$. So there are attached only branches among branches (A′), (B′) and (C′) at u and v in T_p. Since $\mathrm{diam}(T) = 7$, T_p must have at least one branch (C′) attached at u and at least one branch (C′) attached at v. If T_p has no branch (A′) or (B′) attached at u and v, then $S(T_p) \cup \{v\}$ is a γ_g-set of T_p, and the set formed by vertex a of one branch (C′) attached at u plus vertices c of the other branches (C′) attached at u or v is a γ_g-set of T_p^2. It follows that $\gamma_g(T_p^2) = \gamma_g(T_p)-1$, a contradiction. From now on we may assume that T_p has at least one branch (C′) attached at u and at least one branch (C′) attached at v and at least one branch among branches (A′) and (B′) attached at u or v. Therefore, without loss of generality, we distinguish between the following cases.

Case 1. T_p has one branch (A′) attached at u and no branch (B′) attached at u. In this case, we are in one of the following situations.

a) No branch (A′) or (B′) is attached at v. Then $S(T_p)$ is a γ_g-set of T_p, and the set formed by vertex a of one branch (C′) attached at u plus vertices c of the other branches (C′) attached at u or v, is a γ_g-set of T_p^2. Hence $\gamma_g(T_p^2) = \gamma_g(T_p) - 1$, a contradiction.

b) One branch (A′) and no branch (B′) are attached at v. If T_p has exactly two branches (C′), then $S(T_p)$ is a γ_g-set of T_p and the set formed by u plus vertex c of branch (C′) attached at u and vertex a of branch (C′) attached at v is a γ_g-set of T_p^2. It follows that $\gamma_g(T_p^2) = \gamma_g(T_p) - 1$, a contradiction. Thus we assume that T_p has at least three branches (C′). Without loss of generality, we assume that at least two branches (C′) are attached at u. Then $S(T_p)$ is a γ_g-set of T_p and the set formed by vertex c of one of the branches (C′) attached at u plus vertices a of the remaining branches (C′) is a γ_g-set of T_p^2. It follows that $\gamma_g(T_p^2) = \gamma_g(T_p) - 2$, a contradiction.

c) At least one branch (B′) and no branch (A′) are attached at v. Then $S(T_p)$ is a γ_g-set of T_p and the set formed by vertices c of branches (C′) plus vertex v is a γ_g-set of T_p^2. Hence $\gamma_g(T_p^2) \leq \gamma_g(T_p) - 1$, a contradiction.

d) One branch (A′) and at least one branch (B′) are attached at v. Then $S(T_p)$ is a γ_g-set of T_p and the set formed by vertices c of all branches (C′) plus vertex v is a γ_g-set of T_p^2. Therefore $\gamma_g(T_p^2) \leq \gamma_g(T_p) - 2$, a contradiction.

Case 2. T_p has one branch (A′) and at least one branch (B′) attached at u. In this case, we are in one of the following situations.
a) No branch (A′) or (B′) is attached at v. Then $S(T_p)$ is a γ_g-set of T_p and the set formed by vertices c of all branches (C′) plus vertex u is a γ_g-set of T_p^2. Therefore $\gamma_g(T_p^2) \leq \gamma_g(T_p) - 1$, a contradiction.

b) No branch (A′) and at least one branch (B′) are attached at v. Then $S(T_p)$ is a γ_g-set of T_p and the set formed by vertices c of all branches (C′) plus u and v is a γ_g-set of T_p^2. Hence $\gamma_g(T_p^2) \leq \gamma_g(T_p) - 1$, a contradiction.

c) One branch (A′) and at least one branch (B′) are attached at v. Then $S(T_p)$ is a γ_g-set of T_p and the set formed by vertices c of all branches (C′) plus u and v is a γ_g-set of T_p^2. Hence $\gamma_g(T_p^2) \leq \gamma_g(T_p) - 2$, a contradiction.
Up to now by cases 1 and 2 we found that no branch (A′) is attached at u and v. From now on we may assume that no branch (A′) is attached to u and v.

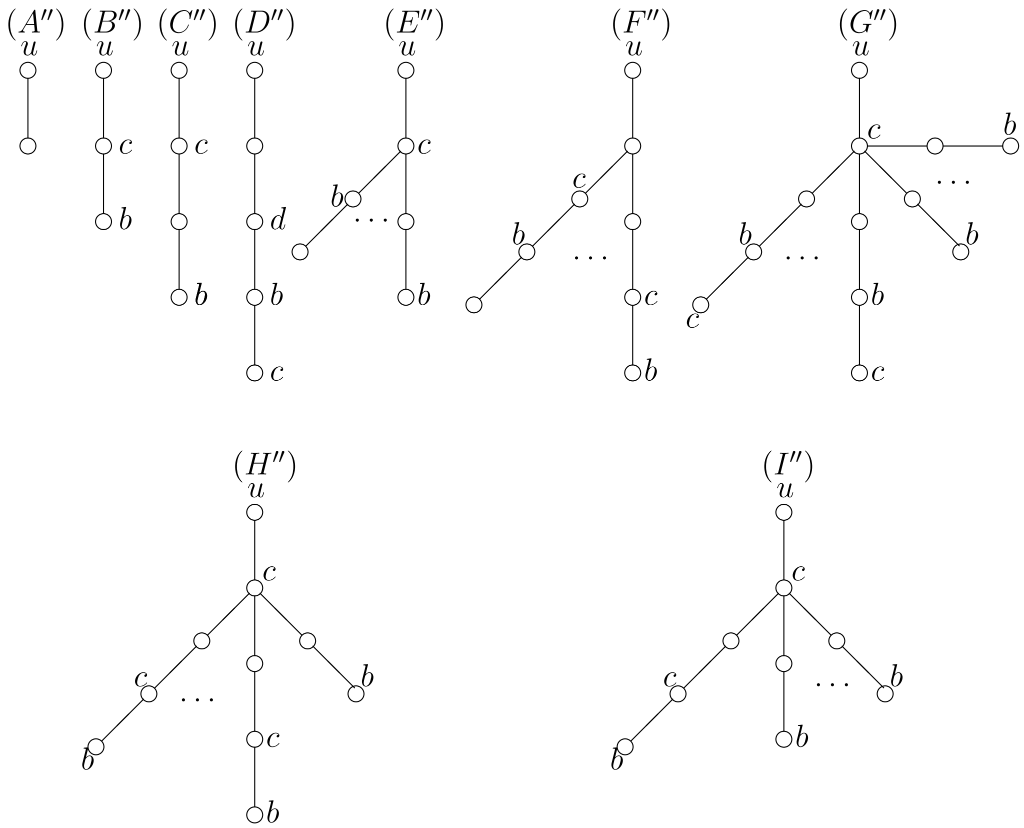
Case 3. At least one branch (B′) is attached at u. In this case, we are in one of the following situations.

a) No branch (B′) is attached at v. Then $S(T_p) \cup \{v\}$ is a γ_g-set of T_p and the set formed by vertices c of all branches (C′) plus vertex u is a γ_g-set of T_p^2. Therefore $\gamma_g(T_p^2) \leq \gamma_g(T_p) - 1$, a contradiction.

b) At least one branch (B′) is attached at v. Then $S(T_p)$ is a γ_g-set of T_p and the set formed by vertices a of all branches (C′) plus u and v is a γ_g-set of T_p^2. It follows that if more than one branch (B′) is attached at each of u or v, then $\gamma_g(T_p^2) < \gamma_g(T_p)$. Equality between $\gamma_g(T_p^2)$ and $\gamma_g(T_p)$ holds when there is attached exactly one branch (B′) at u, and one at v. □

7 Trees with diameter eight

For this section, consider the branches of Figure 8.



Family of figures $Q'' = \{(A''), (B''), (C''), (D''), (E''), (F''), (G''), (H''), (I'')\}$

Figure **8**

According to the definition of family τ, if $T \in \tau$ and $\mathrm{diam}(T) = 8$, then T_p can be a tree of Figure **9** or 10 with u as a center vertex.

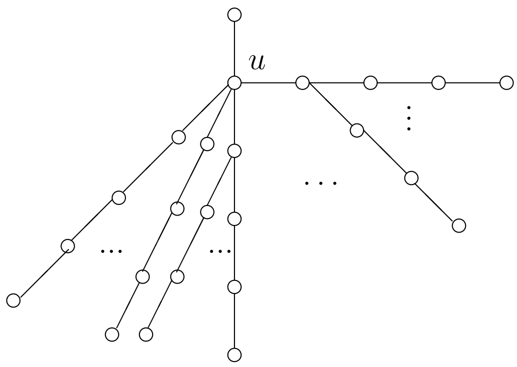


Figure **9**

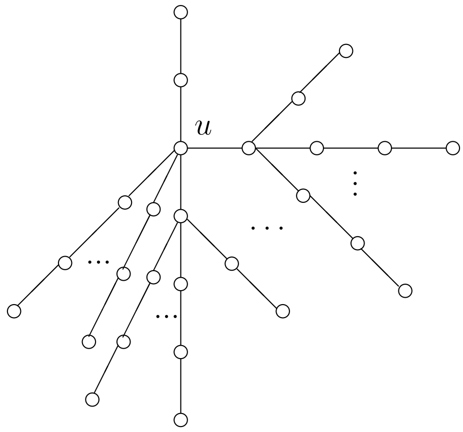


Figure **10**

Figure **9** consist of one branch (A'') and at least two branches among branches (D'') and (F'') attached at u, while Figure **10** consist of one branch (B'') and some branches among branches (C'') and (H'') attached at u. Note that in Figure **10**, since $\mathrm{diam}(T) = 8$, so at least two branches (H'') are attached at u.

Theorem 7.1 *If T is a tree of diameter 8, then $\gamma_g(T^2) = \gamma_g(T)$ if and only if $T \in \tau$.*

Proof. If $T \in \tau$, then $S(T)$ is both a γ_g-set of T and γ_g-set of T^2, and therefore $\gamma_g(T^2) = \gamma_g(T)$.
Conversely, let T be a tree of diameter 8 and center vertex u such that $\gamma_g(T^2) = \gamma_g(T)$. By Theorems 2.4 and 4.3, we have $\gamma_g(T_p^2) = \gamma_g(T_p)$ and since $\mathrm{diam}(T_p) \geq 5$ we have $\gamma_g(T_p) = \gamma(T_p)$. Let $B_1, B_2, \ldots, B_k$ denote the branches of T_p attached at u. Note that since $\mathrm{diam}(T) = 8$, so the branches attached at u in T or T_p are at most in diameter four. Let S_i be the set of support vertices of branch B_i. Note that if $B_i = (A'')$, then $|S_i| = 0$. Let W_i be the set of vertex labeled b if $B_i = (B'')$ and $W_i = \{v \in V(B_i) \mid d_{T_p}(v, u) = 2, d_{T_p}(v) > 1\}$ if $B_i \neq (B'')$, $i \in \{1, 2, \ldots, k\}$. Let $W = \bigcup_{i=1}^{k} W_i$. Clearly, $|W_i| \leq |S_i|$, $i \in \{1, 2, \ldots, k\}$ and $\sum_{i=1}^{k} |S_i| \leq |S(T_p)| \leq \gamma(T_p) = \gamma_g(T_p)$. Since $W \cup \{u\}$ is a global dominating set of T_p^2 we obtain that $\gamma_g(T_p^2) \leq |W| + 1$. We note that if $B_i \in Q''$, then $|W_i| = |S_i|$. However, T_p may contain some branch $B_j \notin Q''$ and for which we have $|S_j| > |W_j|$. Now let us examine the different situations.

Case 1. Either $|S_j| \geq |W_j| + 2$ for some $j \in \{1, 2, \ldots, k\}$ or $|S_r| = |W_r| + 1$ and $|S_t| = |W_t| + 1$ for some $r, t \in \{1, 2, \ldots, k\}$ with $r \neq t$. In this case, we have
$\gamma_g(T_p^2) \leq |W|+1 = (\sum_{i=1}^{k} |W_i|)+1 \leq \left(\sum_{i=1}^{k} |S_i|\right)-2+1 \leq \gamma_g(T_p)-1$, a contradiction.

Case 2. $|S_j| = |W_j| + 1$ for just one $j \in \{1, 2, \ldots, k\}$ and $B_i \in Q''$, for $i \in \{1, 2, \ldots, k\} - \{j\}$. Without loss of generality, assume that $j = 1$. Clearly, since $\mathrm{diam}(T) = 8$, at least one of the branches $(D''), (F''), (G''), (H'')$ and (I'') is attached at u. First suppose that there is a branch (A'') attached at u in T_p, then $S(T) =$

$(\bigcup_{i=1}^{k} S_i) \cup \{u\}$ and so $|S(T)| = \sum_{i=1}^{k} |S_i| + 1$, implying that

$$\begin{aligned}\gamma_g(T_p^2) &\leq |W| + 1 = (\sum_{i=1}^{k} |W_i|) + 1 = |w_1| + (\sum_{i=2}^{k} |W_i|) + 1 \\ &= (|S_1| - 1) + (\sum_{i=2}^{k} |S_i|) + 1 = \sum_{i=1}^{k} |S_i| = |S(T_p)| - 1 \leq \gamma_g(T_p) - 1,\end{aligned}$$

a contradiction. Suppose now that no branch (A'') is attached at u. Then the set M consists of W_1 and the vertices labeled b of branches $B_2, B_3, \ldots, B_k$ is a global dominating set of T^2. The number of such vertices labeled b in each B_i, with $i \neq 1$, equals to $|W_i|$. Therefore we obtain

$$\begin{aligned}\gamma_g(T_p^2) &\leq |M| = |W| = \sum_{i=1}^{k} |W_i| = |W_1| + \sum_{i=2}^{k} |W_i| \\ &= (|S_1| - 1) + \sum_{i=2}^{k} |S_i| = (\sum_{i=1}^{k} |S_i|) - 1 \leq \gamma_g(T_p) - 1,\end{aligned}$$

a contradiction.

Hence from now on we will assume that each branch B_i belongs to Q'', $i = 1, 2, \ldots, k$. Note that according to the definition of pruned subgraph, at most one branch (A'') is attached at u in T_p, and since $\mathrm{diam}(T) = 8$, at least two branches among branches (D''), (F''), (G''), (H'') and (I'') are attached at u.

Case 3. There is a branch among (E''), (G'') or (I'') attached at u in T_p. In this case, the set M consisting of vertices labeled c in branches of $\{(E''), (G''), (I'')\}$ plus vertices of W_i not in branches of $\{(E''), (G''), (I'')\}$ is a global dominating set of T^2. Using the fact that the number of vertices labeled c in branch $B_j \in \{(E''), (G''), (I'')\}$ is less than $|W_j|$, we deduce that

$$\gamma_g(T_p^2) \leq |M| < \sum_{i=1}^{k} |W_i| = \sum_{i=1}^{k} |S_i| \leq |S(T_p)| \leq \gamma(T_p) = \gamma_g(T_p),$$

a contradiction. In the next case, we may consider that each B_i belongs to $\{(A''), (B''), (C''), (D''), (F''), (H'')\}$.

Case 4. $B_i \in \{(C''), (D''), (F''), (H'')\}$ for every $i \in \{1, 2, \ldots, k\}$. Clearly $\bigcup_{i=1}^{k} S_i$ does not dominate u in T_p and $(\bigcup_{i=1}^{k} S_i) \cup \{u\}$ is a dominating set of T. Hence $\gamma(T_p) = 1 + \sum_{i=1}^{k} |S_i|$. Now consider the following two subcases. Suppose that at least one branch among (F'') and (H'') is attached at u. In this case, the set M consists of the vertices labeled c in all branches attached at u is a global dominating set of T_p^2. Since the number of vertices labeled c in each branch attached at u equals to the number of support vertices of that branch, we obtain:

$$\gamma_g(T_p^2) \leq |M| = \sum_{i=1}^{k} |S_i| = \gamma(T_p) - 1 = \gamma_g(T_p) - 1,$$

a contradiction. Now suppose that no branch among branches (F'') and (H'') is attached at u. Hence each B_i belongs to $\{(C''),(D'')\}$. Since $\mathrm{diam}(T) = 8$, at least two branches of (D'') are attached at u in T_p. It follows that the set M that consists of the vertex labeled d in one branch (D'') plus vertices labeled b of the other branches attached at u is a global dominating set of T_p^2. Hence we obtain:

$$\gamma_g(T_p^2) \leq |M| = \sum_{i=1}^{k} |S_i| = \gamma(T_p) - 1 = \gamma_g(T_p) - 1,$$

a contradiction. Therefore there is at least one branch among (A'') and (B'') attached at u in T_p.

Case 5. $B_i = (A'')$ for one $i \in \{1, 2, \ldots, k\}$. Without loss of generality let $i = 1$. Suppose there are some branches among branches (B''),(C'') and (H'') attached at u in T_p. Then $S(T_p) = (\bigcup_{i=2}^{k} S_i) \cup \{u\}$ and so $|S(T)| = 1 + (\sum_{i=2}^{k} |S_i|)$. Also since $\mathrm{diam}(T) = 8$, there exist at least two branches among branches (D''),(F'') and (H'') attached at u in T_p. Hence the set M that consists of the vertices labeled c in all branches attached at u is a global dominating set of T^2. Now since the number of vertices labeled c in each branch $B_i \in \{(B''),(C''),(D''),(F''),(H'')\}$ equals $|S_i|$, we obtain:

$$\gamma_g(T_p^2) \leq |M| = \sum_{i=2}^{k} |S_i| = |S(T_p)| - 1 \leq \gamma(T_p) - 1 = \gamma_g(T_p) - 1,$$

a contradiction. Hence every B_i belongs to $\{(A''),(D''),(F'')\}$. Note that a tree with such branches is a tree of the family τ (see Figure **9**).

Case 6. There are some branches (B'') attached at u in T_p. By case 5, there is no branch (A'') attached at u in T_p. If there are some branches among (D'') and (F'') attached at u in T_p, then the set $\bigcup_{i=1}^{k} S_i$ does not dominate the vertices of $N_{T_p}(u)$ in branches (D'') and (F'') but $(\bigcup_{i=1}^{k} S_i) \cup \{u\}$ is a dominating set of T_p. It follows that $\gamma(T) = 1 + \sum_{i=1}^{k} |S_i|$. On the other hand, the set M that consists of the vertices labeled b of branches (D'') and (F'') plus vertices of W_i of branches $B_i \notin \{(D''),(F'')\}$ attached at u in T_p, is a global dominating set of T_p^2. Now since the number of vertices labeled b in branch $B_i \in \{(D''),(F'')\}$ equals $|W_i|$ we have:

$$\gamma_g(T_p^2) \leq |M| = \sum_{i=1}^{k} |W_i| = \sum_{i=1}^{k} |S_i| = \gamma(T_p) - 1 = \gamma_g(T_p) - 1,$$

a contradiction. Hence all the branches attached at u in T_p are among (B''),(C'') and (H''). We will show that T has exactly one branch (B'') attached at u. Suppose to the contrary, at least two branches (B'') are attached at u. In this case the set M that consists of the vertices W_i of each branch $B_i \in \{(C''),(H'')\}$ plus vertex u is a global dominating set of T^2. Hence

$$\gamma_g(T_p^2) \leq |M| \leq \sum_{i=1}^{k} |S_i| - 1 \leq \gamma(T_p) - 1 = \gamma_g(T_p) - 1,$$

a contradiction. Consequently, T has one branch (B'') attached at u and all the other branches are among (C'') and (H''). Now since $\mathrm{diam}(T) = 8$, at least two branches of (H'') are attached at u in T_p. It is clear then such a tree T belongs to family τ (see Figure **10**). □

We conclude this paper by mentioning that the problem of characterizing all graphs G such that $\gamma_g(G) = \gamma_g(G^2)$ remains open. Although the case of trees was solved in this paper, it is still interesting to see the case of the unicyclic graphs or more generally the cactus graphs.

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