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# Bipieri tableaux

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**Abstract**

We introduce a new class of combinatorial objects, which we call "bipieri tableaux," which arise in a natural way from the evaluation of products consisting of commutative or noncommutative complete homogeneous symmetric functions and elementary symmetric functions through repeated applications of Pieri rules. We prove using sign-reversing involutions on bipieri tableaux an elegant coproduct formula for noncommutative Schur-hooks and an elegant coproduct formula for elements of the shin basis indexed by a reverse hook composition. We show how sign-reversing involutions on commutative bipieri tableaux may be used to construct alternative combinatorial proofs of special cases of the classical Littlewood-Richardson rule.

## 1 Introduction

The classical Pieri rules for the Schur basis $\{s_\lambda\}_{\lambda \in \mathcal{P}}$ of $\mathsf{Sym}$ consist of the Pieri rule for products of the form $s_\lambda h_r$, and the Pieri rule for products of the form $s_\lambda e_r$. There are three well-known recently-introduced Schur-like bases of $\mathsf{NSym}$ which satisfy natural analogues of both classical Pieri rules, namely: the dual quasi-Schur basis $\{\mathcal{S}^*_\alpha\}_{\alpha \in \mathcal{C}}$, the immaculate basis $\{\mathfrak{S}_\alpha\}_{\alpha \in \mathcal{C}}$, and the shin basis $\{\text{ש}_\alpha\}_{\alpha \in \mathcal{C}}$. We refer to these bases as the *canonical Schur-like bases of* $\mathsf{NSym}$.

Tableaux which result from the process of evaluating (commutative or noncommutative) homogeneous or elementary symmetric functions in terms of Schur or Schur-like functions through a repeated application of a Pieri rule, such as semistandard tableaux, are of course very simple and natural combinatorial objects. However, products involving both homogeneous functions and elementary functions give rise to more complicated tableaux through repeated applications of Pieri rules. Informally, a *bipieri tableau* is a tableau of this form. The purpose of this paper is to formalize this definition and to demonstrate how bipieri tableaux may be used to construct combinatorial proofs of simple formulas for evaluating sums of the following form and coproducts of sums of the following form in terms of Schur/Schur-like functions based on sign-reversing involutions:

$$\sum(\text{sign function})(\text{product of homogeneous and elementary functions}). \qquad (1.1)$$

More precisely, we are interested in sums of the above form with terms of constant degree. Given a sum $\Sigma$ of the form indicated in (1.1) such that the terms of $\Sigma$ are all of equal degree, after expanding each term of $\Sigma$ in the Schur basis or a (fixed) Schur-like basis using bipieri tableaux, there is typically a very large amount of subsequent cancellation with respect to the entire alternating sum $\Sigma$, and often such cancellation occurs in a nice and simple combinatorial way. Given a formula which equates an element $\mathscr{S}_\alpha$ of a Schur-like basis of $\mathsf{NSym}$ with a sum $\Sigma$ of the form described above where $\alpha$ belongs to some class of compositions, by taking the coproduct of both sides of such a formula and expanding the resultant summand, one may "recursively" use the original formula $\mathscr{S}_\alpha = \Sigma$ to simplify this expansion, thus yielding a formula for coproducts of the form $\Delta\mathscr{S}_\alpha$ in terms of expressions of the form $\mathscr{S}_\beta \otimes \mathscr{S}_\gamma$. We clarify this idea in Section 3.3 and in Section 4. The symbol $\mathscr{S}$ henceforth denotes an arbitrary canonical Schur-like basis of $\mathsf{NSym}$.

There are many known formulas for Schur/Schur-like functions involving sums of the form (1.1) with constant-degree terms. For example, the elegant "diving board" formula

$$\mathfrak{w}_{(1^n,m,1^r)} = \sum_{\substack{0\leq i\leq r\\ 0\leq j\leq n}} (-1)^{i+j} E_{n-j} H_{m+i} H_j E_{r-i}$$

for the shin basis is proven in [6] using a complicated induction argument. We remark that there are many well-known combinatorial proofs of formulas for commutative Schur functions involving cancellations in alternating sums [9], and there are also several known sign-reversing involution proofs of formulas for the immaculate basis [1, 3].

In this article, we prove using sign-reversing involutions on bipieri tableaux a variety of combinatorial formulas for Schur/Schur-like functions indexed by hook/hook-like compositions. Summations of the form (1.1) often may be expressed in a natural way in terms of commutative/noncommutative Schur functions indexed by diving boards, which is why a large portion of this article deals with such combinatorial objects. Furthermore, it is natural to consider Schur/Schur-like functions indexed by hook/hook-like compositions, because:

(i) Commutative Schur-hooks are very interesting combinatorial objects since there are applications of commutative Schur-hooks in graph theory [8], number theory [11, 12], and the combinatorics of lattice paths [10];

(ii) Schur-like functions indexed by reverse hook compositions are natural combinatorial objects and arise frequently in the theory of noncommutative symmetric functions. For example, the $\Psi$-basis $\{\Psi_\alpha\}_{\alpha\in\mathcal{C}}$ of $\mathsf{NSym}$, which is one of the most common analogues of the power sum basis $\{p_\lambda\}_{\lambda\in\mathcal{P}}$ of $\mathsf{Sym}$, is usually defined in terms of elements of the Schur-like ribbon basis $\{R_\alpha\}_{\alpha\in\mathcal{C}}$ of $\mathsf{NSym}$ indexed by reverse hook compositions; and

(iii) Schur/Schur-like functions indexed by relatively "simple" compositions such as rectangles, products of rectangles, hooks, diving boards, etc. are often "well-behaved" in the sense that such Schur/Schur-like functions often may be expanded in a simple way in terms of standard bases of $\mathsf{Sym}$ or $\mathsf{NSym}$.

It would be interesting to consider formulas for evaluating sums of the form (1.1) in terms of Schur/Schur-like functions such as Schur-rectangles or Schur functions indexed by products of rectangles, or in terms of more general Schur/Schur-like functions; we currently leave it as an open problem to use bipieri tableaux to construct combinatorial formulas for more general classes of Schur/Schur-like functions.

Many of the results introduced in this article are formulas for classes of shin functions. The shin basis arises in a natural context in the sense that the tableaux resulting from repeated applications of the shin-Pieri rule (shin tableaux) are *very* natural analogues of semistandard Young tableaux [6], because:

(i) Shin tableaux are precisely *composition* tableaux with weakly increasing rows and strictly increasing columns; and

(ii) Semistandard Young tableaux are precisely *partition* tableaux with weakly increasing rows and strictly increasing columns.

The shin basis is a natural Schur-like basis to consider when dealing with cancellations in alternating sums of the form (1.1) since shin tableaux are intuitively very similar to semistandard Young tableaux, and thus bipieri shin tableaux seem like natural and intuitive combinatorial objects to use to construct sign-reversing involutions.

In Section 2 we briefly review some basic terminology and notation related to compositions and composition tableaux. In Section 3 we define the term *bipieri shin tableau*, and we prove the reverse hook formula

$$\boldsymbol{\varpi}_{(1^n,m>1)} = \sum_{i=0}^{n}(-1)^i E_{n-i}H_mH_i$$

for the shin basis combinatorially using a sign-reversing involution on a set of bipieri shin tableaux, and we prove a new combinatorial formula

$$\sum_{i=0}^{n}(-1)^i E_{n-i}H_1H_i = \sum_{i=2}^{n}(-1)^i \boldsymbol{\varpi}_{(1^{n+1-i},i)}$$

for a class of shin-quasi-reverse-hooks (see Section 3.2), and we use these formulas to prove an elegant coproduct formula

$$\Delta\boldsymbol{\varpi}_{(1^n,m)} = \sum_{(r_1,r_2)}(-1)^{(r_1,r_2)}\boldsymbol{\varpi}_{r_1}\otimes\boldsymbol{\varpi}_{r_2}$$

for shin functions indexed by a reverse hook composition. Coproduct formulas such as the above formula are interesting because there are no known combinatorial formulas for coproducts of the form $\Delta\mathscr{S}_\gamma$ for arbitrary $\gamma \in \mathcal{C}$ in terms of expressions of the form $\mathscr{S}_\alpha \otimes \mathscr{S}_\beta$.

In Section 4, we prove an elegant coproduct formula

$$\Delta\mathscr{S}_{(n,1^m)} = \sum_{\lambda,\mu} c_{\lambda,\mu}^{(n,1^m)} \mathscr{S}_\lambda \otimes \mathscr{S}_\mu$$

for noncommutative Schur-hooks using the "recursive" technique described above. In Section 5 we introduce the concept of a *bipieri Schur tableau*, and we prove a simple combinatorial formula of the form

$$\sum_\lambda (-1)^\lambda s_\lambda = \sum_{i=0}^{n} (-1)^i h_m e_i e_{n-i}$$

using a sign-reversing involution on a set of bipieri Schur tableaux (where $(-1)^\lambda$ denotes a sign function on partitions which is defined in Section 5) which may be used to construct an alternative combinatorial proof of the following special case of the classical Littlewood-Richardson rule:

$$\Delta s_{(1^r)} = \sum_{\lambda,\mu} c_{\lambda,\mu}^{(1^r)} s_\lambda \otimes s_\mu.$$

Finally, in Section 6, we discuss some open problems related to bipieri tableaux, and avenues for further research.

## 2 Preliminaries on compositions and composition tableaux

Standard background material on compositions, partitions, skew tableaux, etc. is given in [2], [5], and [6]. Since our paper is based on a new class of composition tableaux, it seems natural to review some basic notation and terminology related to combinatorial objects such as compositions and composition tableaux. Background material on the combinatorial Hopf algebras $\mathsf{Sym}$, $\mathsf{QSym}$, and $\mathsf{NSym}$ is also given in [2], [5], and [6], but we omit preliminary discussions on these Hopf algebras for the sake of brevity.

A *weak composition* is a finite sequence of nonnegative integers. The term *composition* used without the qualifier "weak" refers to a finite sequence of natural numbers. We adopt the standard convention whereby the empty sequence $()$ is considered a weak composition. The symbol $\mathcal{C}$ denotes the set of all compositions. The *order* $|\alpha|$ of a (weak) composition $\alpha$ is equal to the sum of the entries of $\alpha$ if $\alpha \neq ()$, and $|\alpha| = 0$ otherwise. A composition of order $n \in \mathbb{N}_0$ is said to be a *composition of* $n$. For $n \in \mathbb{N}_0$ and $\alpha \in \mathcal{C}$, the notation $\alpha \vDash n$ is used to indicate that $\alpha$ is a composition of $n$. The *length* $\ell(\alpha)$ of a weak composition $\alpha$ is the number of entries of $\alpha$, and for indices $i \in \mathbb{N}$ satisfying $i \leq \ell(\alpha)$, $\alpha_i$ denotes the $i^{\text{th}}$ entry of the sequence $\alpha$. As in [6], we adopt the convention whereby for a (weak) composition $\alpha$

and a natural number $i$, if $i > \ell(\alpha)$, then the symbol $\alpha_i$ is defined to be zero. Given (weak) compositions $\alpha$ and $\beta$, the *concatenation* of $\alpha$ and $\beta$ refers to the following sequence: $\alpha \cdot \beta = (\alpha_1, \alpha_2, \cdots, \alpha_{\ell(\alpha)}, \beta_1, \beta_2, \cdots, \beta_{\ell(\beta)})$. Given a composition $\alpha \neq ()$, letting $m \in \mathbb{N}$ denote the maximum entry of the sequence $\alpha$, we define the *content* of $\alpha$ as the weak composition $\beta$ of length $m$ such that for all indices $i \in \mathbb{N}$ satisfying $i \leq m$, the entry $\beta_i$ is equal to the number of entries of $\alpha$ which are equal to $i$.

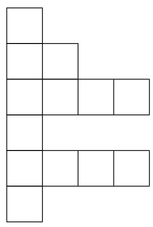
A (weak) composition $\alpha$ is said to be *contained in* a (weak) composition $\beta$, denoted $\alpha \subseteq \beta$, if $\ell(\alpha) \leq \ell(\beta)$ and $\alpha_i \leq \beta_i$ for all indices $i \in \mathbb{N}$ satisfying $i \leq \ell(\alpha)$. The *descent set* $D(\alpha)$ of a composition $\alpha$ is: $D(\alpha) = \{\alpha_1, \alpha_1 + \alpha_2, \cdots, \alpha_1 + \alpha_2 + \cdots + \alpha_{\ell(\alpha)-1}\}$.

A *partition* is a composition with weakly decreasing entries. Observe that the empty composition is vacuously a partition. The symbol $\mathcal{P}$ denotes the set of all partitions. A partition of order $n \in \mathbb{N}_0$ is said to be a *partition of* $n$. For $n \in \mathbb{N}_0$ and $\lambda \in \mathcal{P}$, the notation $\lambda \vdash n$ is used to indicate that $\lambda$ is a partition of $n$.

It is convenient for our purposes to use the term *tableau* in the broadest sense: for our purposes, a *composition tableau* (or simply *tableau*) is a finite (possibly empty) collection of cells which are arranged in left-justified rows such that each such cell is either unlabeled or is labeled with a natural number. The *length* of a tableau $\mathcal{T}$ refers to the number of rows of $\mathcal{T}$ and is denoted by $\ell(\mathcal{T})$. Letting $\mathcal{T}$ and $\mathcal{U}$ be nonempty tableaux, $\mathcal{T}$ and $\mathcal{U}$ are considered to be equal if $\ell(\mathcal{T}) = \ell(\mathcal{U})$ and the number of cells in the $i^{\text{th}}$ lowest row of $\mathcal{T}$ is equal to the number of cells in the $i^{\text{th}}$ lowest row of $\mathcal{U}$ for all indices $i \in \mathbb{N}$ satisfying $i \leq \ell(\mathcal{T}) = \ell(\mathcal{U})$.

We define the *diagram* of a composition $\alpha$ using French notation: the diagram of $\alpha \in \mathcal{C} \setminus \{()\}$ is the unique tableau $\mathcal{T}$ with unlabeled cells such that the $i^{\text{th}}$ lowest row of $\mathcal{T}$ consists of exactly $\alpha_i$ cells for all indices $i \in \mathbb{N}$ satisfying $i \leq \ell(\alpha)$, and the diagram of $() \in \mathcal{C}$ is the empty tableau without any cells. The diagram of a composition $\alpha$ is denoted by: $\text{diag}(\alpha)$.

**Example 2.1.** Letting $\alpha = (1, 4, 1, 4, 2, 1)$, we have that $\text{diag}(\alpha)$ is equal to



using French notation.

We henceforth identify each composition $\alpha$ with $\text{diag}(\alpha)$. Using French notation, we thus define the *shape* (or *outer shape*) of a tableau $\mathcal{T}$ as the unique composition $\alpha \in \mathcal{C}$ such that the tableau obtained by removing any labels of $\mathcal{T}$ is $\text{diag}(\alpha)$. The shape of a tableau $\mathcal{T}$ is denoted by: $\text{shape}(\mathcal{T})$. A *partition tableau* is a composition tableau $\mathcal{T}$ such that $\text{shape}(\mathcal{T}) \in \mathcal{P}$.

Let $\mathcal{T}$ be a configuration of cells, and let $\mathcal{U}$ be a tableau. Then the arrangement $\mathcal{T}$ is said to be a *lower subtableau* of $\mathcal{U}$ if:

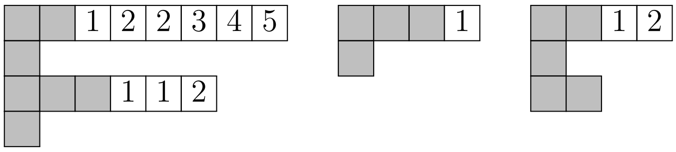
(i) $\mathcal{T}$ forms a tableau;

(ii) shape($\mathcal{T}$) $\subseteq$ shape($\mathcal{U}$); and

(iii) If the lowest row of $\mathcal{T}$ and the lowest row of $\mathcal{U}$ are in the same position, and the first column of $\mathcal{T}$ and the first column of $\mathcal{U}$ are in the same position, then the labels of $\mathcal{T}$ coincide with labels of $\mathcal{U}$, and the unlabeled cells of $\mathcal{T}$ coincide with unlabeled cells of $\mathcal{U}$.

The statement "$\mathcal{T}$ is a lower subtableau of $\mathcal{U}$" is denoted as follows: $\mathcal{T} \leq_L \mathcal{U}$. The term *upper subtableau* may be defined analogously, and the statement "$\mathcal{T}$ is an upper subtableau of $\mathcal{U}$" is denoted as follows: $\mathcal{T} \leq_U \mathcal{U}$.

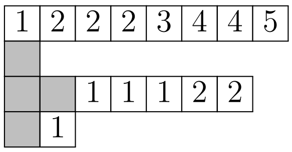
**Example 2.2.** We henceforth denote unlabeled cells in tableaux as gray cells. Denoting tableaux cells with color is often useful to illustrate combinatorial rules such as Pieri rules. Let $\mathcal{U}$, $\mathcal{T}$, and $\mathcal{T}'$ respectively denote the tableaux illustrated below.



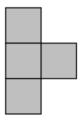

We thus have that $\mathcal{T} \leq_L \mathcal{U}$, $\mathcal{T}' \leq_U \mathcal{U}$, $\mathcal{T} \not\leq_U \mathcal{U}$, and $\mathcal{T}' \not\leq_L \mathcal{U}$.

Letting $\mathcal{T}$ be a tableau, let $\mathcal{T}_0$ denote the arrangement of cells obtained by restricting $\mathcal{T}$ to any unlabeled cells of $\mathcal{T}$. More generally, for all indices $i \in \mathbb{N}_0$, let $\mathcal{T}_i$ denote the configuration of cells obtained by restricting $\mathcal{T}$ to any unlabeled cells of $\mathcal{T}$ and any cells of $\mathcal{T}$ of label $j$ for natural numbers $j$ satisfying $j \leq i$. A *lower skew tableau* is a tableau $\mathcal{T}$ such that $\mathcal{T}_0 \leq_L \mathcal{T}$ and the term *upper skew tableau* may be defined analogously.

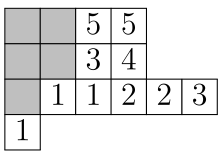
**Example 2.3.** Letting $\mathcal{T}$ denote the tableau



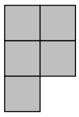

we have that $\mathcal{T}_0$ is equal to the tableau illustrated below.



We thus have that $\mathcal{T}_0 \leq_L \mathcal{T}$, so $\mathcal{T}$ is a lower skew tableau. However, $\mathcal{T}$ is not an upper skew tableau, since it is not the case that $\mathcal{T}_0 \leq_U \mathcal{T}$. Letting $\mathcal{U}$ denote the tableau



we have that $\mathcal{U}_0$ is equal to the tableau illustrated below.



We thus have that $\mathcal{U}$ is an upper skew tableau. However, $\mathcal{U}$ is not a lower skew tableau, since it is not the case that $\mathcal{U}_0 \leq_L \mathcal{U}$.

Given a skew tableau $\mathcal{T}$, the *inner shape* of $\mathcal{T}$ is defined as: shape$(\mathcal{T}_0)$. A *proper tableau* is a tableau $\mathcal{T}$ such that $\mathcal{T}_0$ is trivial. The descent set $D(\mathcal{T})$ of a tableau $\mathcal{T}$ may be informally defined as $\{i : i+1 \text{ lies above } i\}$.

For compositions $\alpha$ and $\beta$ such that $\alpha$ (regarded as a tableau) is a lower subtableau of $\beta$, let $\beta/\alpha$ denote the cells of $\beta$ which do not coincide with any cells of $\alpha$ when the composition tableaux $\alpha$ and $\beta$ are positioned so that the first column of $\alpha$ and the first column of $\beta$ are in the same position and the lowest row of $\alpha$ and the lowest row of $\beta$ are in the same position. For compositions $\alpha$ and $\beta$ such that $\alpha$ (regarded as a tableau) is an upper subtableau of $\beta$, the expression $\beta /\!\!/ \alpha$ is defined analogously as in [5]. Letting $\mathcal{T}$ be a lower skew tableau of shape $\beta \in \mathcal{C}$ and inner shape $\alpha \in \mathcal{C}$, as in [6] we define the *(lower) skew shape* of $\mathcal{T}$ as $\beta/\alpha$. The term *upper skew shape* may be defined analogously.

Let $\alpha, \beta \in \mathcal{C}$. As in [5] we define the term *vertical strip* as a configuration which is of the form $\beta/\alpha$ or of the form $\beta /\!\!/ \alpha$ with at most one cell per row, and we define the term *horizontal strip* as a configuration which is of the form $\beta/\alpha$ or of the form $\beta /\!\!/ \alpha$ with at most one cell per column.

The sign-reversing involution $\phi$ we use in Section 3.1 is such that for all tableaux $\mathcal{T} \in \operatorname{dom}(\phi)$, $\phi(\mathcal{T})$ is defined based on a certain tableau associated with $\mathcal{T}$ which we refer to as the *upper tableau* of $\mathcal{T}$. Given an arbitrary tableau $\mathcal{T}$, if $\ell(\mathcal{T}) \geq 2$, then the upper tableau of $\mathcal{T}$ is the tableau obtained by restricting $\mathcal{T}$ to: the cells of $\mathcal{T}$ in the uppermost row of $\mathcal{T}$ together with the cell with coordinates $(1, \ell(\mathcal{T}) - 1)$. If $\ell(\mathcal{T}) = 1$, then the upper tableau of $\mathcal{T}$ is $\mathcal{T}$.

A natural generalization of Young's lattice $\mathcal{L}_Y$ is constructed in [5] and used in the formulation of the left Pieri rules for the dual quasi-Schur basis given in [5]. Let $\alpha$ and $\beta$ be compositions. The composition $\beta$ is said to *cover* the composition $\alpha$, denoted $\alpha \lessdot_C \beta$, if either $\beta = (1) \cdot \alpha$ or $\beta$ can be obtained from $\alpha$ by adding 1 to the first (leftmost) entry of the sequence $\beta$ of size $k$ for some $k$ [5]. The poset $\mathcal{L}_C$ introduced in [5] may be defined as $(\mathcal{C}, \leq_C)$, where the partial order $\leq_C$ on $\mathcal{C}$ may be defined as the reflexive closure of the transitive closure of the order relation $\lessdot_C$.

## 3 Bipieri shin tableaux

The *shin basis* $\{\boldsymbol{\mathfrak{w}}_\alpha\}_{\alpha \in \mathcal{C}}$ of $\mathsf{NSym}$ was recently introduced in [6]. The shin basis may be defined using a noncommutative analogue of horizontal strips as follows [6]. Letting $\alpha$ and $\beta$ be compositions, $\beta$ differs from $\alpha$ by a *(right) shin-horizontal strip* of size $r \in \mathbb{N}_0$ if:

(i) $\alpha \subseteq \beta$;

(ii) $|\beta| = |\alpha| + r$; and

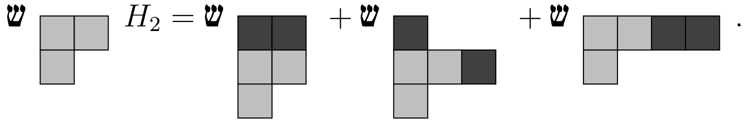
(iii) $\forall\, i, j \in \mathbb{N}\ (\beta_i > \alpha_i, j > i) \Longrightarrow \beta_j \leq \alpha_i$.

This last axiom is referred to as the *overhang axiom*. The function $\boldsymbol{\mathfrak{w}} : \mathcal{C} \to \mathsf{NSym}$ maps an arbitrary composition $\alpha$ to the unique noncommutative symmetric function $\boldsymbol{\mathfrak{w}}_\alpha$ satisfying $\boldsymbol{\mathfrak{w}}_\alpha H_r = \sum_\beta \boldsymbol{\mathfrak{w}}_\beta$, where the sum is over all compositions $\beta$ which differ from $\alpha$ by a shin-horizontal strip of size $r$, letting $r \in \mathbb{N}_0$ be arbitrary. This right Pieri rule (which is multiplicity-free) is referred to as the *(right) shin-Pieri rule.*

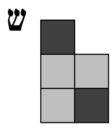
**Example 3.1.** Expanding the product $\boldsymbol{\mathfrak{w}}_{(1,2)} H_2$ in terms of shin functions using the shin-Pieri rule, we have that:

$$\boldsymbol{\mathfrak{w}}_{(1,2)} H_2 = \boldsymbol{\mathfrak{w}}_{(1,2,2)} + \boldsymbol{\mathfrak{w}}_{(1,3,1)} + \boldsymbol{\mathfrak{w}}_{(1,4)}.$$

Shin-horizontal strips are illustrated in the below expansion of the product $\boldsymbol{\mathfrak{w}}_{(1,2)} H_2$:



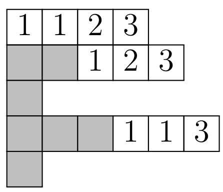
Observe that expressions such as



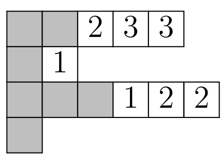
do not appear in the above expansion by the overhang axiom.

Elements of the family $\{\boldsymbol{\mathfrak{w}}_\alpha\}_{\alpha \in \mathcal{C}}$ are referred to as *shin functions*. Letting $\alpha$ and $\beta$ be compositions such that $\alpha \subseteq \beta$, a lower skew tableau $\mathcal{T}$ of inner shape $\alpha$ and outer shape $\beta$ is a *(skew) shin tableau* if: for all indices $i \in \mathbb{N}$, if $\beta_i > \alpha_i$, then for all $j > i$, $\alpha_j \leq \alpha_i$, and $\mathcal{T}$ is weakly increasing in the rows and strictly increasing in the columns [6].

**Example 3.2.** For example, the tableau



is a skew shin tableau of inner shape $\alpha = (1, 3, 1, 2)$ and outer shape $\beta = (1, 6, 1, 5, 4)$. The tableau



of inner shape $\alpha = (1, 3, 1, 2)$ and outer shape $\beta = (1, 6, 2, 5)$ is not a skew shin tableau, since $\beta_3 > \alpha_3$ and since $\alpha_4 > \alpha_3$.

The expression "shin tableau" used without the qualifier "skew" henceforth refers to a (skew) shin tableau $\mathcal{T}$ such that $\mathcal{T}$ is a proper tableau. As discussed in [6], a repeated application of the shin-Pieri rule shows that

$$\mathfrak{W}_\alpha H_\beta = \sum_{\mathcal{T}} \mathfrak{W}_{\text{shape}(\mathcal{T})}$$

where the sum is over all skew shin tableaux $\mathcal{T}$ of inner shape $\alpha$ with $\beta_1$ ones, $\beta_2$ twos, etc. We thus have that the complete homogeneous basis of $\mathsf{NSym}$ has a positive, uni-triangular expansion in terms of the family $\{\mathfrak{W}_\alpha\}_{\alpha \in \mathcal{C}}$, which shows that $\text{im}(\mathfrak{W})$ is a basis of $\mathsf{NSym}$. In particular, we thus have that

$$H_\beta = \sum_{\alpha \geq_\ell \beta} \mathcal{K}^{\mathfrak{W}}_{\alpha,\beta} \mathfrak{W}_\alpha$$

for arbitrary $\beta \in \mathcal{C}$, where $\leq_\ell$ denotes the lexicographic order on $\mathcal{C}$, and $\mathcal{K}^{\mathfrak{W}}_{\alpha,\beta}$ denotes the number of shin tableaux of shape $\alpha$ and content $\beta$ by analogy with classical Kostka coefficients. The definition of the term *bipieri shin tableau* given below may be used to generalize the above combinatorial formulas for expressions such as $\mathfrak{W}_\alpha H_\beta E_{\beta'} H_{\beta''} E_{\beta'''} \cdots H_{\beta^{(n-1)}} E_{\beta^{(n)}}$. The following combinatorial rule for computing expressions of the form $\mathfrak{W}_\alpha R_\beta$ in the shin basis is proven in [6]: for all $\alpha, \beta \in \mathcal{C}$,

$$\mathfrak{W}_\alpha R_\beta = \sum_{\gamma \vDash |\alpha|+|\beta|} \sum_{\mathcal{T}} \mathfrak{W}_\gamma$$

where the inner sum is over all skew shin tableaux $\mathcal{T}$ of (lower) skew shape $\gamma/\alpha$ such that $D(\mathcal{T}) = D(\beta)$. The ribbon multiplication formula given above allows us to evaluate expressions of the form $\mathfrak{W}_\alpha E_\beta$ in terms of the shin basis, using *(right) shin-vertical strips* [6] in the following sense. Since $E_{(n)} = R_{(1^n)}$, we thus have that

$$\mathfrak{W}_\alpha E_n = \sum_{\gamma \vDash |\alpha|+n} \sum_{\mathcal{T}} \mathfrak{W}_\gamma$$

where the inner sum is over all skew shin tableaux $\mathcal{T}$ of (lower) skew shape $\gamma/\alpha$ such that $D(\mathcal{T}) = \{1, 2, \cdots, n-1\}$. We refer to this combinatorial rule as the *(right) shin-elementary-Pieri rule.*

**Definition 3.3.** A *(right) skew bipieri shin tableau* is a tableau $\mathcal{T}$ such that $\mathcal{T}_i \leq_L \mathcal{T}$ for all indices $i \in \mathbb{N}_0$, and either $\text{shape}(\mathcal{T}_{i+1})$ differs from $\text{shape}(\mathcal{T}_i)$ by a (right) shin-horiztonal strip or $\text{shape}(\mathcal{T}_{i+1})$ differs from $\text{shape}(\mathcal{T}_i)$ by a (right) shin-vertical strip for all indices $i \in \mathbb{N}_0$.

The expression "bipieri shin tableau" used without the qualifier "right" and the qualifier "skew" henceforth refers to a right skew bipieri shin tableau $\mathcal{T}$ such that $\mathcal{T}$ is a proper tableau. For example, the tableaux given below are all bipieri shin tableaux:

$$\begin{array}{|c|c|c|}\hline 2 & 3 & \\ \hline 1 & 3 & \\ \hline 1 & 2 & 2 \\ \hline \end{array} \qquad \begin{array}{|c|c|c|}\hline 3 & & \\ \hline 3 & & \\ \hline 2 & 2 & 3 \\ \hline 1 & & \\ \hline \end{array} \qquad \begin{array}{|c|c|c|c|}\hline 3 & 3 & & \\ \hline 2 & & & \\ \hline 1 & 1 & 1 & 2 \\ \hline \end{array} \qquad \begin{array}{|c|c|c|}\hline 2 & 3 & \\ \hline 1 & 3 & \\ \hline 1 & 2 & 3 \\ \hline 1 & & \\ \hline \end{array}$$

Given a (skew) bipieri shin tableau $\mathcal{T}$, it is often useful to indicate whether $\text{shape}(\mathcal{T}_{i+1})/\text{shape}(\mathcal{T}_i)$ forms a shin-horizontal strip or a shin-vertical strip for $i \in \mathbb{N}_0$. Let $w$ denote a fixed word of length $n \in \mathbb{N}$ over the alphabet $\{h, e\}$. Letting $\mathcal{T}$ denote a skew bipieri shin tableau (resp. bipieri shin tableau), $\mathcal{T}$ is a *skew $w$-bipieri shin tableau* (resp. $w$-bipieri shin tableau) if the set of labels of $\mathcal{T}$ is contained in $\{1, 2, \cdots, n\}$, and: for all indices $i \in \mathbb{N}_0$ satisfying $i < n$, if $w_{i+1} = h$ then $\mathcal{T}_{i+1}$ differs from $\mathcal{T}_i$ by a shin-horizontal strip, and if $w_{i+1} = e$ then $\mathcal{T}_{i+1}$ differs from $\mathcal{T}_i$ by a shin-vertical strip. For example, the bipieri tableau

$$\begin{array}{|c|c|c|}\hline 2 & 3 & \\ \hline 1 & 3 & \\ \hline 1 & 2 & 2 \\ \hline \end{array}$$

is an *ehe*-bipieri shin tableau, and the bipieri tableau

$$\begin{array}{|c|c|c|}\hline 3 & & \\ \hline 3 & & \\ \hline 2 & 2 & 3 \\ \hline 1 & & \\ \hline \end{array}$$

is both an *hhe*-bipieri shin tableau and an *ehe*-bipieri shin tableau. The definition of a bipieri shin tableau is very natural, because bipieri shin tableaux are precisely the tableaux which result from the process of expanding products consisting of complete homogeneous functions and elementary functions in the shin basis through a repeated application of the shin-Pieri rule and the shin-elementary-Pieri rule. Definition 3.3 simply formalizes this idea.

### 3.1 The reverse hook formula for the shin basis

A *reverse hook* composition is a composition of the form $(1^n, m)$ where $n, m \in \mathbb{N}_0$. The reverse hook formula for the shin basis given in Section 1 of course follows from the more general diving board formula

$$\mathfrak{w}_{(1^n,m,1^r)} = \sum_{i=0}^{r} \sum_{j=0}^{n} (-1)^{i+j} E_{n-j} H_{m+i} H_j E_{r-i}$$

for the shin basis, which holds for arbitrary $n, r \in \mathbb{N}_0$ and arbitrary $m \in \mathbb{N}_{\geq 2}$. In this subsection, we offer a new combinatorial proof of the reverse hook formula for the shin basis using a sign-reversing involution on a set of bipieri shin tableaux.

**Theorem 3.4.** *The equality*

$$\mathfrak{w}_{(1^n,m)} = \sum_{j=0}^{n} (-1)^j E_{n-j} H_m H_j$$

*holds for $n \in \mathbb{N}_0$ and $m \in \mathbb{N}_{\geq 2}$.*

*Proof.* Rewriting the sum

$$\sum_{j=0}^{n}(-1)^j E_{n-j} H_m H_j$$

as

$$\sum_{j=0}^{n}(-1)^j \boldsymbol{\mathfrak{w}}_{()} E_{n-j} H_m H_j,$$

by the shin-Pieri rule, we thus have that

$$\sum_{j=0}^{n}(-1)^j E_{n-j} H_m H_j = \sum_{\mathcal{T}}(-1)^{\#\text{ of cells labeled 3 in }\mathcal{T}} \boldsymbol{\mathfrak{w}}_{\text{shape}(\mathcal{T})}$$

where the latter sum is over all *ehh*-bipieri shin tableaux $\mathcal{T}$ such that the number of cells labeled 1 in $\mathcal{T}$ plus the number of cells labeled 3 in $\mathcal{T}$ is $n$, and the number of cells labeled 2 in $\mathcal{T}$ is $m$. Let $X = X_{m,n}$ denote the set of all such tableaux, letting $m \in \mathbb{N}_{\geq 2}$ and $n \in \mathbb{N}_0$ be fixed. An example of a set of this form is given in Section 7.1. Note that since $m > 1$, there must be at least two cells labeled 2 in the tableaux under consideration. Observe that for $\mathcal{T} \in X$, the shape of $\mathcal{T}_1$ is of the form $(1^{n-j})$ for some integer $j$ satisfying $0 \leq j \leq n$ since $\mathcal{T}$ is an *ehh*-bipieri shin tableau. We construct a sign-reversing involution $\phi = \phi_{m,n} : X \to X$ on $X$ so that for $\mathcal{T} \in X$, $\phi(\mathcal{T})$ is defined based on the upper tableau of $\mathcal{T}$. The symbol $\mathcal{T}$ henceforth denotes an arbitrary element in $X$ unless otherwise specified, and $j = j_{\mathcal{T}}$ henceforth denotes the number of cells labeled 1 in $\mathcal{T}$ unless otherwise specified. Also, $\boxed{\times}$ henceforth denotes a noncell for the sake of convenience.

We consider four cases separately, based on the upper tableau of $\mathcal{T}$. Before defining the involution $\phi$, we offer an intuitive description of this involution using four informal "rules" corresponding to these four cases. Letting $\mathcal{U}_1$ and $\mathcal{U}_2$ be tableaux of the same shape, the statement "$\mathcal{U}_1 \longleftrightarrow \mathcal{U}_2$" is meant to be informally read as: "if the upper tableau of a tableau $\mathcal{T}$ in $X$ is of the form $\mathcal{U}_1$, then $\phi(\mathcal{T})$ is the tableau obtained from $\mathcal{T}$ by "replacing" the upper tableau of $\mathcal{T}$ with an upper tableau of the form $\mathcal{U}_2$, and vice-versa". We make this notion more precise in the cases considered after the rules given below, which illustrate the involution $\phi$. The cells in the tableaux given below which have labels that are changed according to $\phi$ and $\phi^{-1}$ are colored; the labels in uncolored cells illustrated below are unchanged by $\phi$ and $\phi^{-1}$. Also recall that for each rule given below, the tableaux are of equal shape. Note that **Rule 3** below only applies to two elements in $X$. Also, **Rule 4** only applies to a unique element in $X$ as we discuss below.

**Rule 1:** $\begin{array}{|c|c|c|c|c|}\hline 2 & 3 & 3 & \cdots & 3 \\ \hline 1 \\ \cline{1-1}\end{array} \longleftrightarrow \begin{array}{|c|c|c|c|c|}\hline 3 & 3 & 3 & \cdots & 3 \\ \hline 2 \\ \cline{1-1}\end{array}$

**Rule 2:** $\begin{array}{|c|c|c|c|c|c|c|c|c|c|}\hline 1 & 2 & 2 & \cdots & 2 & 2 & 3 & 3 & \cdots & 3 \\ \hline 1 \\ \cline{1-1}\end{array} \longleftrightarrow \begin{array}{|c|c|c|c|c|c|c|c|c|c|}\hline 2 & 2 & 2 & \cdots & 2 & 3 & 3 & 3 & \cdots & 3 \\ \hline 1 \\ \cline{1-1}\end{array}$

**Rule 3:** $\begin{array}{|c|c|c|c|c|c|c|c|c|c|}\cline{1-1} \times \\ \hline 1 & 2 & 2 & \cdots & 2 & 2 & 3 & 3 & \cdots & 3 \\ \hline\end{array} \longleftrightarrow \begin{array}{|c|c|c|c|c|c|c|c|c|c|}\cline{1-1} \times \\ \hline 2 & 2 & 2 & \cdots & 2 & 3 & 3 & 3 & \cdots & 3 \\ \hline\end{array}$

**Rule 4:** $\begin{array}{cccc} 2 & 2 & \cdots & 2 \\ 1 & & & \end{array} \longleftrightarrow \begin{array}{cccc} 2 & 2 & \cdots & 2 \\ 1 & & & \end{array}$

**Case 1:** First suppose that $\ell(\mathcal{T}) \geq 2$, and that there are no cells of label 1 in row $\ell(\mathcal{T})$ of $\mathcal{T}$, and that there is exactly one cell of label 2 in row $\ell(\mathcal{T})$ of $\mathcal{T}$. In this case, $\phi(\mathcal{T})$ is the tableau obtained by replacing the unique cell of label 2 in row $\ell(\mathcal{T})$ of $\mathcal{T}$ with a cell of label 3, and replacing the cell immediately beneath this cell with a cell of label 2. Since $\phi(\mathcal{T})$ is an *ehh*-bipieri shin tableau such that shape$(\phi(\mathcal{T})_1) = (1^{n-(j+1)})$ with $j+1$ cells labeled 3 and $m$ cells labeled 2, we have that $\phi(\mathcal{T}) \in X$. For example,

$$\mathcal{T} = \begin{array}{ccccc} 2 & 3 & 3 & 3 & \\ 1 & 2 & 2 & 2 & 2 \\ 1 & & & & \\ 1 & & & & \\ 1 & & & & \end{array} \Longrightarrow \phi(\mathcal{T}) = \begin{array}{ccccc} 3 & 3 & 3 & 3 & \\ 2 & 2 & 2 & 2 & 2 \\ 1 & & & & \\ 1 & & & & \\ 1 & & & & \end{array}.$$

Conversely, suppose that $\ell(\mathcal{T}) \geq 2$, and that each cell in row $\ell(\mathcal{T})$ of $\mathcal{T}$ is labeled 3, and that the cell immediately beneath the leftmost cell in the top row is labeled 2. In this case, $\phi(\mathcal{T})$ is the tableau obtained by replacing the leftmost cell in row $\ell(\mathcal{T})$ with a cell of label 2, and replacing the cell immediately beneath this cell with a cell of label 1. Since $\phi(\mathcal{T})$ is an *ehh*-bipieri shin tableau such that shape$(\phi(\mathcal{T})_1) = (1^{n-(j-1)})$ with $(j-1) \geq 0$ cells labeled 3 and $m$ cells labeled 2, we have that $\phi(\mathcal{T}) \in X$, with $\phi(\phi(\mathcal{T})) = \mathcal{T}$.

**Case 2:** Now suppose that $\ell(\mathcal{T}) \geq 2$, and that there is exactly one cell in row $\ell(\mathcal{T})$ of $\mathcal{T}$ labeled 1. Observe that the fact that $X$ is a set of *ehh*-bipieri tableaux together with the fact that $m \geq 2$ ensures that tableaux of the following form are not in $X$:

$$\begin{array}{cccccc} 1 & 2 & 3 & 3 & \cdots & 3 \\ 1 & & & & & \\ \vdots & & & & & \\ 1 & & & & & \end{array} \notin X. \tag{3.1}$$

Therefore, the number of cells labeled 2 in the uppermost row of $\mathcal{T}$ is either equal to 0, or is greater than or equal to 2. If there are no cells labeled 2 in the top row of $\mathcal{T}$, then $\phi(\mathcal{T})$ is the tableau obtained by replacing the unique cell labeled 1 with a cell labeled 3. Otherwise, $\phi(\mathcal{T})$ is the tableau obtained by replacing the rightmost cell labeled 2 in the top row of $\mathcal{T}$ with a cell labeled 3 and replacing the leftmost cell of the top row of $\mathcal{T}$ with a cell labeled 2. Since $\phi(\mathcal{T})$ is an *ehh*-bipieri shin tableau such that shape$(\phi(\mathcal{T})_1) = (1^{n-(j+1)})$ with $j+1$ cells labeled 3 and $m$ cells labeled 2, we have that $\phi(\mathcal{T}) \in X$. Furthermore, from (3.1), we have that **Case 1** does not apply to $\phi(\mathcal{T}) \in X$. For example,

$$\mathcal{T} = \begin{array}{cccccc} 1 & 2 & 2 & 2 & 2 & 3 \\ 1 & & & & & \\ 1 & & & & & \\ 1 & & & & & \\ 1 & & & & & \end{array} \Longrightarrow \phi(\mathcal{T}) = \begin{array}{cccccc} 2 & 2 & 2 & 2 & 3 & 3 \\ 1 & & & & & \\ 1 & & & & & \\ 1 & & & & & \\ 1 & & & & & \end{array}.$$

Conversely, suppose that $\ell(\mathcal{T}) \geq 2$, and that there are no cells in the top row of $\mathcal{T}$ of label 1, and that it is not the case that there is exactly one cell of label 2 in the top row of $\mathcal{T}$, and that the cell immediately beneath the leftmost cell in the top row is labeled 1. It is clear that **Case 1** does not apply to $\mathcal{T}$ in this situation. If there are no cells of label 2 in the top row of $\mathcal{T}$, then $\phi(\mathcal{T})$ is the tableau obtained by replacing the leftmost cell in the top row of $\mathcal{T}$ with a cell labeled 1. Otherwise, $\phi(\mathcal{T})$ is the tableau obtained by replacing the leftmost cell in the top row of $\mathcal{T}$ labeled 3 with a cell labeled 2 and replacing the leftmost cell in the top row of $\mathcal{T}$ with a cell labeled 1. Since $\phi(\mathcal{T})$ is an *ehh*-bipieri tableau such that shape$(\phi(\mathcal{T})_1) = (1^{n-(j-1)})$ with $j-1$ cells labeled 3 and $m$ cells labeled 2, we have that $\phi(\mathcal{T}) \in X$. Moreover, from (3.1), it is clear that $\phi(\phi(\mathcal{T})) = \mathcal{T}$.

**Case 3:** If $\ell(\mathcal{T}) = 1$ and there is a cell labeled 1 in $\mathcal{T}$, then $\phi(\mathcal{T})$ is the tableau obtained by replacing the rightmost cell labeled 2 with a cell labeled 3 and by replacing the unique cell labeled 1 with a cell labeled 2. If $\ell(\mathcal{T}) = 1$ and there is no cell labeled 1 in $\mathcal{T}$, then $\phi(\mathcal{T})$ is the tableau obtained by replacing the leftmost cell labeled 3 with a cell labeled 2 and replacing the leftmost cell with a cell labeled 1.

**Case 4:** Finally, by the shin-Pieri rule, there is a unique element $\mathcal{T} \in X$ such that each cell in the topmost row of $\mathcal{T}$ is labeled 2, namely, the unique element in $X$ of shape $(1^n, m)$ where $m > 1$. In this case we define $\phi(\mathcal{T}) = \mathcal{T}$.

By the shin-Pieri rule, the upper tableau of $\mathcal{T} \in X$ must be of one of the forms given above. Let $\mathcal{U}$ denote the unique element in $X$ of shape $(1^n, m)$. Since $\phi : X \to X$ as given above is an involution on $X$, and since

$$|\#\text{ of cells labeled 3 in } \mathcal{T} - \#\text{ of cells labeled 3 in } \phi(\mathcal{T})| = 1$$

if $\mathcal{T} \neq \phi(\mathcal{T})$, we thus have that $\phi : X \to X$ is a sign-reversing involution with respect to the following sum:

$$\sum_{\mathcal{T}} (-1)^{\#\text{ of cells labeled 3 in } \mathcal{T}} \mathfrak{w}_{\text{shape}(\mathcal{T})}.$$

So, since $\mathcal{U}$ is the unique element in $X$ which is fixed by $\phi$, we thus have that:

$$\sum_{\mathcal{T}} (-1)^{\#\text{ of cells labeled 3 in } \mathcal{T}} \mathfrak{w}_{\text{shape}(\mathcal{T})} = \mathfrak{w}_{\text{shape}(\mathcal{U})} = \mathfrak{w}_{(1^n,m)}.$$

So since

$$\sum_{j=0}^{n} (-1)^j E_{n-j} H_m H_j = \sum_{\mathcal{T}} (-1)^{\#\text{ of cells labeled 3 in } \mathcal{T}} \mathfrak{w}_{\text{shape}(\mathcal{T})}$$

we thus have that

$$\mathfrak{w}_{(1^n,m)} = \sum_{j=0}^{n} (-1)^j E_{n-j} H_m H_j$$

as desired. □

We refer to mappings of the form $\phi_{m,n} : X_{m,n} \to X_{m,n}$ as *reverse hook involutions.* A mapping of this form is illustrated in Section 7.1.

### 3.2 A formula for shin-quasi-reverse-hooks

Recall that the reverse hook formula

$$\text{ш}_{(1^n,m)} = \sum_{i=0}^{n} (-1)^i E_{n-i} H_m H_i$$

for the shin basis only holds for reverse hooks $(1^n, m) \in \mathcal{C}$ such that $m > 1$. For an arbitrary hook $(1^n, m) \in \mathcal{C}$ where $n, m \in \mathbb{N}_0$, we define the *shin-quasi-reverse-hook* $SQRH_{(1^n,m)}$ as follows:

$$SQRH_{(1^n,m)} = \sum_{i=0}^{n} (-1)^i E_{n-i} H_m H_i. \tag{3.2}$$

**Theorem 3.5.** *The identity*

$$SQRH_{(1^n,m)} = \begin{cases} \text{ш}_{(1^n,m)} & \text{if } m > 1 \text{ or } n = 0, \\ \sum_{i=2}^{n} (-1)^i \text{ш}_{(1^{n+1-i},i)} & \text{if } m = 1 \text{ and } n > 1, \\ 0 & \text{if } m = n = 1 \text{ or } (m = 0 \text{ and } n > 0). \end{cases}$$

*holds for an arbitrary hook* $(1^n, m) \in \mathcal{C}$.

*Proof.* The case whereby $m > 1$ holds by Theorem 3.4. The cases whereby $n = 0$ or $m = n = 1$ or $m = 0$ and $m > 0$ hold trivially. So it remains to consider the case whereby $m = 1$ and $n > 1$. For this remaining case, we make use of a sign-reversing involution on bipieri tableaux following the technique used in Section 3.1.

Rewriting the sum

$$\sum_{i=0}^{n} (-1)^i E_{n-i} H_1 H_i$$

as

$$\sum_{i=0}^{n} (-1)^i \text{ш}_{()} E_{n-i} H_1 H_i,$$

by the shin-Pieri rule, we thus have that

$$\sum_{i=0}^{n} (-1)^i E_{n-i} H_1 H_i = \sum_{\mathcal{T}} (-1)^{\#\text{ of cells labeled 3 in } \mathcal{T}} \text{ш}_{\text{shape}(\mathcal{T})}$$

where the latter sum is over all *ehh*-bipieri shin tableaux $\mathcal{T}$ such that the number of cells labeled 1 in $\mathcal{T}$ plus the number of cells labeled 3 in $\mathcal{T}$ is $n$, and there is exactly one cell labeled 2 in $\mathcal{T}$. Let $Y = Y_n$ denote the set of all such tableaux, letting $n > 1$ be fixed. An example of a set of this form is given in Section 7.2. We construct a sign-reversing involution $\psi = \psi_n : Y \to Y$ on $Y$ as follows.

**Rule 1:** Let $\mathcal{T} \in Y$. If shape($\mathcal{T}$) is a (nonstraight) reverse hook, and each cell in the uppermost row of $\mathcal{T}$ is labeled 3, define $\psi(\mathcal{T}) = \mathcal{T}$. Now let $\mathcal{T}$ be a tableau in $Y$ which is not of this form.

**Rule 2:** If the tableau $\mathcal{U}$ obtained by replacing the cell labeled 2 with a cell labeled 3 and replacing an uppermost cell labeled 1 with a cell labeled 2 is in $Y$, but the tableau obtained by replacing the cell labeled 2 with a cell labeled 1 and replacing any lowest-leftmost cell labeled 3 with a cell labeled 2 is not in $Y$, then let $\psi(\mathcal{T}) = \mathcal{U} \in Y$. For example, $\psi$ maps

$$\begin{ytableau} 3 \\ 1 \\ 1 & 2 & 3 & 3 \end{ytableau}$$

to the tableau given below.

$$\begin{ytableau} 3 \\ 2 \\ 1 & 3 & 3 & 3 \end{ytableau}$$

**Rule 3:** If the tableau obtained by replacing the cell labeled 2 with a cell labeled 3 and replacing an uppermost cell labeled 1 with a cell labeled 2 is in $Y$, and the tableau $\mathcal{U}$ obtained by replacing the cell labeled 2 with a cell labeled 1 and replacing any lowest-leftmost cell labeled 3 with a cell labeled 2 is in $Y$, then let $\psi(\mathcal{T}) = \mathcal{U} \in Y$. For example, $\psi$ maps

$$\begin{ytableau} 2 \\ 1 & 3 & 3 & 3 \end{ytableau}$$

to the tableau given below.

$$\begin{ytableau} 1 \\ 1 & 2 & 3 & 3 \end{ytableau}$$

**Rule 4:** Otherwise, if the conditions given above do not hold, and if the tableau $\mathcal{U}$ obtained by replacing the cell labeled 2 with a cell labeled 1 and replacing any lowest-leftmost cell labeled 3 with a cell labeled 2 is in $Y$, then let $\psi(\mathcal{T}) = \mathcal{U} \in Y$. For example, $\psi$ maps

$$\begin{ytableau} 3 \\ 2 & 3 & 3 & 3 \end{ytableau}$$

to the tableau given below.

$$\begin{ytableau} 2 \\ 1 & 3 & 3 & 3 \end{ytableau}$$

**Rule 5:** It is easily seen that a tableau $\mathcal{T}$ in $Y$ does not satisfy the above conditions if and only if $\mathcal{T}$ contains a configuration of the form

$$\begin{ytableau} \cdots & 3 \\ \cdots & \cdots \\ 1 & 2 \end{ytableau}.$$

Let the uppermost cell in the first column of $\mathcal{T}$ be labeled $x \in \{1, 3\}$. In this final case, $\phi(\mathcal{T})$ may be defined as the tableau obtained by "replacing" the label of this cell with $\{1, 3\} \setminus \{x\}$. For example, $\phi$ maps

$$\begin{array}{|c|c|cc}\cline{1-2} 1 & 3 \\ \hline 1 & 2 & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{3} \\ \hline \end{array}$$

to the tableau given below.

$$\begin{array}{|c|c|cc}\cline{1-2} 3 & 3 \\ \hline 1 & 2 & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{3} \\ \hline \end{array}$$

In the case whereby **Rule 1** applies, it is clear we have that $\psi(\mathcal{T}) \in Y$ and $\psi(\psi(\mathcal{T})) = \mathcal{T}$.

Now suppose that **Rule 2** applies to $\mathcal{T}$. First consider the case whereby the configuration $\begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline \end{array}$ appears in the first column of $\mathcal{T}$. But then the configuration $\begin{array}{|c|}\hline 3 \\ \hline 2 \\ \hline 1 \\ \hline \end{array}$ cannot be in the first column, since $\begin{array}{|c|}\hline 3 \\ \hline 3 \\ \hline 2 \\ \hline \end{array}$ is not column-strict. Also, there cannot be any other 3-cells in $\mathcal{T}$ in this case, because otherwise the tableau obtained by replacing the cell labeled 2 with a cell labeled 1 and replacing any lowest-leftmost cell labeled 3 with a cell labeled 2 would be in $Y$. So in this case, $\mathcal{T}$ must be of the form

$$\begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline 1 \\ \hline \cdots \\ \hline 1 \\ \hline 1 \\ \hline \end{array},$$

and we thus have that $\psi(\mathcal{T})$ is equal to

$$\begin{array}{|c|}\hline 3 \\ \hline 2 \\ \hline 1 \\ \hline \cdots \\ \hline 1 \\ \hline 1 \\ \hline \end{array}.$$

**Rule 1**, **Rule 2**, and **Rule 3** do not apply to $\psi(\mathcal{T})$, and by **Rule 4** we have that $\psi(\psi(\mathcal{T})) = \mathcal{T}$ in this case. Now suppose that the configuration $\begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline \end{array}$ does not appear in the first column of $\mathcal{T}$. That is, the unique cell labeled 2 is "to the right" of the first column. In this case, $\mathcal{T}$ is necessarily of one of the following forms:

$$\begin{array}{|c|c|c|c|c|c|}
\hline 3 \\ \hline 1 \\ \hline 1 \\ \hline \cdots \\ \hline 1 & 2 & 3 & 3 & \cdots & 3 \\ \hline 1 \\ \hline 1 \\ \hline \cdots \\ \hline 1 \\ \hline
\end{array}
\qquad
\begin{array}{|c|c|c|c|c|c|}
\hline 1 \\ \hline 1 \\ \hline \cdots \\ \hline 1 & 2 & 3 & 3 & \cdots & 3 \\ \hline 1 \\ \hline 1 \\ \hline \cdots \\ \hline 1 \\ \hline
\end{array}$$

In the first case, **Rule 1** does not apply to $\psi(\mathcal{T})$ since $\mathcal{T}$ cannot be of hook shape, and **Rule 2** and **Rule 3** do not apply to $\psi(\mathcal{T})$, since $\begin{array}{|c|}\hline 3 \\ \hline 3 \\ \hline 2 \\ \hline\end{array}$ is not column-strict. By **Rule 4** we have that $\psi(\psi(\mathcal{T})) = \mathcal{T}$ in this case as desired. In the second case, **Rule 1** does not apply to $\psi(\mathcal{T})$, since $\psi(\mathcal{T})$ cannot have only 3-cells in the top row, and **Rule 2** does not apply to $\psi(\mathcal{T})$ in the second case since the $12 \to 23$ and $23 \to 12$ cell transformations both yield elements in $Y$, so **Rule 3** applies, with $\psi(\psi(\mathcal{T})) = \mathcal{T}$ as desired.

Now suppose that **Rule 3** applies to $\mathcal{T}$. It is easily seen that $\mathcal{T}$ and $\psi(\mathcal{T})$ must be of the following forms:

$$\mathcal{T} = \begin{array}{|c|c|c|c|c|c|}
\hline 2 \\ \hline 1 \\ \hline \cdots \\ \hline 1 & 3 & 3 & 3 & \cdots & 3 \\ \hline 1 \\ \hline 1 \\ \hline \cdots \\ \hline 1 \\ \hline
\end{array}
\Longrightarrow \psi(\mathcal{T}) = \begin{array}{|c|c|c|c|c|c|}
\hline 1 \\ \hline 1 \\ \hline \cdots \\ \hline 1 & 2 & 3 & 3 & \cdots & 3 \\ \hline 1 \\ \hline 1 \\ \hline \cdots \\ \hline 1 \\ \hline
\end{array}.$$

We thus have that **Rule 1** applies to $\psi(\mathcal{T})$. Repeating the arguments given in the preceding paragraph thus shows that $\psi(\psi(\mathcal{T})) = \mathcal{T}$ as desired.

Now suppose that **Rule 4** applies to $\mathcal{T}$. It is easily seen that $\mathcal{T}$ must contain a configuration of the form $\begin{array}{|c|}\hline 3 \\ \hline \cdots \\ \hline 2 \\ \hline\end{array}$ in this case. By **Rule 4**, $\mathcal{T}$ must be of one of the following forms.

$$\begin{array}{|c|c|c|c|c|}
\hline 3 \\ \hline 2 & 3 & 3 & \cdots & 3 \\ \hline 1 \\ \hline 1 \\ \hline \cdots \\ \hline 1 \\ \hline
\end{array}
\qquad
\begin{array}{|c|c|c|c|c|}
\hline 3 \\ \hline 2 \\ \hline 1 & 3 & 3 & \cdots & 3 \\ \hline 1 \\ \hline \cdots \\ \hline 1 \\ \hline
\end{array}$$

So $\psi(\mathcal{T})$ is of one of the following respective forms.

$$\begin{array}{|c|cccc}\hline 3 \\ \hline 1 & \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{\cdots} & \multicolumn{1}{c|}{3} \\ \hline 1 \\ \cline{1-1} 1 \\ \cline{1-1} \cdots \\ \cline{1-1} 1 \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|cccc}\hline 3 \\ \cline{1-1} 1 \\ \hline 1 & \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{\cdots} & \multicolumn{1}{c|}{3} \\ \hline 1 \\ \cline{1-1} \cdots \\ \cline{1-1} 1 \\ \cline{1-1} \end{array}$$

By **Rule 2**, it is clear that we have that $\psi(\psi(\mathcal{T})) = \mathcal{T}$ in both cases.

In the remaining case whereby **Rule 5** applies to $\mathcal{T}$, it is obvious that $\psi(\psi(\mathcal{T})) = \mathcal{T}$. So we have shown that $\psi$ is well-defined, and an involution. It is obvious that $\psi$ is a sign-reversing involution with respect to the summation

$$\sum_{i=0}^{n} (-1)^i E_{n-i} H_1 H_i = \sum_{\mathcal{T}} (-1)^{\#\text{ of cells labeled 3 in }\mathcal{T}} \text{ש}_{\text{shape}(\mathcal{T})},$$

and since $\psi(\mathcal{T}) = \mathcal{T}$ if and only if **Rule 1** applies to $\mathcal{T}$, it is clear that we have that $SQRH_{(1^{n>1},1)} = \sum_{i=2}^{n} (-1)^i \text{ש}_{(1^{n+1-i},i)}$ as desired. $\square$

We refer to mappings of the form $\psi_n : Y_n \to Y_n$ as *SQRH involutions.* An illustration of a mapping of this form is given in Section 7.2.

### 3.3 A coproduct formula for shin functions indexed by a reverse hook

The coproduct of an alternating sum of the form (1.1) with constant-degree terms often may be evaluated "recursively" in the sense outlined in Section 1. We illustrate this idea in this subsection in our proof of the following new combinatorial formula for shin functions of the form $\text{ש}_{(1^n,m)}$ for $m \geq 3$. We adopt the convention whereby $(1^0) = ()$ and $(0) = ()$.

**Theorem 3.6.** *Letting $m \geq 3$, we have that:*

$$\Delta \text{ש}_{(1^n,m)} = \sum_{(r_1,r_2)} (-1)^{(r_1,r_2)} \text{ש}_{r_1} \otimes \text{ש}_{r_2}$$

*where the above sum is over all ordered pairs $(r_1, r_2)$ of reverse hooks $r_1 = (1^w, x)$ and $r_2 = (1^y, z)$ such that either:*

*(i) $w + y = n$ and $x + z = m$ and $w \neq 0 \Longrightarrow x > 1$ and $y \neq 0 \Longrightarrow z > 1$; or*

*(ii) $r_1 = (1^{a+1-i}, i)$ and $r_2 = (1^{n-a}, m-1)$ for some $a \in \mathbb{N}_{\geq 2}$ and some integer $i$ satisfying $2 \leq i \leq a$ or vice-versa (reversing the indices of $r_1$ and $r_2$ given above).*

*In the former case, $(-1)^{(r_1,r_2)} = 1$, and in the latter case, $(-1)^{(r_1,r_2)} = (-1)^i$, letting $i$ be as given above. In Case 2, if $r_1 = r_2$, then the ordered pair $(r_1, r_2)$ is "counted twice".*

*Proof.* Let $m \in \mathbb{N}_{\geq 3}$ and $n \in \mathbb{N}_0$ be fixed. We now show that the coproduct $\Delta \boldsymbol{\varpi}_{(1^n,m)}$ may be expressed in a natural way using shin-quasi-reverse-hooks. By the reverse hook formula for the shin basis, we have that:

$$\begin{aligned}
\Delta \boldsymbol{\varpi}_{(1^n,m)} &= \Delta \left( \sum_{i=0}^{n} (-1)^i E_{n-i} H_m H_i \right) \\
&= \sum_{i=0}^{n} (-1)^i \Delta (E_{n-i} H_m H_i) \\
&= \sum_{i=0}^{n} (-1)^i \Delta E_{n-i} \Delta H_m \Delta H_i.
\end{aligned}$$

Now expand the summand $(-1)^i \Delta E_{n-i} \Delta H_m \Delta H_i$ as follows:

$$\begin{aligned}
&\sum_{i=0}^{n} (-1)^i \sum_{\substack{j_1,j_2 \in \mathbb{N}_0 \\ j_1+j_2=n-i}} E_{j_1} \otimes E_{j_2} \sum_{\substack{k_1,k_2 \in \mathbb{N}_0 \\ k_1+k_2=m}} H_{k_1} \otimes H_{k_2} \sum_{\substack{\ell_1,\ell_2 \in \mathbb{N}_0 \\ \ell_1+\ell_2=i}} H_{\ell_1} \otimes H_{\ell_2} \\
&= \sum_{i=0}^{n} \sum_{\substack{j_1,j_2,k_1,k_2,\ell_1,\ell_2 \in \mathbb{N}_0 \\ j_1+j_2=n-i \\ k_1+k_2=m \\ \ell_1+\ell_2=i}} (-1)^i \left(E_{j_1} \otimes E_{j_2}\right) \left(H_{k_1} \otimes H_{k_2}\right) \left(H_{\ell_1} \otimes H_{\ell_2}\right).
\end{aligned}$$

Therefore, the coproduct $\Delta \boldsymbol{\varpi}_{(1^n,m)}$ is equal to:

$$\sum_{a,b,c,d \in \mathbb{N}_0} (-1)^a E_b H_c H_d \otimes E_{n-a-b} H_{m-c} H_{a-d}.$$

Equivalently,

$$\Delta \boldsymbol{\varpi}_{(1^n,m)} = \sum_{\substack{a,b,c,d \in \mathbb{N}_0 \\ a+b \leq n \\ c \leq m \\ d \leq a}} (-1)^a E_b H_c H_d \otimes E_{n-a-b} H_{m-c} H_{a-d}. \tag{3.3}$$

We now offer an elegant bijective proof that the summation given in (3.3) may be rearranged to yield the sum given below:

$$\Delta \boldsymbol{\varpi}_{(1^n,m)} = \sum_{\substack{a,b,c,d \in \mathbb{N}_0 \\ a+c=n \\ b+d=m}} SQRH_{(1^a,b)} \otimes SQRH_{(1^c,d)}. \tag{3.4}$$

Let $\mathscr{T}_1$ denote the set of all tuples $(a,b,c,d)$ in $\mathbb{N}_0^4$ such that $a+b \leq n$, $c \leq m$, and $d \leq a$. Given a tuple $(a,b,c,d)$ in $\mathscr{T}_1$ define:

$$t_{(a,b,c,d)} = (-1)^a E_b H_c H_d \otimes E_{n-a-b} H_{m-c} H_{a-d}.$$

So the terms of (3.3) (ignoring any possible cancellation) are precisely expressions of the form $t_{(a,b,c,d)}$ for $(a,b,c,d) \in \mathscr{T}_1$. Now consider the summation given in (3.4), and consider the terms (ignoring any possible cancellation) resulting from expanding this sum using (3.2). Let $w, x, y, z \in \mathbb{N}_0$ be such that $w + y = n$ and $x + z = m$. So the expression

$$SQRH_{(1^w,x)} \otimes SQRH_{(1^y,z)}$$

is an arbitrary term in (3.4). Expanding this expression using (3.2), and letting $i \in \mathbb{N}_0$ and $j \in \mathbb{N}_0$ be indices such that $i \leq w$ and $j \leq y$, thus thus have that terms of the form

$$u_{(w,x,y,z,i,j)} = (-1)^i E_{w-i} H_x H_i \otimes (-1)^j E_{y-j} H_z H_j$$

are precisely the terms resulting from expanding the terms of (3.4) using (3.2) (ignoring any possible cancellation). Let $\mathscr{T}_2$ denote the set of all tuples $(w, x, y, z, i, j) \in \mathbb{N}_0^6$ such that $w + y = n$, $x + z = m$, $i \leq w$, and $j \leq y$. Now define the mapping

$$\mathscr{B} : \mathscr{T}_1 \to \mathscr{T}_2$$

so that given an arbitrary tuple $(a, b, c, d)$ in the domain $\mathscr{T}_1$ of $\mathscr{B}$,

$$\mathscr{B}(a, b, c, d) = (b + d, c, n - b - d, m - c, d, a - d). \tag{3.5}$$

We briefly show that $\mathscr{B}$ as given above is well-defined, in the sense that $\mathscr{B}(a, b, c, d)$ is indeed an element in the given codomain of $\mathscr{B}$, again letting

$$(a, b, c, d) \in \mathscr{T}_1$$

be arbitrary. Since $d \leq a$ we have that $b + d \leq a + b$ and since $a + b \leq n$ we thus have that $0 \leq n - b - d$. Similarly, we have that the fourth and last entries of the integer tuple $(b+d, c, n-b-d, m-c, d, a-d)$ are nonnegative since $(a, b, c, d) \in \mathscr{T}_1$. So it is clear that $\mathscr{B}(a, b, c, d) \in \mathbb{N}_0^6$. Also observe that:

$$(\mathscr{B}(a, b, c, d))_1 + (\mathscr{B}(a, b, c, d))_3 = n, \tag{3.6}$$

$$(\mathscr{B}(a, b, c, d))_2 + (\mathscr{B}(a, b, c, d))_4 = m. \tag{3.7}$$

Recall that $\mathscr{T}_2 \subseteq \mathbb{N}_0^6$ is a set of weak compositions. Letting $i$ be an index, $(\mathscr{B}(a, b, c, d))_i$ denotes the $i^{\text{th}}$ entry in the weak composition $\mathscr{B}(a, b, c, d) \in \mathscr{T}_2$, using the notation introduced in Section 2. Since $d \leq b + d$, we have that:

$$(\mathscr{B}(a, b, c, d))_5 \leq (\mathscr{B}(a, b, c, d))_1. \tag{3.8}$$

Since $a \leq n - b$, we have that $a - d \leq n - b - d$, and we thus have that:

$$(\mathscr{B}(a, b, c, d))_6 \leq (\mathscr{B}(a, b, c, d))_3. \tag{3.9}$$

From (3.6), (3.7), (3.8), and (3.9) together with the fact that $\mathscr{B}(a, b, c, d) \in \mathbb{N}_0^6$, we thus have that $\mathscr{B}(a, b, c, d) \in \mathscr{T}_2$ as desired. We claim that $\mathscr{B} : \mathscr{T}_1 \to \mathscr{T}_2$ is injective. Let $(a_1, b_1, c_1, d_1)$ and $(a_2, b_2, c_2, d_2)$ be arbitrary tuples in the domain $\mathscr{T}_1$ of $\mathscr{B}$. Suppose that $\mathscr{B}(a_1, b_1, c_1, d_1) = \mathscr{B}(a_2, b_2, c_2, d_2)$. So the tuple

$$(b_1 + d_1, c_1, n - b_1 - d_1, m - c_1, d_1, a_1 - d_1)$$

equals

$$(b_2 + d_2, c_2, n - b_2 - d_2, m - c_2, d_2, a_2 - d_2)$$

and we thus have that $c_1 = c_2$ and $d_1 = d_2$. Since $b_1 + d_1 = b_2 + d_2$ we thus have that $b_1 = b_2$, and since $a_1 - d_1 = a_2 - d_2$ we thus have that $a_1 = a_2$, thus proving that $\mathscr{B} : \mathscr{T}_1 \to \mathscr{T}_2$ is injective. We claim that $\mathscr{B} : \mathscr{T}_1 \to \mathscr{T}_2$ is surjective. Let $(w, x, y, z, i, j) \in \mathbb{N}_0^6$ be an arbitrary element in the codomain of $\mathscr{B}$. We claim that $(i+j, w-i, x, i)$ is in the domain of $\mathscr{B}$. Since $i \leq w$ we have that $(i+j, w-i, x, i) \in \mathbb{N}_0^4$. Since $j \leq y$, we have that $j+w \leq y+w$, and we thus have that $j+w \leq n$. Therefore, $(i + j) + (w - i) \leq n$, i.e.:

$$(i + j, w - i, x, i)_1 + (i + j, w - i, x, i)_2 \leq n. \tag{3.10}$$

Since $x + z = m$, we have that $x \leq m$, i.e.:

$$(i + j, w - i, x, i)_3 \leq m. \tag{3.11}$$

Clearly, we have that $i \leq i + j$, i.e.:

$$(i + j, w - i, x, i)_4 \leq (i + j, w - i, x, i)_1. \tag{3.12}$$

So from (3.10), (3.11), and (3.12) together with the fact that $(i+j, w-i, x, i) \in \mathbb{N}_0^4$, we thus have that $(i+j, w-i, x, i) \in \mathscr{T}_1$ as desired. Now evaluate $\mathscr{B}(i+j, w-i, x, i)$ using (3.5):

$$\mathscr{B}(i + j, w - i, x, i) = (w, x, n - w, m - x, i, j).$$

Since $(w, x, y, z, i, j) \in \mathscr{T}_2$, we have that $y = n - w$ and $z = m - x$, thus proving that

$$\mathscr{B}(i + j, w - i, x, i) = (w, x, y, z, i, j)$$

and thus proving that $\mathscr{B} : \mathscr{T}_1 \to \mathscr{T}_2$ is surjective as desired. We thus have that $\mathscr{B} : \mathscr{T}_1 \to \mathscr{T}_2$ is bijective as desired.

Again let $(a, b, c, d)$ be an arbitrary tuple in the domain $\mathscr{T}_1$ of $\mathscr{B}$. Now evaluate the term $u_{\mathscr{B}(a,b,c,d)}$:

$$\begin{aligned}
u_{\mathscr{B}(a,b,c,d)} &= u_{(b+d,c,n-b-d,m-c,d,a-d)} \\
&= (-1)^d E_b H_c H_d \otimes (-1)^{a-d} E_{n-b-a} H_{m-c} H_{a-d} \\
&= (-1)^a E_b H_c H_d \otimes E_{n-b-a} H_{m-c} H_{a-d} \\
&= t_{(a,b,c,d)}.
\end{aligned}$$

So we have shown that for an arbitrary tuple $\alpha \in \mathscr{T}_1$,

$$u_{\mathscr{B}(\alpha)} = t_\alpha.$$

Since $\mathscr{B} : \mathscr{T}_1 \to \mathscr{T}_2$ is bijective, we may rewrite the summation

$$\sum_{\beta \in \mathscr{T}_2} u_\beta = \sum_{\substack{a,b,c,d \in \mathbb{N}_0 \\ a+c=n \\ b+d=m}} SQRH_{(1^a,b)} \otimes SQRH_{(1^c,d)}$$

as

$$\sum_{\beta \in \mathscr{T}_2} u_\beta = \sum_{\alpha \in \mathscr{T}_1} u_{\mathscr{B}(\alpha)}$$

and since $u_{\mathscr{B}(\alpha)} = t_\alpha$ we thus have that

$$\sum_{\beta \in \mathscr{T}_2} u_\beta = \sum_{\alpha \in \mathscr{T}_1} u_{\mathscr{B}(\alpha)} = \sum_{\alpha \in \mathscr{T}_1} t_\alpha = \sum_{\substack{a,b,c,d \in \mathbb{N}_0 \\ a+b \leq n \\ c \leq m \\ d \leq a}} (-1)^a E_b H_c H_d \otimes E_{n-a-b} H_{m-c} H_{a-d}$$

thus proving (3.4). Now rewrite the summation given in (3.4) as follows:

$$\sum_{\substack{a,b,c,d \in \mathbb{N}_0 \\ a+c=n \\ b+d=m \\ b=0 \Longrightarrow a=0 \\ d=0 \Longrightarrow c=0 \\ b=1 \Longrightarrow a=0 \\ d=1 \Longrightarrow c=0}} \text{ש}_{(1^a,b)} \otimes \text{ש}_{(1^c,d)} + \sum_{a \in \mathbb{N}_{\geq 2}} SQRH_{(1^a,1)} \otimes SQRH_{(1^{n-a},m-1)} +$$
$$\sum_{c \in \mathbb{N}_{\geq 2}} SQRH_{(1^{n-c},m-1)} \otimes SQRH_{(1^c,1)}.$$

Expanding the latter two sums in terms of shin functions in the case whereby $m-1>1$ yields the combinatorial formula given in the above theorem. $\square$

**Example 3.7.** Using the above combinatorial rule for shin-reverse-hooks, we have that: $\Delta \text{ש}_{(1,1,1,3)} = \text{ש}_{()} \otimes \text{ש}_{(1,1,1,3)} + \text{ש}_{(1)} \otimes \text{ש}_{(1,1,1,2)} + \text{ש}_{(1,1,1,2)} \otimes \text{ש}_{(1)} + \text{ש}_{(1,1,1,3)} \otimes \text{ש}_{()} + \text{ש}_{(1,1,2)} \otimes \text{ש}_{(2)} + 2\text{ש}_{(1,2)} \otimes \text{ש}_{(1,2)} - \text{ש}_{(1,3)} \otimes \text{ש}_{(2)} + \text{ש}_{(2)} \otimes \text{ש}_{(1,1,2)} - \text{ש}_{(2)} \otimes \text{ש}_{(1,3)}$.

**Example 3.8.** Using the above combinatorial rule for shin-reverse-hooks, we have that: $\Delta \text{ש}_{(1,1,1,1,4)} = \text{ש}_{()} \otimes \text{ש}_{(1,1,1,1,4)} + \text{ש}_{(1)} \otimes \text{ש}_{(1,1,1,1,3)} + \text{ש}_{(1,1,1,1,2)} \otimes \text{ש}_{(2)} + \text{ש}_{(1,1,1,1,3)} \otimes \text{ש}_{(1)} + \text{ש}_{(1,1,1,1,4)} \otimes \text{ש}_{()} + \text{ש}_{(1,1,1,2)} \otimes \text{ש}_{(1,2)} + \text{ש}_{(1,1,1,2)} \otimes \text{ש}_{(3)} + \text{ש}_{(1,1,2)} \otimes \text{ש}_{(1,1,2)} + \text{ש}_{(1,1,2)} \otimes \text{ש}_{(1,3)} + \text{ש}_{(1,1,3)} \otimes \text{ש}_{(1,2)} - \text{ש}_{(1,1,3)} \otimes \text{ש}_{(3)} + \text{ש}_{(1,2)} \otimes \text{ש}_{(1,1,1,2)} + \text{ש}_{(1,2)} \otimes \text{ש}_{(1,1,3)} + \text{ש}_{(1,3)} \otimes \text{ש}_{(1,1,2)} - 2\text{ש}_{(1,3)} \otimes \text{ש}_{(1,3)} + \text{ש}_{(1,4)} \otimes \text{ש}_{(3)} + \text{ש}_{(2)} \otimes \text{ש}_{(1,1,1,1,2)} + \text{ש}_{(3)} \otimes \text{ש}_{(1,1,1,2)} - \text{ש}_{(3)} \otimes \text{ש}_{(1,1,3)} + \text{ש}_{(3)} \otimes \text{ש}_{(1,4)}$.

Our technique used to prove this formula may be applied more generally to expressions of the form $\text{ש}_{\mathscr{d}}$ where $\mathscr{d} \in \mathcal{C}$ is a diving board composition, using the diving board formula for the shin basis (see Section 4).

## 4 A coproduct formula for noncommutative Schur-hooks

The *immaculate basis* $\{\mathfrak{S}_\alpha\}_{\alpha \in \mathcal{C}}$ of $\mathsf{NSym}$ was recently introduced in [2]. One particularly appealing aspect of the immaculate basis is that this Schur-like basis may be defined using a very natural noncommutative analogue of the classical Jacobi-Trudi formula [2]. Alternatively, the immaculate basis may be defined using simple analogues of the classical Pieri rules [2]. Letting $\alpha, \beta \in \mathcal{C}$, the composition $\beta$ differs from $\alpha$ by a *(right) immaculate-horizontal strip* of size $r \in \mathbb{N}_0$ if:

(i) $\alpha \subseteq \beta$;

(ii) $|\beta| = |\alpha| + r$; and

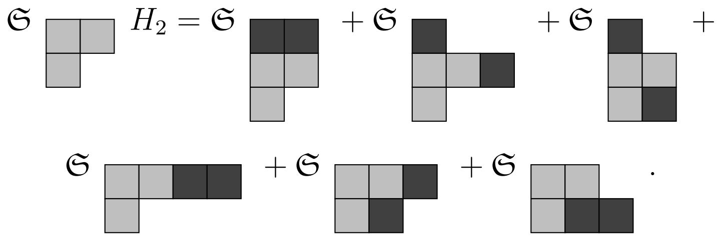
(iii) $\ell(\beta) \leq \ell(\alpha) + 1$.

The immaculate function $\mathfrak{S}_\alpha$ is the unique noncommutative symmetric function satisfying $\mathfrak{S}_\alpha H_r = \sum_\beta \mathfrak{S}_\beta$, where the sum is over all compositions $\beta$ which differ from $\alpha$ by an immaculate-horizontal strip of size $r$, letting $r \in \mathbb{N}_0$ be arbitrary [2]. This multiplicity-free right Pieri rule is referred to as the *(right) immaculate-Pieri rule.*

**Example 4.1.** Expanding the product $\mathfrak{S}_{(1,2)} H_2$ in terms of immaculate functions using the immaculate-Pieri rule, we have that:

$$\mathfrak{S}_{(1,2)} H_2 = \mathfrak{S}_{(1,2,2)} + \mathfrak{S}_{(1,3,1)} + \mathfrak{S}_{(2,2,1)} + \mathfrak{S}_{(1,4)} + \mathfrak{S}_{(2,3)} + \mathfrak{S}_{(3,2)}.$$

Immaculate-horizontal strips are illustrated in the below expansion of the product $\mathfrak{S}_{(1,2)} H_2$:



The immaculate-Pieri rule is less "restrictive" compared to the shin-Pieri rule in the sense that immaculate-horizontal strip cells are permitted to be underneath "overhangs" (see Example 3.1).

By the ribbon multiplication rule for the immaculate basis, for all $\alpha, \beta \in \mathcal{C}$,

$$\mathfrak{S}_\alpha R_\beta = \sum_{\gamma \vDash |\alpha|+|\beta|} \sum_{\mathcal{T}} \mathfrak{S}_\gamma$$

where the inner sum is over all skew immaculate tableaux $\mathcal{T}$ of skew shape $\gamma/\alpha$ such that $D(\mathcal{T}) = D(\beta)$ [2]. We may thus define the *(right) immaculate-elementary-Pieri rule* using *(right) immaculate-vertical strips* by analogy with the Pieri rule for expressions of the form $\mathfrak{S}_\alpha E_n$. We define the term *(right) bipieri immaculate tableau* by analogy with Definition 3.3.

The *dual quasi-Schur basis* $\{\mathcal{S}^*_\alpha\}_{\alpha \in \mathcal{C}}$ of $\mathsf{NSym}$ was recently introduced in [5]. We refer to elements of the family $\{\mathcal{S}^*_\alpha\}_{\alpha \in \mathcal{C}}$ as *dual quasi-Schur functions*. We remark that elements of the dual quasi-Schur basis are often referred to as *noncommutative Schur functions* [5], but the expression "noncommutative Schur function" also has a different meaning with respect to the theory of Schur functions in noncommuting variables developed by Fomin and Greene in [7]. For our purposes, the expression "noncommutative Schur functions" refers to the elements in the canonical Schur-like bases of $\mathsf{NSym}$ (and similarly, the expression "noncommutative Schur-hooks" refers to elements in the canonical Schur-like bases indexed by hook compositions).

There are known left- and right-Pieri rules for the dual quasi-Schur basis for left- and right-multiplication by a homogeneous/elementary generator. As proven in [5],

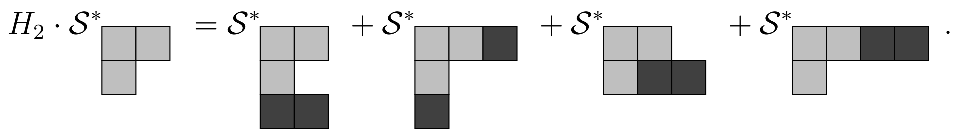
$$H_n \mathcal{S}^*_\alpha = \sum_\beta \mathcal{S}^*_\beta$$

where the above sum is over all compositions $\beta \geq_C \alpha$ such that $|\beta| = |\alpha| + n$, and $\beta /\!\!/ \alpha$ is a horizontal strip where the cells have been added from left to right. We thus define the term *(left) dual quasi-Schur-horizontal strip* accordingly.

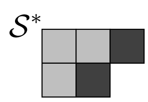
**Example 4.2.** Expanding the product $H_2 \cdot \mathcal{S}^*_{(1,2)}$ in terms of the dual quasi-Schur basis using the dual quasi-Schur-Pieri rule, we have that:

$$H_2 \cdot \mathcal{S}^*_{(1,2)} = \mathcal{S}^*_{(2,1,2)} + \mathcal{S}^*_{(1,1,3)} + \mathcal{S}^*_{(3,2)} + \mathcal{S}^*_{(1,4)}.$$

Left dual quasi-Schur-horizontal strips are illustrated in the below expansion of the product $H_2 \cdot \mathcal{S}^*_{(1,2)}$.



Observe that an expression of the form



does not appear in the above expansion, as is easily verified using the definition of the order relation $\lessdot_C$ given in Section 2 together with the fact that cells must be added from left to right with respect to dual quasi-Schur-horizontal strips.

As proven in [5],

$$E_n \mathcal{S}^*_\alpha = \sum_\beta \mathcal{S}^*_\beta$$

where the above sum is over all compositions $\beta \geq_C \alpha$ such that $|\beta| = |\alpha| + n$, and $\beta /\!\!/ \alpha$ is a vertical strip where the cells have been added from right to left. We thus define the term *(left) dual quasi-Schur-vertical strip* accordingly. We define the term *(left) skew bipieri dual quasi-Schur tableau* by analogy with Definition 3.3. Right Pieri rules for the dual quasi-Schur basis are proven in [13] and we thus define the term *right skew bipieri dual quasi-Schur tableau* by analogy with Definition 3.3.

We have thus far introduced several Schur-like bases of $\mathsf{NSym}$ in this article. Since these Schur-like bases may all be defined using analogues of the classical Pieri rules, it is natural to consider the transition matrices between these Schur-like bases. We present some transition matrices between Schur-like bases of $\mathsf{NSym}$ in Section 7.3.

Now that we have introduced the three canonical Schur-like bases of $\mathsf{NSym}$, we consider a useful sum of the form (1.1) which has a simple evaluation in terms of each of these Schur-like bases. We are referring to the following simple formula for noncommutative Schur-hooks:

**Lemma 4.3.** *The equality*

$$\mathscr{S}_{(m,1^n)} = \sum_{i=0}^{n} (-1)^i H_{m+i} E_{n-i}$$

*holds for arbitrary $m \in \mathbb{N}$ and $n \in \mathbb{N}_0$.*

*Proof.* Letting $i \in \mathbb{N}_0$ and $j \in \mathbb{N}$ be arbitrary indices, by the Pieri rules for the canonical Schur-like bases of $\mathsf{NSym}$ we have that $H_i E_j = \mathscr{S}_{(i,1^j)} + \mathscr{S}_{(i+1,1^{j-1})}$, so the sum $\sum_{i=0}^{n}(-1)^i H_{m+i} E_{n-i}$ telescopes in an obvious way. □

The indices of the terms in the expression $\mathscr{S}_{(i,1^j)} + \mathscr{S}_{(i+1,1^{j-1})}$ may be regarded in a natural way as bipieri tableaux, so it is natural to interpret Lemma 4.3 combinatorially using sign-reversing involutions on bipieri tableaux by analogy with the strategy used in the previous section. Lemma 4.3 is interesting mainly because this formula may be used to prove an elegant formula

$$\Delta \mathscr{S}_{(m,1^n)} = \sum_{\hbar_1, \hbar_2} \mathscr{S}_{\hbar_1} \otimes \mathscr{S}_{\hbar_2}$$

for the coproduct of a noncommutative Schur-hook, where the above summation is over all hook compositions $\hbar_1$ and $\hbar_2$ given by Littlewood-Richardson coefficients of the form $c^{(m,1^n)}_{\hbar_1,\hbar_2}$. This formula nicely illustrates that coproducts of sums of the form (1.1) often may be expressed in a simple way in terms of Schur-like bases of $\mathsf{NSym}$, and that formulas for coproducts of this form may be interpreted combinatorially using bipieri tableaux. Combinatorial formulas such as $\Delta \mathcal{S}^*_{(m,1^n)} = \sum_{\hbar_1,\hbar_2} \mathcal{S}^*_{\hbar_1} \otimes \mathcal{S}^*_{\hbar_2}$ are interesting because there is no known combinatorial formula for coproducts of the form $\Delta \mathcal{S}^*_\gamma$ such as the Littlewood-Richardson rule

$$\Delta \mathcal{S}_\gamma = \sum_{\alpha,\beta} S^{\gamma}_{\alpha,\beta} \mathcal{S}_\alpha \otimes \mathcal{S}_\beta$$

proven in [5]. Combinatorial formulas such as $\Delta \mathcal{S}^*_{(m,1^n)} = \sum_{\hbar_1,\hbar_2} \mathcal{S}^*_{\hbar_1} \otimes \mathcal{S}^*_{\hbar_2}$ are also interesting because coproducts of the form $\Delta \mathcal{S}^*_\gamma$ do not always expand positively in terms of expressions of the form $\mathcal{S}^*_\alpha \otimes \mathcal{S}^*_\beta$. Although there is a Littlewood-Richardson rule of the form

$$\mathfrak{S}_\alpha \mathfrak{S}_\lambda = \sum_{\beta \vDash |\alpha|+|\lambda|} G^\beta_{\alpha,\lambda} \mathfrak{S}_\beta$$

for $\alpha \in \mathcal{C}$ and $\lambda \in \mathcal{P}$ which is proven in [3], there is no known Littlewood-Richardson rule for arbitrary products of elements of the dual immaculate basis of $\mathsf{QSym}$, so by duality there is no known general combinatorial formula for coproducts of the form $\Delta\mathfrak{S}_\gamma$. Also, coproducts of the form $\Delta\mathfrak{S}_\gamma$ do not always expand positively in terms of expressions of the form $\mathfrak{S}_\alpha \otimes \mathfrak{S}_\beta$. Although the dual shin basis $\{\mathfrak{w}^*_\alpha\}_{\alpha \in \mathcal{C}}$ of $\mathsf{QSym}$ satisfies the classical Littlewood-Richardson rule with respect to elements of this basis indexed by partitions, in general it is not the case that products of the form $\mathfrak{w}^*_\alpha \mathfrak{w}^*_\beta$ epand positively in the dual shin basis as discussed in [6]. So combinatorial formulas such as the coproduct formula $\Delta\mathfrak{w}_{(m,1^n)} = \sum_{\hbar_1,\hbar_2} \mathfrak{w}_{\hbar_1} \otimes \mathfrak{w}_{\hbar_2}$ are interesting because in general by duality the coefficients in a formula of the form

$$\Delta\mathfrak{w}_\gamma = \sum_{\alpha,\beta} W^\gamma_{\alpha,\beta} \mathfrak{w}_\alpha \otimes \mathfrak{w}_\beta$$

are not always positive.

**Theorem 4.4.** *The equality*

$$\Delta\mathscr{S}_{(m,1^n)} = \sum_{\lambda,\mu} c^{(m,1^n)}_{\lambda,\mu} \mathscr{S}_\lambda \otimes \mathscr{S}_\mu$$

*holds for an arbitrary hook* $(m, 1^n) \in \mathcal{C}$.

*Proof.* From Lemma 4.3 we have that:

$$\begin{aligned}
\Delta(\mathscr{S}_{(m,1^n)}) &= \Delta\left(\sum_{i=0}^{n} (-1)^i H_{m+i} E_{n-i}\right) \\
&= \sum_{i=0}^{n} (-1)^i \Delta(H_{m+i} E_{n-i}) \\
&= \sum_{i=0}^{n} (-1)^i \Delta(H_{m+i}) \Delta(E_{n-i}).
\end{aligned}$$

Now expand the above summand as follows:

$$\begin{aligned}
&\sum_{i=0}^{n} (-1)^i \left(\sum_{j=0}^{m+i} H_j \otimes H_{m+i-j}\right) \left(\sum_{k=0}^{n-i} E_k \otimes E_{n-i-k}\right) \\
&= \sum_{i=0}^{n} \sum_{j=0}^{m+i} \sum_{k=0}^{n-i} (-1)^i (H_j \otimes H_{m+i-j})(E_k \otimes E_{n-i-k}) \\
&= \sum_{i=0}^{n} \sum_{j=0}^{m+i} \sum_{k=0}^{n-i} (-1)^i (H_j E_k \otimes H_{m+i-j} E_{n-i-k}).
\end{aligned}$$

By the Littlewood-Richardson rule, it is easily seen that if a Littlewood-Richardson coefficient of the form $c_{\lambda,\mu}^{(m,1^n)}$ does not vanish, then $\lambda$ and $\mu$ both must be hook partitions. By the Littlewood-Richardson rule, we have that

$$\Delta s_{(m,1^n)} = \sum_{\hbar_1,\hbar_2} c_{\hbar_1,\hbar_2}^{(m,1^n)} s_{\hbar_1} \otimes s_{\hbar_2}$$

where the above summation is over all hook partitions $\hbar_1, \hbar_2 \in \mathcal{C}$. Expanding the above summation using the commutative analogue of Lemma 4.3, we have that:

$$\Delta s_{(m,1^n)} = \sum_{\hbar_1,\hbar_2} c_{\hbar_1,\hbar_2}^{(m,1^n)} \sum \mathrm{sgn}_{i,j}^{\hbar_1} h_i e_j \otimes \sum \mathrm{sgn}_{k,\ell}^{\hbar_2} h_k e_\ell,$$

where the summations in the above summand are given by Lemma 4.3. Again by Lemma 4.3, the above sum may be written as

$$\Delta s_{(m,1^n)} = \sum_{i=0}^{n} \sum_{j=0}^{m+i} \sum_{k=0}^{n-i} (-1)^i (h_j e_k \otimes h_{m+i-j} e_{n-i-k}),$$

which shows that the sum

$$\sum_{\hbar_1,\hbar_2} c_{\hbar_1,\hbar_2}^{(m,1^n)} \sum \mathrm{sgn}_{i,j}^{\hbar_1} H_i E_j \otimes \sum \mathrm{sgn}_{k,\ell}^{\hbar_2} H_k E_\ell$$

may be written as

$$\sum_{i=0}^{n} \sum_{j=0}^{m+i} \sum_{k=0}^{n-i} (-1)^i (H_j E_k \otimes H_{m+i-j} E_{n-i-k}),$$

thus completing our proof. $\square$

## 5 Bipieri Schur tableaux

Letting $\lambda \in \mathcal{P}$ be arbitrary, the two classical Pieri rules indicated in Section 1 may be formulated as follows: $s_\lambda h_r = \sum_\mu s_\mu$ where the sum is over all partitions $\mu$ such that $\lambda/\mu$ forms a horizontal strip of size $r$, and $s_\lambda e_r = \sum_\mu s_\mu$ where the sum is over all partitions $\mu$ such that $\lambda/\mu$ forms a vertical strip of size $r$. We define the term *(skew) bipieri Schur tableau* by analogy with Definition 3.3. The term *bipieri Schur tableau* used without the qualifier "skew" refers to a skew bipieri Schur tableau $\mathcal{T}$ such that $\mathcal{T}$ is a proper tableau. For example, semistandard tableaux are bipieri Schur tableaux. Recall that a *semistandard tableau* is a tableau of partition shape with no unlabeled cells and weakly increasing rows and strictly increasing columns. The tableaux given below are bipieri Schur tableaux which are not semistandard tableaux.

$$\begin{array}{|c|ccc} \hline 2 \\ \cline{1-1} 2 \\ \hline 1 & \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{1} & \multicolumn{1}{c|}{2} \\ \hline \end{array} \qquad \begin{array}{|c|c|c|} \hline 3 & 5 & 7 \\ \hline 3 & 5 & 7 \\ \hline 2 & 5 & 6 \\ \hline 1 & 2 & 4 \\ \hline \end{array} \qquad \begin{array}{lll} \cline{1-2} \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{4} \\ \cline{1-2} \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{4} \\ \hline \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{3} \\ \hline \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{2} \\ \hline \end{array} \qquad \begin{array}{|c|ccc} \cline{1-1} 1 \\ \cline{1-1} 1 \\ \cline{1-3} 1 & \multicolumn{1}{|c|}{4} & \multicolumn{1}{c|}{4} \\ \hline 1 & \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{3} \\ \hline \end{array} \qquad \begin{array}{|c|} \hline 8 \\ \hline 8 \\ \hline \end{array}$$

Letting $w$ denote a finite word over the alphabet $\{h, e\}$, we define the term *$w$-bipieri Schur tableau* as in Section 3. Note that a tableau $\mathcal{T}$ is a bipieri Schur tableau if and only if $\mathcal{T}$ is a proper tableau of partition shape with weakly increasing rows and columns such that for all $i \in \mathbb{N}$: if there exists a pattern of the form $\boxed{i}\boxed{i}$ then there does not exist a pattern of the form $\begin{array}{|c|}\hline i \\ \hline i \\ \hline\end{array}$ and vice-versa.

### 5.1 A sign-reversing involution on bipieri Schur tableaux

Recall that $\mathsf{Sym}$ is self-dual as a graded Hopf algebra with respect to the Hall inner product $\langle \cdot, \cdot \rangle$ on $\mathsf{Sym} \otimes \mathsf{Sym}$, and recall that the Schur basis of $\mathsf{Sym}$ is self-dual. So by duality, the famous Littlewood-Richardson rule

$$s_\lambda \cdot s_\mu = \sum_\nu c^\nu_{\lambda,\mu} s_\nu$$

is equivalent to the coproduct formula

$$\Delta s_\nu = \sum_{\lambda,\mu} c^\nu_{\lambda,\mu} s_\lambda \otimes s_\mu$$

for arbitrary commutative Schur functions. Sign-reversing involutions on bipieri Schur tableaux may be used to construct alternative combinatorial proofs of special cases of the classical Littlewood-Richardson rule for coproducts of the form $\Delta s_\nu$. For example, the strategy used in Section 4 may be used to construct an alternative combinatorial proof of the coproduct formula

$$\Delta s_{(n,1^m)} = \sum_{\hbar_1,\hbar_2} c^{(n,1^m)}_{\hbar_1,\hbar_2} s_{\hbar_1} \otimes s_{\hbar_2}$$

for commutative Schur-hooks in terms of bipieri Schur tableaux. Bipieri Schur tableaux are also interesting combinatorial objects because alternating sums such as

$$\sum_{j=0}^{n} (-1)^j h_m e_j e_{n-j}$$

often may be evaluated using bipieri tableaux in a natural way. In this section, to illustrate this idea, we prove using bipieri tableaux the combinatorial formula given in Theorem 5.1 below. This combinatorial rule may be used to prove that

$$s_{(1^r)} = \sum_{0 \le j \le i \le r} (-1)^j h_{r-i} e_j e_{i-j}$$

for arbitrary $r \in \mathbb{N}_0$. Taking the coproduct

$$\Delta s_{(1^r)} = \sum_{0 \le j \le i \le r} (-1)^j \Delta h_{r-i} \Delta e_j \Delta e_{i-j}$$

of both sides of this formula and applying the "recursive" technique exhibited in Section 3.3 and in Section 4, one may thus construct an alternative proof of the following special case of the Littlewood-Richardson rule: $\Delta s_{(1^r)} = \sum_{\lambda,\mu} c^{(1^r)}_{\lambda,\mu} s_\lambda \otimes s_\mu$.

**Theorem 5.1.** *For $m \in \mathbb{N}$ and $n \in 2\mathbb{N}$,*

$$\sum_{\lambda} (-1)^{\lambda} s_{\lambda} = \sum_{j=0}^{n} (-1)^j h_m e_j e_{n-j},$$

*where the former sum is over all partitions $\lambda \vdash m+n$ such that $\lambda$ is either of the form $(m, 2^j, 1^k)$ or of the form $(m+2, 2^j, 1^k)$, and the sign function given in the former sum is such that $(-1)^{\lambda} = (-1)^{(\lambda^*)_2 - 1}$ if $\lambda_1 = m$ and $(-1)^{\lambda} = (-1)^{(\lambda^*)_2}$ otherwise.*

*Proof.* Rewriting the sum $\sum_{j=0}^{n}(-1)^j h_m e_j e_{n-j}$ as $\sum_{j=0}^{n}(-1)^j s_{()} h_m e_j e_{n-j}$ and expanding expressions of the form $s_{()} h_m e_j e_{n-j}$ using the classical Pieri rules, we thus have that

$$\sum_{j=0}^{n} (-1)^j h_m e_j e_{n-j} = \sum_{\mathcal{T}} (-1)^{\#\text{ of cells labeled 2 in } \mathcal{T}} s_{\text{shape}(\mathcal{T})}$$

where the latter sum is over all *hee*-bipieri Schur tableaux $\mathcal{T}$ such that: $\text{shape}(\mathcal{T}) \vdash n+m$, and the number of 2-cells plus the number of 3-cells is $n$. Let $X = X_{m,n}$ denote the set of all such bipieri tableaux. We now construct a sign-reversing involution on this set. Let $\mathcal{T} \in X$, and let $\lambda = \text{shape}(\mathcal{T})$. The sequence of labels in the first column of $\mathcal{T}$, read from top to bottom, is of the form

$$(\underbrace{3, 3, \cdots, 3}_{i}, \underbrace{2, 2, \cdots, 2}_{\ell(\lambda)-i-1}, 1)$$

for some index $i$. The number of cells in the second column of $\lambda$ is equal to $(\lambda^*)_2$. By the Schur-elementary-Pieri rule, the labels of cells in the second column of $\mathcal{T}$ strictly above the first row of $\mathcal{T}$ are all equal to 3. Define $\mathcal{T}^{(0)}$ to be the tableau obtained from $\mathcal{T}$ by replacing any cells labeled 3 in the first column of $\mathcal{T}$ with cells labeled 2. Since the labels of cells in the second column of $\mathcal{T}$ strictly above the first row of $\mathcal{T}$ are all equal to 3, we thus have that $\mathcal{T}^{(0)} \in X$. For example,

$$\mathcal{T} = \begin{array}{|c|cc} \hline 3 & & \\ \cline{1-1} 2 & & \\ \cline{1-2} 2 & \multicolumn{1}{c|}{3} & \\ \hline 1 & \multicolumn{1}{c|}{1} & \multicolumn{1}{c|}{1} \\ \hline \end{array} \Longrightarrow \mathcal{T}^{(0)} = \begin{array}{|c|cc} \hline 2 & & \\ \cline{1-1} 2 & & \\ \cline{1-2} 2 & \multicolumn{1}{c|}{3} & \\ \hline 1 & \multicolumn{1}{c|}{1} & \multicolumn{1}{c|}{1} \\ \hline \end{array} \in X.$$

Let $\mathcal{T}^{(1)}$ denote the tableau obtained by replacing the topmost label in the first column of $\mathcal{T}^{(0)}$ with a 3, let $\mathcal{T}^{(2)}$ denote the tableau obtained by replacing the second topmost label in the first column of $\mathcal{T}^{(1)}$ with a 3, and so forth. Since the labels of cells in the second column of $\mathcal{T}$ strictly above the first row of $\mathcal{T}$ are all equal to 3, we thus have that:

$$\mathcal{T}^{(0)} \in X, \cdots, \mathcal{T}^{(\ell(\lambda)-(\lambda^*)_2)} \in X, \mathcal{T}^{(\ell(\lambda)-(\lambda^*)_2+1)} \notin X, \mathcal{T}^{(\ell(\lambda)-(\lambda^*)_2+2)} \notin X, \text{etc.}$$

The original tableau $\mathcal{T} \in X$ is thus contained within the chain

$$\mathcal{T}^{(0)} \leq \mathcal{T}^{(1)} \leq \cdots \leq \mathcal{T}^{(\ell(\lambda)-(\lambda^*)_2)}$$

of tableaux in $X$. For example, letting

$$\mathcal{T} = \begin{array}{|c|cc} \cline{1-1} 3 \\ \cline{1-1} 2 \\ \cline{1-1} 2 \\ \cline{1-1} 2 \\ \hline 1 & \multicolumn{1}{c|}{1} & \multicolumn{1}{c|}{1} \\ \hline \end{array}$$

then the chain

$$\mathcal{T}^{(0)} \leq \mathcal{T}^{(1)} \leq \mathcal{T}^{(2)} \leq \mathcal{T}^{(3)} \leq \mathcal{T}^{(4)} = \mathcal{T}^{(\ell(\lambda)-(\lambda^*)_2)}$$

is given as follows, with $\mathcal{T} = \mathcal{T}^{(1)}$:

$$\begin{array}{|c|cc} \cline{1-1} 2 \\ \cline{1-1} 2 \\ \cline{1-1} 2 \\ \cline{1-1} 2 \\ \hline 1 & \multicolumn{1}{c|}{1} & \multicolumn{1}{c|}{1} \\ \hline \end{array} \leq \begin{array}{|c|cc} \cline{1-1} 3 \\ \cline{1-1} 2 \\ \cline{1-1} 2 \\ \cline{1-1} 2 \\ \hline 1 & \multicolumn{1}{c|}{1} & \multicolumn{1}{c|}{1} \\ \hline \end{array} \leq \begin{array}{|c|cc} \cline{1-1} 3 \\ \cline{1-1} 3 \\ \cline{1-1} 2 \\ \cline{1-1} 2 \\ \hline 1 & \multicolumn{1}{c|}{1} & \multicolumn{1}{c|}{1} \\ \hline \end{array} \leq \begin{array}{|c|cc} \cline{1-1} 3 \\ \cline{1-1} 3 \\ \cline{1-1} 3 \\ \cline{1-1} 2 \\ \hline 1 & \multicolumn{1}{c|}{1} & \multicolumn{1}{c|}{1} \\ \hline \end{array} \leq \begin{array}{|c|cc} \cline{1-1} 3 \\ \cline{1-1} 3 \\ \cline{1-1} 3 \\ \cline{1-1} 3 \\ \hline 1 & \multicolumn{1}{c|}{1} & \multicolumn{1}{c|}{1} \\ \hline \end{array}$$

Chains of bipieri tableaux of this form alternate with respect to the signs of corresponding terms in the sum $\sum_{\mathcal{T}}(-1)^{\mathcal{T}} s_{\mathrm{shape}(\mathcal{T})}$, writing

$$(-1)^{\mathcal{T}} = (-1)^{\#\text{ of cells labeled 2 in }\mathcal{T}}$$

for the sake of convenience. We thus have that if $\ell(\lambda) - (\lambda^*)_2$ is odd, the expression

$$(-1)^{\mathcal{T}^{(0)}} s_{\mathrm{shape}(\mathcal{T}^{(0)})} + \cdots + (-1)^{\mathcal{T}^{(\ell(\lambda)-(\lambda^*)_2)}} s_{\mathrm{shape}(\mathcal{T}^{(\ell(\lambda)-(\lambda^*)_2)})}$$

vanishes, and if $\ell(\lambda) - (\lambda^*)_2$ is even, we have that the above expression is equal to $\pm s_{\mathrm{shape}(\mathcal{T}^{(0)})}$, selecting the tableau $\mathcal{T}^{(0)}$ as a representative of the chain $\mathcal{T}^{(0)} \leq \mathcal{T}^{(1)} \leq \cdots \leq \mathcal{T}^{(\ell(\lambda)-(\lambda^*)_2)}$. We thus have that

$$\sum_{j=0}^{n}(-1)^j h_m e_j e_{n-j} = \sum_{\mathcal{U}} \mathrm{sign}(\mathcal{U}) s_{\mathrm{shape}(\mathcal{U})}$$

for some appropriate sign function which we later define, where the latter sum is over all biperi Schur tableaux $\mathcal{U}$ satisfying the conditions given above together with the conditions that $\ell(\lambda) - (\lambda^*)_2$ is even and $\mathcal{U} = \mathcal{U}^{(0)}$. We are thus implicitly using a sign-reversing involution $\phi = \phi_{m,n}$ on $X$. By the Schur-elementary-Pieri rule, we have that $\lambda_1 \in \{m, m+1, m+2\}$. Letting $\mathcal{U}$ be a tableau satisfying the conditions given above, suppose that $\lambda_1 = m+1$, letting $\lambda$ denote the shape of $\mathcal{U}$. The label of the rightmost cell of the first row of $\mathcal{U}$ is either 2 or 3. We may assume without loss of generality that the label of this cell is 2. Let $\mathcal{V} \in X$ denote the bipieri tableau obtained by replacing this label with 3. We thus have that $(-1)^{\mathcal{U}} s_{\mathrm{shape}(\mathcal{U})} = (-1)^{\mathcal{U}} s_\lambda$ and $(-1)^{\mathcal{V}} s_{\mathrm{shape}(\mathcal{V})} = (-1)^{\mathcal{V}} s_\lambda$ are of opposite signs, thus proving that

$$\sum_{j=0}^{n}(-1)^j h_m e_j e_{n-j} = \sum_{\mathcal{U}} \mathrm{sign}(\mathcal{U}) s_{\mathrm{shape}(\mathcal{U})}$$

where the latter sum is over all biperi Schur tableaux $\mathcal{U}$ satisfying the conditions given above together with the condition that $(\text{shape}(\mathcal{U}))_1 \in \{m, m+2\}$. The sign function given in this sum is determined in an obvious way by whether $\lambda_1 = m$ or $\lambda_1 = m+2$. In particular, if $\lambda_1 = m$, then $(-1)^{\lambda} = (-1)^{(\lambda^*)_2 - 1}$ and if $\lambda_1 = m+2$, then $(-1)^{\lambda} = (-1)^{(\lambda^*)_2}$.

Now, given a partition $\lambda \vdash m+n$ such that (1) $\lambda_1 \in \{m, m+2\}$, (2) for all indices $i$ satisfying $1 < i \leq \ell(\lambda)$, we have that $\lambda_i \in \{1, 2\}$, and (3) $\ell(\lambda) - (\lambda^*)_2$ is even, there is a unique tableau $\mathcal{U}$ corresponding to the sum $\sum_{\mathcal{U}} \text{sign}(\mathcal{U}) s_{\text{shape}(\mathcal{U})}$ of shape $\lambda$, i.e. the tableau obtained by labeling the cells of $\lambda$ in an obvious manner. Conversely, given a tableau $\mathcal{U}$ given by the sum $\sum_{\mathcal{U}} \text{sign}(\mathcal{U}) s_{\text{shape}(\mathcal{U})}$, there is a (unique) partition satisfying conditions (1), (2) and (3) such that $\text{shape}(\mathcal{U}) = \lambda$. It is easily seen that given a partition $\lambda \vdash m+n$ such that $\lambda_1 \in \{m, m+2\}$ and $1 < i \leq \ell(\lambda) \implies \lambda_i \in \{1, 2\}$ for all indices $i$ in $\mathbb{N}$, it necessarily follows that $\ell(\lambda) - (\lambda^*)_2$ is even since $n$ is even, thus completing our proof. □

# 6 Conclusion

We conclude by describing some problems related to the results given in this paper. The problem of constructing left Pieri rules for the shin basis is an interesting combinatorial problem. There are no known general combinatorial formulas for expressions of the form $H_n \boldsymbol{ש}_\alpha$ or expressions of the form $E_n \boldsymbol{ש}_\alpha$, so it is not clear how to define a left bipieri shin tableau. There are no known identities such Theorem 3.4 for immaculate functions indexed by a reverse hook or dual quasi-Schur functions indexed by a reverse hook, and furthermore, there are no known combinatorial formulas for immaculate functions indexed by a diving board composition or dual quasi-Schur functions indexed by a diving board composition. We currently leave finding such formulas as an open problem. A left Pieri rule for expressions of the form $H_n \mathfrak{S}_\alpha$ is proven in [4], although there is no known left Pieri rule for expressions of the form $E_n \mathfrak{S}_\alpha$, so it is not clear how to define a left bipieri immaculate tableau. We currently leave it as an open problem to use right bipieri dual quasi-Schur tableaux to evaluate sums of the form (1.1) and coproducts of sums of this form in terms of the dual quasi-Schur basis using the results given in [13]. We are interested in using noncommutative bipieri tableaux to prove formulas which equate elements of a canonical Schur-like basis $\{\mathscr{S}_\alpha\}_{\alpha \in \mathcal{C}}$ of $\mathsf{NSym}$ and sums of the form (1.1) with constant-degree terms which may be used to construct coproduct formulas as in Section 3.3 and Section 4, since there are no known general Littlewood-Richardson rules for coproducts of the following forms: $\Delta \boldsymbol{ש}_\gamma$, $\Delta \mathfrak{S}_\gamma$, and $\Delta \mathcal{S}^*_\gamma$.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# 7 Appendix

As discussed in Section 1, given an alternating sum of products of constant degree consisting of complete homogeneous and elementary symmetric functions, by expanding each such product in the Schur basis or a Schur-like basis, there is typically a large amount of cancellation with respect to the entire alternating sum. In this section we illustrate this idea with respect to the reverse hook formula for the shin basis, and with respect to our formula for expressions of the form $SQRH_{(1^{n>1},1)}$. In this section, we also present transition matrices between the canonical Schur-like bases of $\mathsf{NSym}$.

## 7.1 An illustration of a reverse hook involution

We illustrate our combinatorial interpretation of the reverse hook formula for the shin basis in this section by showing how the reverse hook sign-reversing involution $\phi_{2,3}$ operates on the set $X_{2,3}$. Expanding each term in the alternating sum

$$\sum_{j=0}^{3}(-1)^j E_{n-j}H_2H_j$$

in the shin basis separately, there are a total of 37 resultant terms, with only one term remaining after adding these terms. This illustrates our claim that there is typically a large amount of cancellation with respect to alternating sums of the form (1.1) when each term of such a summation is expanded in terms of Schur/Schur-like functions. The tableaux in $X_{2,3}$ are given below.

$$\begin{ytableau} 2 & 2 \\ 1 \\ 1 \\ 1 \end{ytableau}\quad \begin{ytableau} 2 \\ 1 & 2 \\ 1 \\ 1 \end{ytableau}\quad \begin{ytableau} 2 \\ 1 \\ 1 & 2 \\ 1 \end{ytableau}\quad \begin{ytableau} 2 \\ 1 \\ 1 \\ 1 & 2 \end{ytableau}\quad \begin{ytableau} 1 & 2 & 2 \\ 1 \\ 1 \end{ytableau}\quad \begin{ytableau} 1 \\ 1 & 2 & 2 \\ 1 \end{ytableau}\quad \begin{ytableau} 1 \\ 1 \\ 1 & 2 & 2 \end{ytableau}$$

$$\begin{ytableau} 3 \\ 2 & 2 \\ 1 \\ 1 \end{ytableau}\quad \begin{ytableau} 2 & 2 & 3 \\ 1 \\ 1 \end{ytableau}\quad \begin{ytableau} 3 \\ 2 \\ 1 & 2 \\ 1 \end{ytableau}\quad \begin{ytableau} 2 & 3 \\ 1 & 2 \\ 1 \end{ytableau}\quad \begin{ytableau} 2 \\ 1 & 2 & 3 \\ 1 \end{ytableau}\quad \begin{ytableau} 3 \\ 2 \\ 1 \\ 1 & 2 \end{ytableau}\quad \begin{ytableau} 2 & 3 \\ 1 \\ 1 & 2 \end{ytableau}$$

$$\begin{ytableau} 2 \\ 1 & 3 \\ 1 & 2 \end{ytableau}\quad \begin{ytableau} 2 \\ 1 \\ 1 & 2 & 3 \end{ytableau}\quad \begin{ytableau} 3 \\ 1 & 2 & 2 \\ 1 \end{ytableau}\quad \begin{ytableau} 1 & 2 & 2 & 3 \\ 1 \end{ytableau}\quad \begin{ytableau} 3 \\ 1 \\ 1 & 2 & 2 \end{ytableau}\quad \begin{ytableau} 1 & 3 \\ 1 & 2 & 2 \end{ytableau}\quad \begin{ytableau} 1 \\ 1 & 2 & 2 & 3 \end{ytableau}$$

$$\begin{ytableau} 3 & 3 \\ 2 & 2 \\ 1 \end{ytableau}\quad \begin{ytableau} 3 \\ 2 & 2 & 3 \\ 1 \end{ytableau}\quad \begin{ytableau} 2 & 2 & 3 & 3 \\ 1 \end{ytableau}\quad \begin{ytableau} 3 & 3 \\ 2 \\ 1 & 2 \end{ytableau}\quad \begin{ytableau} 3 \\ 2 & 3 \\ 1 & 2 \end{ytableau}\quad \begin{ytableau} 3 \\ 2 \\ 1 & 2 & 3 \end{ytableau}\quad \begin{ytableau} 2 & 3 & 3 \\ 1 & 2 \end{ytableau}$$

$$\begin{ytableau} 2 & 3 \\ 1 & 2 & 3 \end{ytableau}\quad \begin{ytableau} 2 \\ 1 & 2 & 3 & 3 \end{ytableau}\quad \begin{ytableau} 3 & 3 \\ 1 & 2 & 2 \end{ytableau}\quad \begin{ytableau} 3 \\ 1 & 2 & 2 & 3 \end{ytableau}\quad \begin{ytableau} 1 & 2 & 2 & 3 & 3 \end{ytableau}\quad \begin{ytableau} 3 & 3 & 3 \\ 2 & 2 \end{ytableau}\quad \begin{ytableau} 3 & 3 \\ 2 & 2 & 3 \end{ytableau}$$

$$\begin{array}{llll} 3 & & & \\ 2 & 2 & 3 & 3 \end{array} \qquad \begin{array}{lllll} 2 & 2 & 3 & 3 & 3 \end{array}$$

We now illustrate the sign-reversing involution $\phi_{2,3}$ using the informal "rules" given above. We define the (symmetric) binary relation $\blacklozenge$ on $X_{2,3}$ so that given tableaux $\mathcal{T}_1$ and $\mathcal{T}_2$ in $X_{2,3}$, $\mathcal{T}_1 \blacklozenge \mathcal{T}_2$ if and only if $\phi(\mathcal{T}_1) = \mathcal{T}_2$ (or equivalently $\phi(\mathcal{T}_2) = \mathcal{T}_1$).

**Rule 1:** $\begin{array}{lllll} 2 & 3 & 3 & \cdots & 3 \\ 1 & & & & \end{array} \longleftrightarrow \begin{array}{lllll} 3 & 3 & 3 & \cdots & 3 \\ 2 & & & & \end{array}$

$$\begin{array}{ll} 2 & \\ 1 & 2 \\ 1 & \\ 1 & \end{array} \blacklozenge \begin{array}{ll} 3 & \\ 2 & 2 \\ 1 & \\ 1 & \end{array} \qquad \begin{array}{ll} 2 & \\ 1 & \\ 1 & 2 \\ 1 & \end{array} \blacklozenge \begin{array}{ll} 3 & \\ 2 & \\ 1 & 2 \\ 1 & \end{array} \qquad \begin{array}{ll} 2 & \\ 1 & \\ 1 & \\ 1 & 2 \end{array} \blacklozenge \begin{array}{ll} 3 & \\ 2 & \\ 1 & \\ 1 & 2 \end{array}$$

$$\begin{array}{lll} 2 & & \\ 1 & 2 & 3 \\ 1 & & \end{array} \blacklozenge \begin{array}{lll} 3 & & \\ 2 & 2 & 3 \\ 1 & & \end{array} \qquad \begin{array}{ll} 2 & \\ 1 & 3 \\ 1 & 2 \end{array} \blacklozenge \begin{array}{ll} 3 & \\ 2 & 3 \\ 1 & 2 \end{array} \qquad \begin{array}{lll} 2 & & \\ 1 & & \\ 1 & 2 & 3 \end{array} \blacklozenge \begin{array}{lll} 3 & & \\ 2 & & \\ 1 & 2 & 3 \end{array}$$

$$\begin{array}{ll} 2 & 3 \\ 1 & 2 \\ 1 & \end{array} \blacklozenge \begin{array}{ll} 3 & 3 \\ 2 & 2 \\ 1 & \end{array} \qquad \begin{array}{ll} 2 & 3 \\ 1 & \\ 1 & 2 \end{array} \blacklozenge \begin{array}{ll} 3 & 3 \\ 2 & \\ 1 & 2 \end{array} \qquad \begin{array}{lll} 2 & 3 & 3 \\ 1 & 2 & \end{array} \blacklozenge \begin{array}{lll} 3 & 3 & 3 \\ 2 & 2 & \end{array}$$

$$\begin{array}{lll} 2 & 3 & \\ 1 & 2 & 3 \end{array} \blacklozenge \begin{array}{lll} 3 & 3 & \\ 2 & 2 & 3 \end{array} \qquad \begin{array}{llll} 2 & & & \\ 1 & 2 & 3 & 3 \end{array} \blacklozenge \begin{array}{llll} 3 & & & \\ 2 & 2 & 3 & 3 \end{array}$$

**Rule 2:** $\begin{array}{llllllllll} 1 & 2 & 2 & \cdots & 2 & 2 & 3 & 3 & \cdots & 3 \\ 1 & & & & & & & & & \end{array} \longleftrightarrow \begin{array}{llllllllll} 2 & 2 & 2 & \cdots & 2 & 3 & 3 & 3 & \cdots & 3 \\ 1 & & & & & & & & & \end{array}$

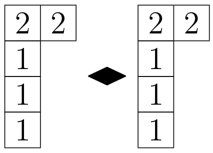
$$\begin{array}{lll} 1 & & \\ 1 & 2 & 2 \\ 1 & & \end{array} \blacklozenge \begin{array}{lll} 3 & & \\ 1 & 2 & 2 \\ 1 & & \end{array} \qquad \begin{array}{lll} 1 & & \\ 1 & & \\ 1 & 2 & 2 \end{array} \blacklozenge \begin{array}{lll} 3 & & \\ 1 & & \\ 1 & 2 & 2 \end{array} \qquad \begin{array}{llll} 1 & & & \\ 1 & 2 & 2 & 3 \end{array} \blacklozenge \begin{array}{llll} 3 & & & \\ 1 & 2 & 2 & 3 \end{array}$$

$$\begin{array}{lll} 1 & 2 & 2 \\ 1 & & \\ 1 & & \end{array} \blacklozenge \begin{array}{lll} 2 & 2 & 3 \\ 1 & & \\ 1 & & \end{array} \qquad \begin{array}{llll} 1 & 2 & 2 & 3 \\ 1 & & & \end{array} \blacklozenge \begin{array}{llll} 2 & 2 & 3 & 3 \\ 1 & & & \end{array} \qquad \begin{array}{lll} 1 & 3 & \\ 1 & 2 & 2 \end{array} \blacklozenge \begin{array}{lll} 3 & 3 & \\ 1 & 2 & 2 \end{array}$$

**Rule 3:** $\begin{array}{llllllllll} \times & & & & & & & & & \\ 1 & 2 & 2 & \cdots & 2 & 2 & 3 & 3 & \cdots & 3 \end{array} \longleftrightarrow \begin{array}{llllllllll} \times & & & & & & & & & \\ 2 & 2 & 2 & \cdots & 2 & 3 & 3 & 3 & \cdots & 3 \end{array}$

$$\begin{array}{lllll} 1 & 2 & 2 & 3 & 3 \end{array} \blacklozenge \begin{array}{lllll} 2 & 2 & 3 & 3 & 3 \end{array}$$

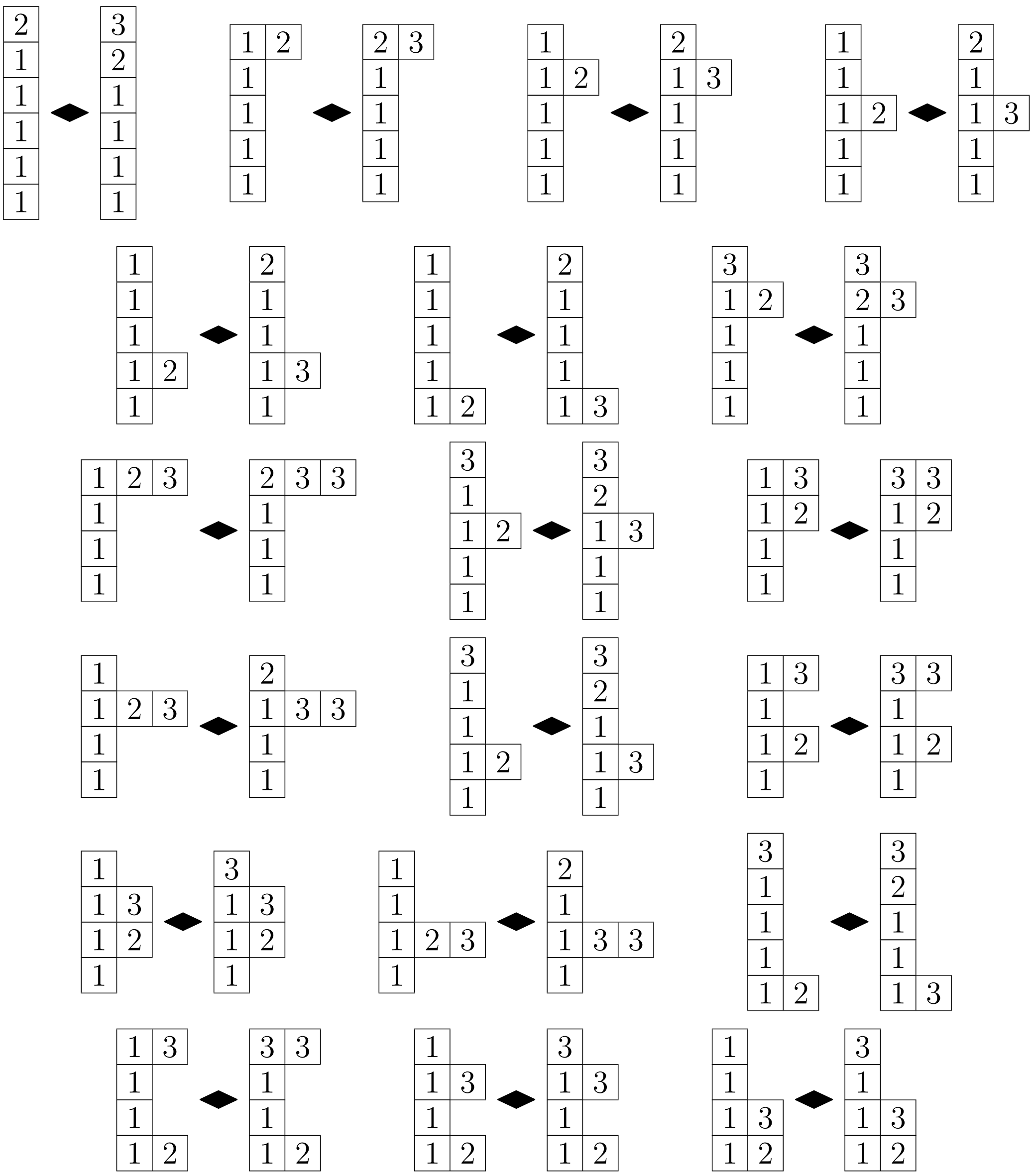
**Rule 4:** $\begin{array}{llll} 2 & 2 & \cdots & 2 \\ 1 & & & \end{array} \longleftrightarrow \begin{array}{llll} 2 & 2 & \cdots & 2 \\ 1 & & & \end{array}$

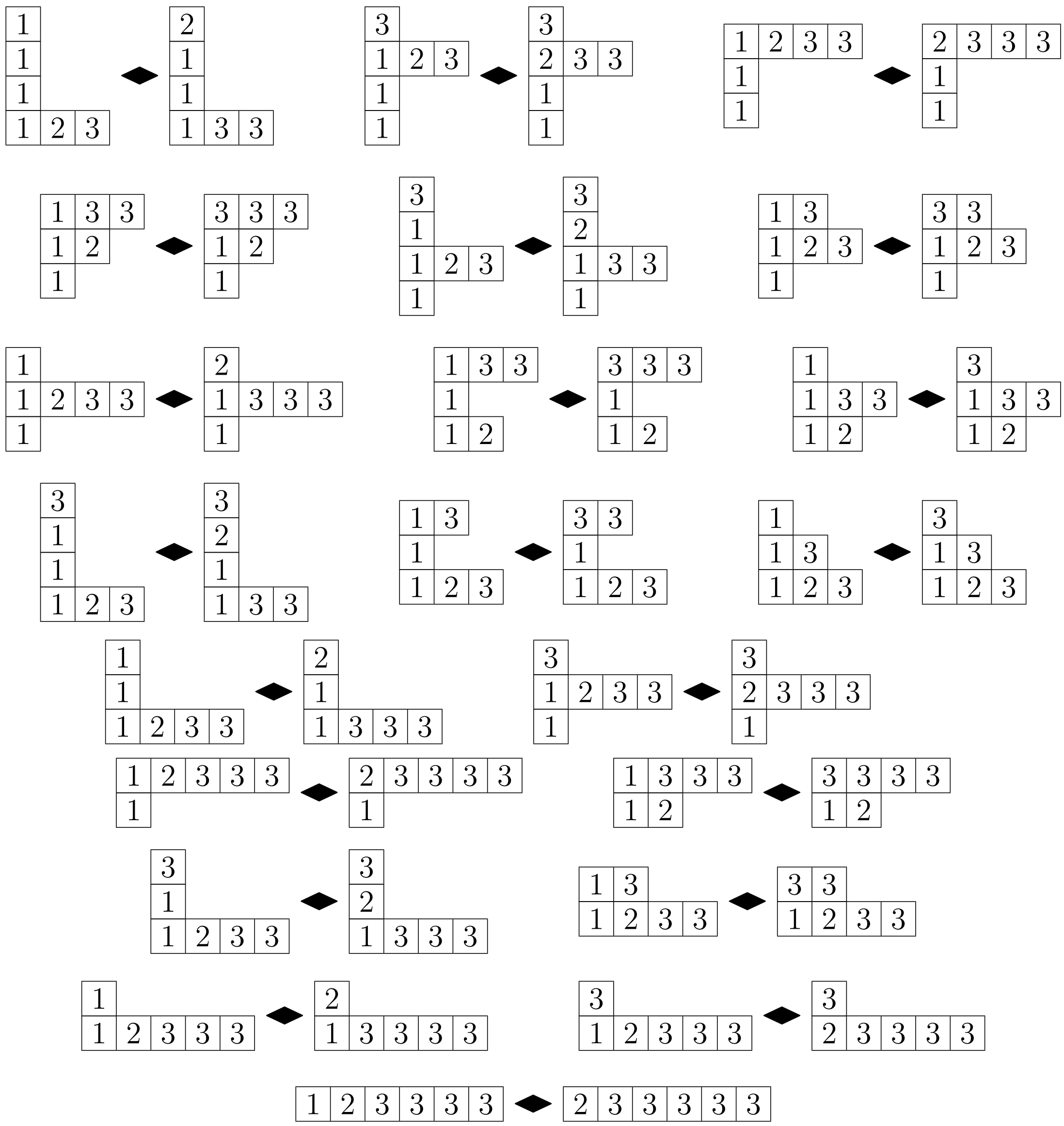


We thus have that the unique term indexed by the composition $(1, 1, 1, 2)$ remains, with respect to the sign-reversing involution $\phi_{2,3}$.

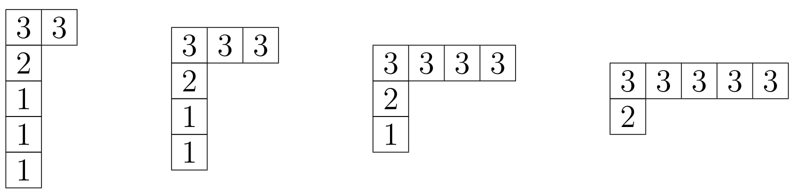
## 7.2 An illustration of an SQRH involution

Letting $Y_n$ denote the domain of the sign-reversing involution $\psi_n$ defined in Section 3.2, elements $\mathcal{T}$ in $Y_5$ such that $\psi(\mathcal{T}) \neq \mathcal{T}$ are illustrated below, letting the binary relation $\blacklozenge$ be such that $\mathcal{T}_1 \blacklozenge \mathcal{T}_2$ if and only if $\psi(\mathcal{T}_1) = \mathcal{T}_2$ (or equivalently $\psi(\mathcal{T}_2) = \mathcal{T}_1$).





The elements in the domain of $\psi_5$ that are fixed by $\psi_5$ are illustrated below:



(7.1)

Separately expanding each term of the sum $\sum_{i=0}^{5}(-1)^i E_{n-i}H_1H_i$ using bipieri shin tableaux, there are a total of 84 terms. However, there are only 4 terms remaining after simplifying the resultant sum consisting of 84 terms, namely the bipieri shin tableaux given in (7.1). This illustrates the following idea which was discussed in Section 1: given a sum of the form indicated in (1.1) (with terms of equal degree), after expanding the terms of this sum in a Schur-like basis using bipieri tableaux, there is typically a very large amount of subsequent cancellation.

### 7.3 Transition matrices between the canonical Schur-like bases

We conclude by showing some examples of transition matrices between Schur-like bases of NSym. As in [2], $M(A,B) = M_4(A,B)$ denotes the transition matrix between bases $A$ and $B$, with rows and columns indexed by compositions of 4 in lexicographic order.

$$M(\mathcal{S}^*, \mathfrak{S}) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$M(\mathfrak{S}, \mathcal{S}^*) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$M(\mathcal{S}^*, \text{ש}) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$M(\text{ש}, \mathcal{S}^*) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$M(\mathfrak{S},\text{ש}) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$M(\text{ש},\mathfrak{S}) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Since the immaculate-Pieri rule and the shin-Pieri rule are intuitively "similar", it seems natural that the transition matrices between the immaculate basis and the shin basis should be "close" to identity matrices in some sense. We leave it as an open problem to formalize this notion.

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