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Partitions according to multiplicities and part sizes

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Abstract

In this paper we study the largest parts in integer partitions according to multiplicities and part sizes. Firstly we investigate various properties of the multiplicities of the largest parts. We then consider the sum of the m largest parts - first as distinct parts and then including multiplicities. Finally, we find the generating function for the sum of the m largest parts of a partition, i.e., the first m parts of a weakly decreasing sequence of parts.

1 Introduction

A partition of n is a representation of n as a sum of positive integers where the summands are arranged left to right in weakly decreasing order. The summands are

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called parts of the partition. Thus a partition λ of n into $k > 0$ parts will generally be expressed as

$$\lambda = \lambda_1\lambda_2\cdots\lambda_k, \text{ where } \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0. \tag{1.1}$$

Standard properties of partitions can be found in Andrews (see [1]). In this paper we are interested in statistics relating to the largest parts in partitions, such as those in [2–7]. In some of these references statistics related to the largest part and multiplicities thereof (in partitions) have been considered. In this paper, we extend these results to the mth largest parts - not previously considered - by using the framework below.

Multiplicities and part sizes of partitions are well-studied statistics. These statistics have invariably been studied in terms of generating functions which have been obtained by a method involving functional equations which relate a larger case of the statistic to smaller cases. A recursion is generated upwards based on the smallest cases (smallest to largest).

Here, however, we provide a different conceptual framework in order to develop our generating functions. Namely, we begin the analysis by defining the generating function on the basis of the largest term in the partitions. To obtain an explicit version of this generating function, we use the functional recursion which moves from the largest term to smaller terms. This device enables generating functions for previously unstudied statistics in partition theory to be investigated.

In Section 2, we start by discussing the multiplicities of the largest parts in a partition in three different ways. In Section 3 we consider the sum of the m largest (distinct/unequal) parts in a partition. We include multiplicities in Section 4 and calculate the sum of the m largest parts in a partition where each part is added as many times as it occurs in the partition (its multiplicity). The last case is dealt with in Section 5, where we sum the first m parts of a partition (expressed again as a (weakly) decreasing sequence of parts).

For partitions where all parts are distinct, all three of these variations of counting largest parts are the same and for all of our results, the special case $m = 1$ has been previously studied. Here we use the above method and extend the results to $m > 1$.

2 Multiplicity of the largest parts in a partition

We start with the following generating function:

$$D_{a_1}(t) = D_{a_1}(t|x_1, x_2, \ldots) = \sum_{\pi = a_1^{s_1} a_2^{s_2} \cdots a_j^{s_j}} t^{s_1a_1+s_2a_2+\cdots+s_ja_j} \prod_{i=1}^{j} x_i^{s_i},$$

where the sum is over all partitions $\pi = a_1^{s_1}a_2^{s_2}\cdots a_j^{s_j}$ with $a_1 > a_2 > \cdots > a_j > 0$ and $s_1, \ldots, s_j > 0$, where s_i counts the multiplicity of part a_i. Thus the x_i's sequentially count the multiplicity of the next largest part in the partition. Note that $x_1, x_2, \ldots$ in $D_{a_1}(t|x_1, x_2, \ldots)$ is not an infinite sequence and has at most a_1 terms.

Since each partition with first part a_1 can be written either as $a_1^{s_1}$ or as $a_1^{s_1}\pi$ with π a partition such that its first part is at most $a_1 - 1$, we obtain

$$D_{a_1}(t|x_1, x_2, \ldots) = \frac{x_1 t^{a_1}}{1 - x_1 t^{a_1}} + \frac{x_1 t^{a_1}}{1 - x_1 t^{a_1}} \sum_{j=1}^{a_1 - 1} D_j(t|x_2, x_3, \ldots). \tag{2.1}$$

Define

$$D(t, u) = D(t, u|x_1, x_2, \ldots) = 1 + \sum_{a_1 \geq 1} D_{a_1}(t|x_1, x_2, \ldots)u^{a_1}.$$

By multiplying (2.1) by $(1 - x_1 t^{a_1})u^{a_1}$ and summing over $a_1 \geq 1$, we get

$$\begin{aligned}\sum_{a_1 \geq 1} D_{a_1}(t|x_1, x_2, \ldots)u^{a_1} - x_1 \sum_{a_1 \geq 1} D_{a_1}(t|x_1, x_2, \ldots)(tu)^{a_1}\\ = x_1 \sum_{a_1 \geq 1} (tu)^{a_1} + \frac{x_1 tu}{1 - ut} \sum_{j \geq 1} D_j(t|x_2, x_3, \ldots)(tu)^j,\end{aligned}$$

which implies the following result.

Theorem 2.1 *We have*

$$D(t, u|x_1, x_2, \ldots) = 1 - x_1 + x_1 D(t, tu|x_1, x_2, \ldots) + \frac{x_1 tu}{1 - ut} D(t, tu|x_2, x_3, \ldots).$$

Proposition 2.2 *Theorem 2.1 with $x_j = 1$ for all $j \geq 1$ yields*

$$D(t, u|1, 1, \ldots) = \frac{1}{1 - tu} D(t, tu|1, 1, \ldots),$$

which implies that

$$D(t, u|1, 1, \ldots) = \frac{1}{\prod_{j \geq 1} (1 - ut^j)},$$

in accordance with the well-known partition generating function.

2.1 The multiplicity of the largest part in partitions

In this subsection, we rederive the generating function for the number of partitions of n according to the multiplicity of the largest part (see [4]) by a different method, using Theorem 2.1. That is, we study the generating function $F_1(t, u, q) = D(t, u|q, 1, 1, \ldots)$.

Theorem 2.3 *We have*

$$F_1(t, u, q) = 1 + \sum_{j \geq 1} \frac{q^j t^j u}{\prod_{i \geq j} (1 - ut^i)}.$$

Moreover, the generating function for the number of partitions of n such that the largest part appears exactly $r \geq 1$ times is given by

$$\frac{t^r}{\prod_{i \geq r} (1 - t^i)}.$$

Proof: By Theorem 2.1 with $x_1 = q$ and $x_j = 1$ for all $j \geq 2$, we have

$$F_1(t,u,q) = 1 - q + qF_1(t,tu,q) + \frac{qtu}{1-ut} D(t,tu|1,1,\ldots).$$

Thus, by Proposition 2.2, we obtain

$$F_1(t,u,q) = 1 - q + qF_1(t,tu,q) + \frac{qtu}{\prod\limits_{j\geq 1}(1-ut^j)}.$$

By iterating infinitely many times (we assumed that $|q| < 1$), we have

$$F_1(t,u,q) = \sum_{j\geq 0} q^j \left(1 - q + \frac{qt^{j+1}u}{\prod\limits_{i\geq j+1}(1-ut^i)}\right),$$

which is equivalent to

$$F_1(t,u,q) = 1 + \sum_{j\geq 1} \frac{q^j t^j u}{\prod\limits_{i\geq j}(1-ut^i)},$$

as required. □

Example 2.4 Theorem 2.3 with $q = -1$ yields

$$F_1(t,1,-1) = 1 + \sum_{j\geq 1} \frac{(-1)^j t^j}{\prod\limits_{i\geq j}(1-t^i)}.$$

Thus, by Proposition 2.2, we see that the generating function for the number of partitions of n such that the multiplicity of the largest part is an even number is given by

$$\frac{1}{2}\left(\frac{1}{\prod\limits_{j\geq 1}(1-t^j)} + 1 + \sum_{j\geq 1} \frac{(-1)^j t^j}{\prod\limits_{i\geq j}(1-t^i)}\right),$$

and the generating function for the number of partitions of n such that the multiplicity of the largest part is an odd number is given by

$$\frac{1}{2}\left(\frac{1}{\prod\limits_{j\geq 1}(1-t^j)} - 1 - \sum_{j\geq 1} \frac{(-1)^j t^j}{\prod\limits_{i\geq j}(1-t^i)}\right).$$

Corollary 2.5 *The generating function for the total multiplicity of the largest part in all partitions of n is given by*

$$\sum_{j\geq 1} \frac{jt^j}{\prod\limits_{i\geq j}(1-t^i)} = \frac{1}{\prod_{i\geq 1}(1-t^i)} \sum_{j\geq 1} jt^j \prod_{i=1}^{j-1}(1-t^i).$$

PROOF: By Theorem 2.3 we have that

$$\frac{\partial}{\partial q} F_1(t,1,q)\Big|_{q=1} = \sum_{j\geq 1} \frac{jt^j}{\prod_{i\geq j}(1-t^i)},$$

as required. □

2.2 The multiplicity of the m-th largest part in partitions

Here we focus on new results, namely the generating function for the number of partitions of n according to the multiplicity of the m-th largest part. That is, we study the generating function $F_m(t,u,q) = D(t,u|x_1,x_2,\ldots)$ with $x_m = q$ and $x_j = 1$ for all $j \neq m$.

Theorem 2.6 *Let $m \geq 1$; then*

$$F_m(t,u,q) = \sum_{j_m,j_{m-1},\ldots,j_2\geq 1} \prod_{i=2}^{m} \frac{ut^{j_i+j_{i+1}+\cdots+j_m}}{1-ut^{j_i+j_{i+1}+\cdots+j_m}} \left(1+\sum_{j_1\geq 1} \frac{q^{j_1}t^{j_1+j_2+\cdots+j_m}}{\prod_{\ell\geq j_1+j_2+\cdots+j_m}(1-ut^\ell)}\right).$$

PROOF: By Theorem 2.1 with $x_m = q$ and $x_j = 1$ for all $j \neq m$, we have

$$F_m(t,u,q) = F_m(t,tu,q) + \frac{tu}{1-ut} F_{m-1}(t,tu,q). \tag{2.2}$$

Iterating (2.2) on u infinitely many times and summing, we obtain

$$F_m(t,u,q) = \sum_{j_m\geq 1} \frac{ut^{j_m}}{1-ut^{j_m}} F_{m-1}(t,ut^{j_m},q).$$

Hence, by induction on m,

$$F_m(t,u,q) = \sum_{j_m,j_{m-1},\ldots,j_2\geq 1} \prod_{i=2}^{m} \frac{ut^{j_i+j_{i+1}+\cdots+j_m}}{1-ut^{j_i+j_{i+1}+\cdots+j_m}} F_1(t,ut^{j_2+j_3+\cdots+j_m},q).$$

By Theorem 2.3, we have

$$F_m(t,u,q) = \sum_{j_m,j_{m-1},\ldots,j_2\geq 1} \prod_{i=2}^{m} \frac{ut^{j_i+j_{i+1}+\cdots+j_m}}{1-ut^{j_i+j_{i+1}+\cdots+j_m}} \left(1+\sum_{j_1\geq 1} \frac{q^{j_1}t^{j_1+j_2+\cdots+j_m}}{\prod_{\ell\geq j_1+j_2+\cdots+j_m}(1-ut^\ell)}\right),$$

as claimed. □

2.3 The sum of the multiplicities of the first m largest parts

We now study the generating function for multiplicities of the first m largest parts, that is, we study the generating function $G_m(t,u,q) = D(t,u|x_1,x_2,\ldots)$ with $x_1 = x_2 = \cdots = x_m = q$ and $x_j = 1$ for all $j > m$.

Theorem 2.7 *For all $m \geq 1$,*

$$G_m(t,u,q) = 1 + \sum_{j \geq 1} \frac{q^j t^j u}{1-ut^j} G_{m-1}(t, t^j u, q),$$

with $G_0(t,u,q) = \frac{1}{\prod\limits_{j \geq 1}(1-ut^j)}$.

Proof: In a similar fashion to the proof of Theorem 2.3, we obtain

$$G_m(t,u,q) = 1 - q + qG_m(t,tu,q) + \frac{qtu}{1-ut} G_{m-1}(t,tu,q),$$

which leads to

$$G_m(t,u,q) = 1 + \sum_{j \geq 1} \frac{q^j t^j u}{1-ut^j} G_{m-1}(t, t^j u, q).$$

□

Example 2.8 Putting $m = 1$ in Theorem 2.7 yields Theorem 2.3. Putting $m = 2$ in Theorem 2.7 yields

$$G_2(t,u,q) = 1 + \sum_{j_2 \geq 1} \frac{q^{j_2} t^{j_2} u}{1-ut^{j_2}} \left(1 + \sum_{j_1 \geq 1} \frac{q^{j_1} t^{j_1+j_2} u}{1-ut^{j_1+j_2}} \frac{1}{\prod\limits_{j_0 \geq 1}(1-ut^{j_0+j_1+j_2})} \right).$$

2.4 The difference between the multiplicities of the second largest part and the largest part

Now we focus on the generating function $M(t,u,q) := D(t,u|q,1/q,1,1,\ldots)$. By Theorem 2.1, we have

$$M(t,u,q) = 1 - q + qM(t,tu,q) + \frac{qtu}{1-ut} D(t,tu|1/q,1,1,\ldots).$$

Using Theorem 2.3, we obtain

$$M(t,u,q) = 1 - q + qM(t,tu,q) + \frac{qtu}{1-ut} \left(1 + \sum_{j \geq 1} \frac{q^{-j} t^{j+1} u}{\prod_{i \geq j+1}(1-ut^i)} \right).$$

Solving this recursion gives

$$M(t,u,q) = 1 + \sum_{s \geq 1} \frac{q^s t^s u}{1-t^s u} \left(1 + \sum_{j \geq 1} \frac{q^{-j} t^{j+s} u}{\prod_{i \geq j+1}(1-ut^{i+s-1})} \right).$$

The usual approach to summing the quantities under consideration is to compute $\frac{\partial}{\partial q} M(t,u,q)|_{q=1}$. However, a simpler expression (2.4) can be obtained by the following approach.

In order to calculate the difference between the multiplicities of the second largest and largest part, we use the conjugate of a Ferrers diagram of a partition. From this we see that the sum of the multiplicities of the largest and second largest parts is equal to the sum of the second smallest part of all partitions of n. Also

$$\begin{aligned}&\sum_{\pi\in\mathcal{P}(n)} \text{(multiplicity of second largest part - multiplicity of largest part)}\\ &= \sum_{\pi\in\mathcal{P}(n)} \text{(second smallest part size - 2} \times \text{smallest part size)} \qquad (2.3)\end{aligned}$$

or, if there is only one part size this is

$$-\sum_{\pi\in\mathcal{P}(n)} \text{(smallest part size)}.$$

For partitions with only one part size, the sum of the multiplicities of the largest and second largest parts equals the sum of the smallest part in partitions of this type, i.e.,

$$\sum_{\pi=rr\cdots r} r = \sum_{r|n} r =: \sigma(n),$$

with generating function

$$\sum_{k\geq 1} \frac{kz^k}{1-z^k}.$$

For the sum of second smallest parts in partitions of n, let the second smallest part be $k \geq 2$. The generating function for the sum of second smallest part sizes is

$$\sum_{k\geq 2}\sum_{l=1}^{k-1} \frac{z^l}{1-z^l}\frac{kz^k}{1-z^k}\prod_{i\geq k+1}\frac{1}{1-z^i}.$$

Thus the generating function obtained from (2.3) is

$$\sum_{k\geq 2}\sum_{l=1}^{k-1} \frac{z^l}{1-z^l}\frac{kz^k}{1-z^k}\prod_{i\geq k+1}\frac{1}{1-z^i} - 2\sum_{j\geq 1}\prod_{i\geq j}\frac{jt^j}{1-t^i} + \sum_{k\geq 1}\frac{kz^k}{1-z^k}. \qquad (2.4)$$

3 Sum of the m largest (distinct) parts

In this section we sum the m largest (distinct) parts in a partition. For example, given the partition 666533211111 the sum for $m = 1$ would be 6, the sum for $m = 2$ would be $6+5 = 11$ and the sum for $m = 3$ would be $6+5+3 = 14$. The $m = 1$ case can also be seen as the number of parts if you consider the conjugate of the Ferrers diagram. This was studied by Erdös and Lehner in [3] in 1941.

We define the following function for a fixed a_1, where a_1 is the size of the first (largest) part. Let x_i mark the size of the next largest distinct part in the sequence of parts. The variable t marks the size of the partition (counted by n). Let

$$\tilde{D}_{a_1}(t) := \tilde{D}_{a_1}(t|x_1, x_2 \dots) = \sum_{\pi = a_1^{s_1} a_2^{s_2} \dots a_j^{s_j}} t^n \prod_{i=1}^{j} x_i^{a_i}, \tag{3.1}$$

where $n = s_1 a_1 + s_2 a_2 + \cdots s_j a_j$. Then

$$\tilde{D}_{a_1}(t|x_1, x_2, \dots) = \frac{(x_1 t)^{a_1}}{1 - t^{a_1}} + \frac{(x_1 t)^{a_1}}{1 - t^{a_1}} \sum_{j=1}^{a_1 - 1} \tilde{D}_j(t|x_2, x_3, \dots). \tag{3.2}$$

Now define

$$\tilde{D}(t, u) := \tilde{D}(t, u|x_1, x_2, \dots) = 1 + \sum_{a_1 \ge 1} \tilde{D}_{a_1}(t|x_1, x_2 \dots) u^{a_1}.$$

We multiply (3.2) by $(1 - t^{a_1}) u^{a_1}$ and sum on $a_1 \ge 1$:

$$\begin{aligned}
\sum_{a_1 \ge 1} \tilde{D}_{a_1}(t|x_1, x_2, \dots) u^{a_1} - \sum_{a_1 \ge 1} \tilde{D}_{a_1}(t|x_1, x_2, \dots) t^{a_1} u^{a_1} \\
= \sum_{a_1 \ge 1} (x_1 t u)^{a_1} + \sum_{j \ge 1} \tilde{D}_j(t|x_2, x_3, \dots) \sum_{a_1 \ge j+1} (x_1 t u)^{a_1} \\
= \frac{x_1 t u}{1 - x_1 t u} + \sum_{j \ge 1} \tilde{D}_j(t|x_2, x_3, \dots) \frac{(x_1 t u)^{j+1}}{1 - x_1 t u}.
\end{aligned}$$

Thus we can write

$$\begin{aligned}
[\tilde{D}(t, u|x_1, x_2, \dots) - 1] - [\tilde{D}(t, tu|x_1, x_2, \dots) - 1] \\
= \frac{x_1 t u}{1 - x_1 t u} \left[1 + \sum_{j \ge 1} \tilde{D}_j(t|x_2, x_3, \dots)(x_1 t u)^j \right],
\end{aligned}$$

or

$$\tilde{D}(t, u) - \tilde{D}(t, tu) = \frac{x_1 t u}{1 - x_1 t u} \tilde{D}(t, x_1 t u|x_2, x_3, \dots). \tag{3.3}$$

3.1 Sum of the sizes of the largest (distinct) part

In particular, we can find the generating function for the size of the largest part. For example in partition 666533211111 the largest part is 6.

Let $x_1 = q$ and let $x_2, x_3, \dots = 1$ in (3.3) so that

$$\tilde{D}(t, u|q, 1, 1, \dots) - \tilde{D}(t, tu|q, 1, \dots) = \frac{qtu}{1 - qtu} \tilde{D}(t, qtu|1, 1, \dots).$$

Define

$$\tilde{D}_1(t,u,q) := \tilde{D}(t,u|q,1,1,\dots),$$

then (see Proposition 2.2)

$$\tilde{D}_1(t,u,q) - \tilde{D}_1(t,tu,q) = \frac{qtu}{1-qtu}\tilde{D}_1(t,qtu,1) = qtu\frac{1}{\prod_{j\ge 1}(1-uqt^j)}.$$

We iterate this infinitely many times to obtain

$$\tilde{D}_1(t,u,q) = \sum_{i\ge 1}\frac{qt^iu}{\prod_{j\ge i}(1-uqt^j)}. \tag{3.4}$$

We no longer need the variable u, so we replace it by 1 and then differentiate with respect to q:

$$\frac{\partial}{\partial q}\tilde{D}_1(t,1,q) = \sum_{i\ge 1}\frac{\prod_{j\ge i}(1-qt^j)t^i - qt^i\sum_{k\ge i}\frac{\prod_{j\ge i}(1-qt^j)}{1-qt^k}(-t^k)}{\prod_{j\ge i}(1-qt^j)^2}.$$

Finally, replace q by 1 to get:

Proposition 3.1 *The generating function for the sum of the largest part sizes over all partitions of n is*

$$\frac{\partial}{\partial q}\tilde{D}_1(t,1,q)|_{q=1} = \sum_{i\ge 1}t^i\left(1+\sum_{k\ge i}\frac{t^k}{1-t^k}\right)\frac{1}{\prod_{j\ge i}(1-t^j)}.$$

3.2 Recursion for the sum of the m largest (distinct) part sizes

We now generalise and let $x_1 = x_2 = \cdots = x_m = q$ and $x_j = 1$ for all $j > m$ in (3.3) to obtain the recursion

$$\tilde{D}_m(t,u,q) - \tilde{D}_m(t,tu,q) = \frac{qtu}{1-qtu}\tilde{D}_{m-1}(t,qtu,q),$$

where we define

$$\tilde{D}_m(t,u,q) := \tilde{D}(t,u,q|x_1,x_2,\dots)$$

with $x_1 = x_2 = \cdots = x_m = q$ and $x_j = 1$ for all $j > m$. For $m \ge 1$, the latter recursion can be solved as before to obtain

$$\tilde{D}_m(t,u,q) = \sum_{i\ge 1}\frac{qut^i}{1-qut^i}\tilde{D}_{m-1}(t,qut^i,q). \tag{3.5}$$

3.3 Sum of the two largest (distinct) part sizes

Now we sum the two largest (distinct) part sizes. For example in partition 666533211111 the sum would be $6+5=11$. For the particular case where $m=2$ we have (from (3.5) and (3.4))

$$\begin{aligned}\tilde{D}_2(t,u,q) &= \sum_{i\geq 1}\frac{qut^i}{1-qut^i}\tilde{D}_1(t,qut^i,q)\\ &= \sum_{i\geq 1}\sum_{k\geq 1}\frac{q^3u^2t^{k+2i}}{(1-qut^i)\prod_{j\geq k}(1-q^2ut^{j+i})}.\end{aligned}$$

We then let $u=1$

$$\tilde{D}_2(t,1,q) = \sum_{i\geq 1}\sum_{k\geq 1}\frac{q^3t^{k+2i}}{(1-qt^i)\prod_{j\geq k}(1-q^2t^{j+i})}.$$

Differentiate with respect to q, and then let $q=1$ to obtain:

Proposition 3.2 *The generating function for the sum of the two largest distinct part sizes over all partitions of n is*

$$\frac{\partial}{\partial q}\tilde{D}_2(t,1,q)\Big|_{q=1} = \sum_{i\geq 1}\sum_{k\geq 1}\prod_{j\geq k}\frac{1}{1-t^{j+i}}\left[\frac{3t^{k+2i}-2t^{k+3i}}{(1-t^i)^2}+\sum_{l\geq k}\frac{2t^{l+k+3i}}{(1-t^i)(1-t^{l+i})}\right].$$

Here we do not count the case where there is only one distinct part.

3.4 Sum of the three largest (distinct) part sizes

For the case where $m=3$ (in partition 666533211111 this sum would be $6+5+3=14$) we do not count the case if there are only 1 or 2 distinct parts. We obtain:

$$\begin{aligned}\tilde{D}_3(t,u,q) &= \sum_{l\geq 1}\frac{qut^l}{1-qut^l}\tilde{D}_2(t,qut^l,q)\\ &= \sum_{l\geq 1}\sum_{i\geq 1}\sum_{k\geq 1}\frac{q^6u^3t^{3l+2i+k}}{(1-qut^l)(1-q^2ut^{l+i})\prod_{j\geq k}(1-q^3ut^{l+j+i})}.\end{aligned}$$

Then let $u=1$, which gives

Proposition 3.3 *The generating function for the sum of the three largest distinct part sizes over all partitions of n is*

$$\tilde{D}_3(t,1,q) = \sum_{l\geq 1}\sum_{i\geq 1}\sum_{k\geq 1}\frac{q^6t^{3l+2i+k}}{(1-qt^l)(1-q^2t^{l+i})\prod_{j\geq k}(1-q^3t^{l+j+i})}.$$

4 Sum of the m largest parts including their respective multiplicities

In this section we sum the m largest parts including their respective multiplicities. For example, in partition 666533211111 the sum for $m = 1$ is $6+6+6 = 18$ and the sum for $m = 2$ is $6 + 6 + 6 + 5 = 23$.

This variation requires a slight change from the recursion in (3.2), where here x_i counts the sum of all the ith largest parts. We still fix a_1, where a_1 is the size of the first part. The variables x_i and t are as in Section 3.

$$\hat{D}_{a_1}(t|x_1, x_2, \dots) = \frac{(x_1t)^{a_1}}{1-(x_1t)^{a_1}} + \frac{(x_1t)^{a_1}}{1-(x_1t)^{a_1}} \sum_{j=1}^{a_1-1} \hat{D}_j(t|x_2, x_3, \dots). \tag{4.1}$$

Define

$$\hat{D}(t,u) := \hat{D}(t,u|x_1,x_2,\dots) := 1 + \sum_{a_1\ge 1} \hat{D}_{a_1}(t|x_1,x_2,\dots)u^{a_1}.$$

Multiply Equation (4.1) by $(1-(x_1t)^{a_1})u^{a_1}$ and sum as before

$$\begin{aligned}\sum_{a_1\ge1} \hat{D}_{a_1}(t|x_1,x_2,\dots)u^{a_1} - \sum_{a_1\ge1} \hat{D}_{a_1}(t|x_1,x_2,\dots)(x_1t)^{a_1}u^{a_1}& \\ = \sum_{a_1\ge1}(x_1tu)^{a_1} + \sum_{a_1\ge1}(x_1tu)^{a_1}\sum_{j=1}^{a_1-1}\hat{D}_j(t|x_2,x_3,\dots)& \\ = \frac{x_1tu}{1-x_1tu} + \sum_{j\ge1}\hat{D}_j(t|x_2,x_3,\dots)\frac{(x_1tu)^{j+1}}{1-x_1tu}&.\end{aligned}$$

We can express this as

$$\hat{D}(t,u|x_1,x_2,\dots) - \hat{D}(t,x_1tu|x_1,x_2,\dots) = \frac{x_1tu}{1-x_1tu}\hat{D}(t,x_1tu|x_2,x_3,\dots). \tag{4.2}$$

4.1 Sum of all largest parts

Using the above recursion we can now sum all of the parts with largest size. For example in partition 666533211111 the sum is $6 + 6 + 6 = 18$. To do this, we let $x_1 = q$ and let $x_i = 1$ if $i \neq 1$ as before. This gives us

$$\hat{D}(t,u|q,1,1,\dots) - \hat{D}(t,qtu|q,1,1,\dots) = \frac{qtu}{\prod_{j\ge1}(1-uqt^j)}$$

from Proposition (2.2). We define

$$\hat{D}_1(t,u,q) := \hat{D}(t,u|q,1,1,\dots) \tag{4.3}$$

and iterate this function as before to obtain the trivariate function

$$\hat{D}_1(t,u,q) = \sum_{k\geq 1} \frac{uq^k t^k}{\prod_{j\geq k}(1-uq^k t^j)}. \tag{4.4}$$

Now,

$$\frac{\partial}{\partial q}\hat{D}_1(t,1,q) = \sum_{k\geq 1} \frac{kq^{k-1}t^k - q^k t^k \sum_{i\geq k} \frac{1}{1-q^k t^i}(-kq^{k-1}t^i)}{\prod_{j\geq k}(1-q^k t^j)}.$$

By letting $q=1$ in the above we have

Proposition 4.1 *The generating function for the total sum of the largest parts over all partitions of n is*

$$\frac{\partial}{\partial q}\hat{D}_1(t,1,q)\Big|_{q=1} = \sum_{k\geq 1} \frac{kt^k\left(1+\sum_{i\geq k}\frac{t^i}{1-t^i}\right)}{\prod_{j\geq k}(1-t^j)}.$$

The sequences of these sums for $n=1,2,3,\ldots$ is given in [8] as A092321, where an alternative generating function is provided:

$$\sum_{n\geq 1}\left(n\frac{t^n}{1-t^n}\prod_{k=1}^{n}\frac{1}{1-t^k}\right) = \sum_{n\geq 1}\frac{nt^n}{(1-t^n)\prod_{k=1}^{n}(1-t^k)}.$$

4.2 Sum of the two largest part sizes with multiplicities

In the partition 666533211111 the sum of the two largest part sizes with multiplicities is $6+6+6+5=23$. Let $x_1=x_2=q$ in (4.2) and let $x_i=1$ if $i>2$ as before. This yields

$$\hat{D}(t,u|q,q,1,\ldots) - \hat{D}(t,qtu|q,q,1,\ldots) = \frac{qtu}{1-qtu}\sum_{k\geq 1}\frac{uq^{k+1}t^{k+1}}{\prod_{j\geq k}(1-uq^{k+1}t^{j+1})}.$$

Now define

$$\hat{D}_2(t,u,q) := \hat{D}(t,u|q,q,1,\ldots)$$

and after iterating on u, we get

$$\begin{aligned}\hat{D}_2(t,u,q) &= \sum_{i\geq 1}\frac{q^i ut^i}{1-q^i ut^i}\sum_{k\geq 1}\frac{uq^k t^k}{\prod_{j\geq 1}(1-uq^k t^{j+k-1})}\\ &= \sum_{k\geq 1}\frac{uq^{k+i}t^{k+i}}{\prod_{j\geq k}(1-uq^{k+i}t^{j+i})}.\end{aligned}$$

After differentiating with respect to q and then setting $q=u=1$ this yields

Proposition 4.2 *The generating function of the total sum of the two largest part sizes with multiplicities over all partitions of n is*

$$\frac{\partial}{\partial q}\hat{D}_2(t,1,q)|_{q=1} = \sum_{i\geq 1}\sum_{k\geq 1}\frac{(1-t^i)(k+2i)t^{k+2i} + it^{k+3i} + (1-t^i)(k+i)\sum_{l\geq k}\frac{t^{k+l+3i}}{(1-t^{l+i})}}{(1-t^i)^2\prod_{j\geq k}(1-t^{j+i})}. \tag{4.5}$$

5 Sum of the first (largest) m parts

Here we sum the largest m parts whether or not they are distinct. For the previous partition 666533211111 the sum for $m=2$ is $6+6=12$.

In this case we have another version of the generating function D where it is important that we note that x_i marks the size of the i-th part from the left in a weakly decreasing partition of n. That is, in this case it is possible to have equality of the exponents x_i and x_j even if $i \neq j$. The definitions of the other variables remain the same.

$$\bar{D}_{a_1}(t|x_1,x_2,\dots) = (x_1t)^{a_1} + (x_1t)^{a_1}\sum_{j=1}^{a_1}\bar{D}_j(t|x_2,x_3,\dots). \tag{5.1}$$

Define

$$\bar{D}(t,u) := \bar{D}(t,u|x_1,x_2,\dots) := \sum_{a_1\geq 1}\bar{D}_{a_1}(t|x_1,x_2,\dots)u^{a_1}.$$

Now we multiply Equation (5.1) by u^{a_1} and sum on a_1 to get

$$\begin{aligned}\sum_{a_1\geq 1}\bar{D}_{a_1}(t|x_1,x_2,\dots)u^{a_1} &= \sum_{a_1\geq 1}(x_1t)^{a_1}u^{a_1} + \sum_{a_1\geq 1}(x_1t)^{a_1}\sum_{j=1}^{a_1}\bar{D}_j(t|x_2,x_3,\dots)u^{a_1}\\ &= \frac{1}{1-x_1ut}\left(x_1ut + \sum_{j\geq 1}\bar{D}_j(t|x_2,x_3,\dots)(x_1ut)^j\right),\end{aligned}$$

which translates to

$$\bar{D}(t,u|x_1,x_2,\dots) = \frac{1}{1-x_1ut}\left(x_1ut + \bar{D}(t,x_1ut|x_2,x_3,\dots)\right).$$

Recall that m is fixed. Set $x_1,x_2,\dots,x_m = q$ and $x_{m+1},x_{m+2},\dots = 1$ and define $\bar{D}_m(t,u,q) := \bar{D}(t,u|x_1,x_1,\dots)$ in this case. Then we have the recurrence

$$\bar{D}_m(t,u,q) = \frac{1}{1-qut}\left(qut + \bar{D}_{m-1}(t,qut,q)\right).$$

5.1 The first part

In the special case where $m = 1$ we have the same generating function for the largest part from Section 3, namely

$$\bar{D}_1(t,u,q) = \tilde{D}_1(t,u,q) = \sum_{i\geq 1} \frac{uqt^i}{\prod\limits_{j\geq i}(1-uqt^j)},$$

(see Equation (3.4)).

5.2 Sum of the first two parts

If $m = 2$ we have

$$\begin{aligned}\bar{D}_2(t,u,q) &= \frac{1}{1-qut}\left(qut + \bar{D}_1(t,qut,q)\right)\\ &= \frac{1}{1-qut}\left(qut + \sum_{i\geq 1}\frac{uq^2t^{i+1}}{\prod\limits_{j\geq i}(1-uq^2t^{j+1})}\right). \qquad (5.2)\end{aligned}$$

The next step is to let $u = 1$ and then differentiate with respect to q to obtain

$$\begin{aligned}&\frac{\partial}{\partial q}\bar{D}_2(t,1,q)\\ &= \frac{t}{(1-qt)^2} + \sum_{i\geq 1}\frac{(1-qt)2qt^{i+1} - q^2t^{i+1}\left[(-t) + (1-qt)\sum\limits_{k\geq i}\frac{1}{1-q^2t^{k+1}}(-2qt^{k+1})\right]}{(1-qt)^2\prod\limits_{j\geq i}(1-q^2t^{j+1})}.\end{aligned}$$

Letting $q = 1$, we have

Proposition 5.1 *The generating function for the sum of the first two parts over all partitions of n is*

$$\frac{\partial}{\partial q}\bar{D}_2(t,1,q)\Big|_{q=1} = \frac{t}{(1-t)^2} + \sum_{i\geq 1}\frac{t^{i+2} + 2t^{i+1}(1-t)\left(1 + \sum\limits_{k\geq i}\frac{t^{k+1}}{1-t^{k+1}}\right)}{(1-t)^2\prod\limits_{j\geq i}(1-t^{j+1})}.$$

In this case we count the case when there is only one part size.

5.3 Sum of the first three parts

For $m = 3$ we have:

$$\begin{aligned}&\bar{D}_3(t,u,q) = \frac{1}{1-qut}\left(qut + \bar{D}_2(t,qut,q)\right)\\ &= \frac{qut}{1-qut} + \frac{q^2ut^2}{(1-qut)(1-uq^2t^2)} + \frac{1}{(1-qut)(1-uq^2t^2)}\sum_{i\geq 1}\frac{uq^3t^{i+2}}{\prod\limits_{j\geq i}(1-uq^3t^{j+2})}.\end{aligned}$$

Now let $u = 1$ and differentiate with respect to q

$$\frac{\partial}{\partial q}\bar{D}_3(t,1,q)$$
$$= \frac{(1-qt)t - qt(-t)}{(1-qt)^2} + \frac{(1-qt-q^2t^2+q^3t^3)2qt^2 - q^2t^2(-t-2qt^2+3q^2t^3)}{(1-qt-q^2t^2+q^3t^3)^2}$$
$$- \frac{(1-qt)(-2qt^2)+(1-q^2t^2)(-t)}{(1-qt)^2(1-q^2t^2)^2}\sum_{i\ge 1}\frac{q^3t^{i+2}}{\prod_{j\ge i}(1-q^3t^{j+2})}$$
$$+ \frac{1}{(1-qt)(1-q^2t^2)}\sum_{i\ge 1}\frac{\prod_{j\ge i}(1-q^3t^{j+2})3q^2t^{i+2} - q^3t^{i+2}\sum_{k\ge i}\frac{\prod_{j\ge i}(1-q^3t^{j+2})}{1-q^3t^{k+2}}(-3q^2t^{k+2})}{\left(\prod_{j\ge i}(1-q^3t^{j+2})\right)^2}.$$

After substituting $q = 1$ we obtain

Proposition 5.2 *The generating function for the sum of the first three parts over all partitions of n is*

$$\frac{\partial}{\partial q}\bar{D}_3(t,1,q)\Big|_{q=1} =$$
$$\frac{t(1+3t)}{(1-t)^3(1+t)^2}\left(1+\sum_{i\ge 1}\frac{t^{i+2}}{\prod_{j\ge i}(1-t^{j+2})}\right) + \frac{1}{(1-t)^2(1+t)}\sum_{i\ge 1}\frac{3t^{i+2}\left(1+\sum_{k\ge i}\frac{t^{k+2}}{1-t^{k+2}}\right)}{\prod_{j\ge i}(1-t^{j+2})}.$$

Here we include the cases where there are only one or two part sizes.

6 Conclusion

In this paper we have exploited a method that specifies the generating function of partitions from the largest part down (rather than from the smallest up). This allows certain previously unstudied statistics to be investigated. Also, it adds to the repertoire of tools used in the study of integer partitions and the paper consists of several different examples involving variants on the method.

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