AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **65(2)** (2016), Pages 170–182

A class of totally antimagic total graphs

Jaroslav Ivančo

Institute of Mathematics
P. J. Šafárik University
Jesenná 5, 041 54 Košice
Slovakia
jaroslav.ivanco@upjs.sk

Abstract

A total labeling of a graph G is a bijection from the vertex set and edge set of G onto the set $\{1, 2, \ldots, |V(G)| + |E(G)|\}$. Such a labeling ξ is vertex-antimagic (edge-antimagic) if all vertex-weights $wt_\xi(v) = \xi(v) + \sum_{vu \in E(G)} \xi(vu)$, $v \in V(G)$, (all edge-weights $wt_\xi(vu) = \xi(v) + \xi(vu) + \xi(u)$, $vu \in E(G)$) are pairwise distinct. If a labeling is simultaneously vertex-antimagic and edge-antimagic it is called a totally antimagic total labeling. A graph that admits a totally antimagic total labeling is called a totally antimagic total graph. In this paper we will introduce a large class of totally antimagic total graphs.

1 Introduction

We consider finite undirected graphs without loops and multiple edges. If G is a graph, then $V(G)$ and $E(G)$ stand for the vertex set and edge set of G, respectively. The subgraph of a graph G induced by $U \subseteq V(G)$ is denoted by $G[U]$. The set of vertices of G adjacent to a vertex $v \in V(G)$ is denoted by $N_G(v)$. The cardinality of this set, denoted $\deg_G(v)$, is called the degree of v. As usual $\Delta(G)$ and $\delta(G)$ stand for the maximum and minimum degree among vertices of G. For integers p, q we denote by $[p, q]$ the set of all integers z satisfying $p \leq z \leq q$.

A *total labeling* of a graph G is a bijection ξ from $V(G) \cup E(G)$ onto the set $[1, |V(G)| + |E(G)|]$. The associated *vertex-weight* of a vertex $v \in V(G)$ is defined by

$$wt_\xi(v) = \xi(v) + \sum_{u \in N_G(v)} \xi(vu),$$

and the associated *edge-weight* of an edge $uv \in E(G)$ is defined by

$$wt_\xi(uv) = \xi(u) + \xi(uv) + \xi(v).$$

A total labeling is called *vertex-antimagic total* (*edge-antimagic total*), for short *VAT* (*EAT*), if all vertex-weights (edge-weights) are pairwise distinct. A total labeling that is simultaneously vertex-antimagic total and edge-antimagic total is called *totally antimagic total* (*TAT*). A graph that admits a VAT (EAT, TAT) labeling is called a *VAT* (*EAT, TAT*) *graph*.

In [5] it is proved that every graph is VAT. Using a similar method one can check that every graph is EAT. The TAT graphs were defined by Bača et al. in [1], where there were also presented some examples of TAT graphs. The definition of totally antimagic total labeling is an antipodal version of the concept of totally magic labeling defined by Exoo et al. in [2] (see also [6]). The TAT labeling is also an analogy of well known antimagic (edge) labeling defined by Hartsfield and Ringel in [4]. We refer the reader to [3] for comprehensive references.

In this paper we will introduce a large class of graphs which admit TAT labelings.

2 TAT graphs

An *overlaying* of a graph G is a bijection π from $V(G)$ onto $[1, |V(G)|]$ such that for any two vertices u, $v \in V(G)$ satisfying $\pi(v) - \pi(u) = 1$, there is an injective mapping $\sigma : N_G(u) - \{v\} \to N_G(v) - \{u\}$ such that

$$\pi(\sigma(w)) \geq \pi(w) \quad \text{for each } w \in N_G(u) - \{v\}.$$

A graph that admits an overlaying is called an *overlaid graph*.

Note that any two vertices u, v of a complete graph K_n satisfy:

$$N_{K_n}(u) - \{v\} = V(K_n) - \{u, v\} = N_{K_n}(v) - \{u\}.$$

Thus, every bijection from $V(K_n)$ onto $[1, n]$ is an overlaying of K_n.

Observation 1. *The complete graph K_n is overlaid.*

Similarly, $N_{D_n}(u) - \{v\} = \emptyset = N_{D_n}(v) - \{u\}$ for any two vertices u, v of a totally disconnected graph D_n (i.e., $\overline{K}_n$). Therefore, every bijection from $V(D_n)$ onto $[1, n]$ is an overlaying of D_n.

Observation 2. *The totally disconnected graph D_n is overlaid.*

We present a connection between overlaid graphs and TAT graphs. First we suggest that in [1] there were also defined special types of TAT graphs. A total labeling ξ of a graph G is called *super*, if the vertices are labeled with the smallest possible numbers, i.e., $\{\xi(u) : u \in V(G)\} = [1, |V(G)|]$. Similarly, ξ is called to be *sharp ordered* if $wt_\xi(u) < wt_\xi(v)$ holds for every pair of vertices u, v of G such that $\xi(u) < \xi(v)$. A graph that admits a super (sharp ordered) labeling is called a *super* (*sharp ordered*) graph.

Now, we are able to prove the crucial result of the paper.

Proposition 1. *Let G be an overlaid graph. Then G is a sharp ordered super TAT graph.*

Proof. Set $p = |V(G)|$, $q = |E(G)|$, and suppose that π is an overlaying of G. For every edge $e = uv$ of G we denote by $s(e)$ the sum of labels of its end vertices, i.e., $s(e) = \pi(u) + \pi(v)$. Now, denote the edges of G by $e_1, e_2, \dots, e_q$ in such a way that $s(e_i) \le s(e_j)$ holds for every $i < j$. Consider the mapping ξ from $V(G) \cup E(G)$ to $[1, p+q]$ defined by

$$\xi(x) = \begin{cases} \pi(x) & \text{if } x \in V(G), \\ p+i & \text{if } x = e_i. \end{cases}$$

Clearly, ξ is a super total labeling of G.

Suppose that e_i and e_j are distinct edges of G. Without loss of generality, let $i < j$. As $s(e_i) \le s(e_j)$, we have

$$\begin{aligned} wt_\xi(e_i) = s(e_i) + \xi(e_i) = s(e_i) + p + i &< s(e_j) + p + j \\ &= s(e_j) + \xi(e_j) = wt_\xi(e_j). \end{aligned}$$

Therefore, $wt_\xi(e_1) < wt_\xi(e_2) < \cdots < wt_\xi(e_q)$, i.e., ξ is an EAT labeling.

Now suppose that u and v are two vertices of G such that $\pi(v) - \pi(u) = 1$. Then there is an injective mapping σ from $N_G(u) - \{v\}$ to $N_G(v) - \{u\}$ such that $\pi(\sigma(x)) \ge \pi(x)$ for every $x \in N_G(u) - \{v\}$. This implies

$$s(ux) = \pi(u) + \pi(x) < \pi(v) + \pi(\sigma(x)) = s(v\sigma(x))$$

and consequently $\xi(ux) < \xi(v\sigma(x))$. Therefore,

$$\sum_{x \in N_G(u)-\{v\}} \xi(ux) \le \sum_{x \in N_G(u)-\{v\}} \xi(v\sigma(x)) \le \sum_{y \in N_G(v)-\{u\}} \xi(vy).$$

Let $\xi^*(u,v)$ be equal to $\xi(uv)$ when uv is an edge of G, and 0 otherwise. Then we have

$$\begin{aligned} wt_\xi(u) &= \xi(u) + \sum_{x \in N_G(u)} \xi(ux) \\ &= \xi(u) + \sum_{x \in N_G(u)-\{v\}} \xi(ux) + \xi^*(u,v) \\ &< \xi(v) + \sum_{y \in N_G(v)-\{u\}} \xi(vy) + \xi^*(u,v) \\ &= \xi(v) + \sum_{y \in N_G(v)} \xi(vy) = wt_\xi(v). \end{aligned}$$

Thus, $wt_\xi(\pi^{-1}(1)) < wt_\xi(\pi^{-1}(2)) < \cdots < wt_\xi(\pi^{-1}(p))$, i.e., ξ is a sharp ordered VAT labeling, which completes the proof. □

According to Observations 1 and 2 we immediately have

Corollary 1. *The complete graph K_n and the totally disconnected graph D_n are sharp ordered super TAT graphs.*

3 Overlaid graphs

In this section we determine some basic properties of overlaid graphs. We also present some examples of overlaid graphs, i.e., sharp ordered super TAT graphs.

Lemma 1. *Let π be an overlaying of a graph G with n vertices. Then*

$$\deg_G\left(\pi^{-1}(i)\right) \leq \deg_G\left(\pi^{-1}(i+1)\right)$$

for every $i \in [1, n-1]$.

Proof. Set $u = \pi^{-1}(i)$ and $v = \pi^{-1}(i+1)$, for $i \in [1, n-1]$. As $\pi(v) - \pi(u) = 1$, there is an injective mapping $\sigma : N_G(u) - \{v\} \to N_G(v) - \{u\}$. Then $|N_G(u) - \{v\}| \leq |N_G(v) - \{u\}|$ and consequently $|N_G(u)| \leq |N_G(v)|$, i.e., $\deg_G(u) \leq \deg_G(v)$. $\square$

Theorem 1. *Let G and H be overlaid graphs. If $\Delta(G) \leq \delta(H)$ then the disjoint union $G \cup H$ is also an overlaid graph.*

Proof. Let π (ν) be an overlaying of a graph G (H) with p (n) vertices. Consider the mapping $\mu : V(G \cup H) \to [1, p+n]$ defined by

$$\mu(w) = \begin{cases} \pi(w) & \text{if } w \in V(G), \\ p + \nu(w) & \text{if } w \in V(H). \end{cases}$$

Evidently, μ is a bijection. Now suppose that u and v are two vertices of $G \cup H$ satisfying $\mu(v) - \mu(u) = 1$. Distinguish the following cases.

If $\mu(u) \leq p - 1$ then u and v are vertices of G. Moreover, $\pi(v) - \pi(u) = \mu(v) - \mu(u) = 1$. Thus, there is an injection σ from $N_G(u) - \{v\}$ to $N_G(v) - \{u\}$ such that $\pi(\sigma(w)) \geq \pi(w)$ for each $w \in N_G(u) - \{v\}$. Clearly, σ is an injective mapping from $N_{G\cup H}(u) - \{v\}$ to $N_{G\cup H}(v) - \{u\}$ such that $\mu(\sigma(w)) = \pi(\sigma(w)) \geq \pi(w) = \mu(w)$ for each $w \in N_{G\cup H}(u) - \{v\}$.

If $\mu(u) = p$ then u is a vertex of G and v is a vertex of H. Moreover, $\pi(u) = p$, $\nu(v) = 1$, and according to Lemma 1, $\deg_G(u) = \Delta(G)$ and $\deg_H(v) = \delta(H)$. As $\Delta(G) \leq \delta(H)$, there is an injective mapping ρ from $N_G(u) = N_{G\cup H}(u) - \{v\}$ to $N_H(v) = N_{G\cup H}(v) - \{u\}$. Since $N_G(u) \subseteq V(G)$ and $N_H(v) \subseteq V(H)$, $\mu(\rho(w)) \geq p + 1 > p \geq \mu(w)$, for any $w \in N_G(u)$.

If $\mu(u) \geq p + 1$ then u and v are vertices of H. Moreover, $\nu(v) - \nu(u) = \mu(v) - \mu(u) = 1$. Thus, there is an injection σ' from $N_H(u) - \{v\}$ to $N_H(v) - \{u\}$ such that $\nu(\sigma'(w)) \geq \nu(w)$ for each $w \in N_H(u) - \{v\}$. Clearly, σ' is an injective mapping from $N_{G\cup H}(u) - \{v\}$ to $N_{G\cup H}(v) - \{u\}$ such that $\mu(\sigma'(w)) = p + \nu(\sigma'(w)) \geq p + \nu(w) = \mu(w)$ for each $w \in N_{G\cup H}(u) - \{v\}$.

Therefore, μ is an overlaying of $G \cup H$. $\square$

Corollary 2. *The disjoint union of regular overlaid graphs is an overlaid graph. Especially, the disjoint union of complete graphs is an overlaid graph.*

Proof. Let $G = \bigcup_{i=1}^{k} G_i$, where G_i is an overlaid regular graph of degree d_i. Without loss of generality we can assume that $d_1 \leq d_2 \leq \cdots \leq d_k$. For every $m \in [1, k]$, let $H_m = \bigcup_{i=1}^{m} G_i$. $H_1 = G_1$ is an overlaid graph. Now suppose that H_m is an overlaid graph. As $\Delta(H_m) = d_m \leq d_{m+1} = \delta(G_{m+1})$, by Theorem 1, $H_m \cup G_{m+1} = H_{m+1}$ is also an overlaid graph. Therefore, by induction, $H_k = G$ is an overlaid graph.

Any complete graph is regular and, by Observation 1, it is overlaid. Therefore, $G = \bigcup_{i=1}^{k} K_{n_i}$ is also an overlaid graph. □

Let mG denote the disjoint union of m copies of a graph G. According to Corollary 2, we immediately have

Corollary 3. *If G is a regular overlaid graph then mG is an overlaid graph. Especially, mK_n is an overlaid graph.*

Let M be a subset of the vertex set of a graph G. The graph $G(M\triangleright)$ is obtained from G by adding a new vertex w and edges $\{wu : u \in M\}$. Note that $G(M\triangleright)$ is isomorphic to the disjoint union $G \cup K_1$ when $M = \emptyset$, and it is isomorphic to the join $G \oplus K_1$ when $M = V(G)$.

Lemma 2. *Let π be an overlaying of a graph G. Let k be a positive integer satisfying $k + \Delta(G) \leq |V(G)|$ and let $M = \{u \in V(G) : \pi(u) \in [k, |V(G)|]\}$. Then $G(M\triangleright)$ is an overlaid graph.*

Proof. Set $n = |V(G)|$ and consider the mapping $\mu : V\big(G(M\triangleright)\big) \to [1, 1+n]$ defined by

$$\mu(x) = \begin{cases} \pi(x) & \text{if } x \in V(G), \\ 1+n & \text{if } x \notin V(G). \end{cases}$$

Evidently, μ is a bijection. Now suppose that u and v are two vertices of $G(M\triangleright)$ satisfying $\mu(v) - \mu(u) = 1$. Distinguish the following cases.

If $\mu(u) \leq k - 1$ then u and v are vertices of G. Moreover, $\pi(v) - \pi(u) = \mu(v) - \mu(u) = 1$. Thus, there is an injection σ from $N_G(u) - \{v\}$ to $N_G(v) - \{u\}$ such that $\pi(\sigma(x)) \geq \pi(x)$ for each $x \in N_G(u) - \{v\}$. Clearly, σ is an injective mapping from $N_{G(M\triangleright)}(u) - \{v\}$ to $N_{G(M\triangleright)}(v) - \{u\}$ such that $\mu(\sigma(x)) = \pi(\sigma(x)) \geq \pi(x) = \mu(x)$ for each $x \in N_{G(M\triangleright)}(u) - \{v\}$.

If $k \leq \mu(u) < n$ then u and v are vertices of G. Moreover, $\pi(v) - \pi(u) = \mu(v) - \mu(u) = 1$. Thus, there is an injection σ from $N_G(u) - \{v\}$ to $N_G(v) - \{u\}$ such that $\pi(\sigma(x)) \geq \pi(x)$ for each $x \in N_G(u) - \{v\}$. Clearly, the mapping σ', given by

$$\sigma'(x) = \begin{cases} \sigma(x) & \text{if } x \in N_G(u) - \{v\}, \\ w & \text{if } x = w, \end{cases}$$

is an injective mapping from $N_{G(M\triangleright)}(u) - \{v\}$ to $N_{G(M\triangleright)}(v) - \{u\}$ such that $\mu(\sigma'(x)) = \pi(\sigma(x)) \geq \pi(x) = \mu(x)$ for each $x \in N_{G(M\triangleright)}(u) - \{v, w\}$ and $\mu(\sigma'(w)) = 1 + n = \mu(w)$.

If $\mu(u) = n$ then u is a vertex of G and $v = w$. Moreover, $\pi(u) = n$ and according to Lemma 1, $\deg_G(u) = \Delta(G)$, so $\deg_{G(M\triangleright)}(u) = 1 + \Delta(G)$. As $\deg_{G(M\triangleright)}(w) = 1 + n - k \geq 1 + \Delta(G)$ and $\{\mu(x) : x \in N_{G(M\triangleright)}(w)\} = [k, n]$, there is an injective mapping ρ from $N_{G(M\triangleright)}(u) - \{w\}$ to $N_{G(M\triangleright)}(w) - \{u\}$ such that $\mu(\rho(x)) \geq \mu(x)$, for any $x \in N_{G(M\triangleright)}(u) - \{w\}$. □

Theorem 2. *Let G be an overlaid graph and let n be a positive integer. Then the join $G \oplus K_n$ is an overlaid graph.*

Proof. According to Lemma 2, $G \oplus K_1$ is an overlaid graph. As $G \oplus K_m = (G \oplus K_{m-1}) \oplus K_1$, $m \in [2, n]$, by induction, $G \oplus K_n$ is an overlaid graph. □

According to Corollary 3, the graph mK_r is overlaid. Therefore, we immediately have

Corollary 4. *Let m, r and n be positive integers. Then the graph $mK_r \oplus K_n$ is overlaid.*

Thus, the complete $(n+1)$-partite graph $K_{m,1,\dots,1} = mK_1 \oplus K_n$, the star $K_{m,1}$ and the friendship graph $mK_2 \oplus K_1$ are overlaid.

Lemma 3. *Let π be an overlaying of a graph G. Let k be a positive integer satisfying $k \leq 1 + \delta(G)$ and $\deg_G\big(\pi^{-1}(k+1)\big) > \deg_G\big(\pi^{-1}(k)\big)$. Then $G(M\triangleright)$ is an overlaid graph when $M = \{u \in V(G) : \pi(u) \in [1, k]\}$.*

Proof. Set $n = |V(G)|$ and consider the mapping $\mu : V\big(G(M\triangleright)\big) \to [1, 1+n]$ defined by

$$\mu(x) = \begin{cases} 1 + \pi(x) & \text{if } x \in V(G), \\ 1 & \text{if } x \notin V(G). \end{cases}$$

Evidently, μ is a bijection. Now suppose that u and v are two vertices of $G(M\triangleright)$ satisfying $\mu(v) - \mu(u) = 1$. Distinguish the following cases.

If $\mu(u) = 1$ then u is the added vertex, i.e., $u = w$. Moreover, $\pi(v) = 1$ and according to Lemma 1, $\deg_G(v) = \delta(G)$, so $\deg_{G(M\triangleright)}(v) = 1 + \delta(G) \geq k$. As $\deg_{G(M\triangleright)}(w) = k$ and $\{\mu(x) : x \in N_{G(M\triangleright)}(w)\} = [2, 1+k]$, there is an injective mapping ρ from $N_{G(M\triangleright)}(w) - \{v\}$ to $N_{G(M\triangleright)}(v) - \{w\}$ such that $\mu(\rho(x)) \geq \mu(x)$, for any $x \in N_{G(M\triangleright)}(w) - \{v\}$.

If $1 < \mu(u) \leq k$ then u and v are vertices of G. Moreover, $\pi(v) - \pi(u) = \mu(v) - \mu(u) = 1$. Thus, there is an injection σ from $N_G(u) - \{v\}$ to $N_G(v) - \{u\}$ such that $\pi(\sigma(x)) \geq \pi(x)$ for each $x \in N_G(u) - \{v\}$. Clearly, the mapping σ', given by

$$\sigma'(x) = \begin{cases} \sigma(x) & \text{if } x \in N_G(u) - \{v\}, \\ w & \text{if } x = w, \end{cases}$$

is an injective mapping from $N_{G(M\triangleright)}(u) - \{v\}$ to $N_{G(M\triangleright)}(v) - \{u\}$ such that $\mu(\sigma'(x)) = 1 + \pi(\sigma(x)) \geq 1 + \pi(x) = \mu(x)$ for each $x \in N_{G(M\triangleright)}(u) - \{v, w\}$ and $\mu(\sigma'(w)) = 1 = \mu(w)$.

If $\mu(u) = k + 1$ then u and v are vertices of G. Moreover, $\pi(v) - \pi(u) = \mu(v) - \mu(u) = 1$. Thus, there is an injection σ from $N_G(u) - \{v\}$ to $N_G(v) - \{u\}$ such that $\pi(\sigma(x)) \geq \pi(x)$ for each $x \in N_G(u) - \{v\}$. As $u = \pi^{-1}(k)$, $v = \pi^{-1}(k+1)$,

$\deg_G(v) > \deg_G(u)$, and there is a vertex $y \in N_G(v) - \{u\}$ such that $\sigma(x) \neq y$ for each $x \in N_G(u) - \{v\}$. Clearly, the mapping σ', given by

$$\sigma'(x) = \begin{cases} \sigma(x) & \text{if } x \in N_G(u) - \{v\}, \\ y & \text{if } x = w, \end{cases}$$

is an injective mapping from $N_{G(M\triangleright)}(u) - \{v\}$ to $N_{G(M\triangleright)}(v) - \{u\}$ such that $\mu(\sigma(x)) = 1 + \pi(\sigma(x)) \geq 1 + \pi(x) = \mu(x)$ for each $x \in N_{G(M\triangleright)}(u) - \{v, w\}$ and $\mu(\sigma'(w)) = \mu(y) > 1 = \mu(w)$.

If $\mu(u) > k+1$ then u and v are vertices of G. Moreover, $\pi(v) - \pi(u) = \mu(v) - \mu(u) = 1$. Thus, there is an injection σ from $N_G(u) - \{v\}$ to $N_G(v) - \{u\}$ such that $\pi(\sigma(x)) \geq \pi(x)$ for each $x \in N_G(u) - \{v\}$. Clearly, σ is an injective mapping from $N_{G(M\triangleright)}(u) - \{v\}$ to $N_{G(M\triangleright)}(v) - \{u\}$ such that $\mu(\sigma(x)) = 1 + \pi(\sigma(x)) \geq 1 + \pi(x) = \mu(x)$ for each $x \in N_{G(M\triangleright)}(u) - \{v\}$. □

Theorem 3. *Let G be an overlaid graph on p vertices. Let n and m be positive integers satisfying $n \geq p - 1 - \delta(G)$ and $m \leq p - 1 - \Delta(G)$. Then the join $G \oplus (K_n \cup K_m)$ is an overlaid graph.*

Proof. m is a positive integer and so $\Delta(G) < p - 1$. According to Theorem 2, $G \oplus K_n$ is an overlaid graph. Therefore, there is an overlaying π of $G \oplus K_n$. As $\deg_{G\oplus K_n}(v) = n + \deg_G(v)$ for $v \in V(G)$ and $\deg_{G\oplus K_n}(v) = n - 1 + p$ for $v \in V(K_n)$, according to Lemma 1, $\{\pi(v) : v \in V(G)\} = [1, p]$ and $\{\pi(v) : v \in V(K_n)\} = [p+1, p+n]$. Since

$$1 + \delta(G \oplus K_n) = 1 + n + \delta(G) \geq p$$

and

$$\begin{aligned} deg_{G\oplus K_n}\big(\pi^{-1}(p)\big) &= n + \Delta(G) \\ &< n - 1 + p = \deg_{G\oplus K_n}\big(\pi^{-1}(p+1)\big), \end{aligned}$$

by Lemma 3, $\big(G \oplus K_n\big)\big(V(G)\triangleright\big) = G \oplus (K_n \cup K_1)$ is an overlaid graph.

As $G \oplus (K_n \cup K_t) = \big(G \oplus (K_n \cup K_{t-1})\big)\big((V(G) \cup V(K_{t-1}))\triangleright\big)$, $t \in [2, m]$, by induction, $G \oplus (K_n \cup K_m)$ is an overlaid graph. □

By Corollary 3, the graph sK_r is overlaid. Thus, we immediately have

Corollary 5. *Let s, r, n and m be positive integers satisfying $n \geq (s-1)r \geq m$. Then the join $sK_r \oplus (K_n \cup K_m)$ is an overlaid graph.*

Let G and H be disjoint graphs. Let h be a mapping from $V(H)$ to $V(G)$. By $G \cup_h H$ we denote the graph $G \cup H$ together with all edges joining each vertex $u \in V(H)$ and $h(u) \in V(G)$. Note that if G is a graph on n vertices and $h : V(nH) \to V(G)$ is a mapping such that the image of any vertex of ith copy of H is the ith vertex of G, then $G \cup_h nH$ is well-known *corona* of G with H, denoted by $G \odot H$.

Lemma 4. *Let π and ν be overlayings of graphs G and H, respectively. Let h be a mapping from $V(H)$ to $V(G)$ satisfying:*

(a1) $\big(\forall u, v \in V(H)\big)$ $\nu(u) < \nu(v) \Longrightarrow \pi(h(u)) \le \pi(h(v))$;

(a2) $\big(\forall u, v \in V(G)\big)$ $\pi(u) < \pi(v) \Longrightarrow \deg_G(u) + |\{w : h(w) = u\}| \le \deg_G(v) + |\{w : h(w) = v\}|$;

(a3) $\big(\exists z \in N_G(\pi^{-1}(1))\big)$ $\pi(h(\nu^{-1}(|V(H)|))) \le \pi(z)$.

If $\Delta(H) < \delta(G)$ *then* $G \cup_h H$ *is an overlaid graph.*

Proof. Set $p = |V(G)|$ and $n = |V(H)|$. Consider the mapping μ from $V(G \cup_h H)$ to $[1, n+p]$ defined by

$$\mu(w) = \begin{cases} \nu(w) & \text{if } w \in V(H), \\ n + \pi(w) & \text{if } w \in V(G). \end{cases}$$

Evidently, μ is a bijection. Now suppose that u and v are two vertices of $G \cup_h H$ satisfying $\mu(v) - \mu(u) = 1$. Distinguish the following cases.

If $\mu(u) < n$ then u and v are vertices of H. Moreover, $\nu(v) - \nu(u) = \mu(v) - \mu(u) = 1$. Thus, there is an injection σ from $N_H(u) - \{v\}$ to $N_H(v) - \{u\}$ such that $\nu(\sigma(x)) \ge \nu(x)$ for each $x \in N_H(u) - \{v\}$. Clearly, the mapping σ', given by

$$\sigma'(x) = \begin{cases} \sigma(x) & \text{if } x \in N_H(u) - \{v\}, \\ h(v) & \text{if } x = h(u), \end{cases}$$

is an injective mapping from $N_{G \cup_h H}(u) - \{v\}$ to $N_{G \cup_h H}(v) - \{u\}$ such that $\mu(\sigma'(x)) = \nu(\sigma(x)) \ge \nu(x) = \mu(x)$ for each $x \in N_{G \cup_h H}(u) - \{v, h(u)\}$, and by (a1), $\mu(\sigma'(h(u))) = \mu(h(v)) = n + \pi(h(v)) \ge n + \pi(h(u)) = \mu(h(u))$.

If $\mu(u) = n$ then u is a vertex of H and v is a vertex of G. Moreover, $\nu(u) = n$, $\pi(v) = 1$, and according to Lemma 1, $\deg_H(u) = \Delta(H)$ and $\deg_G(v) = \delta(G)$. As $\Delta(H) < \delta(G)$, there is an injective mapping ρ from $N_H(u)$ to $N_G(v) - \{z\}$. Since $N_H(u) \subseteq V(H)$ and $N_G(v) \subseteq V(G)$, $\mu(\rho(w)) \ge n + 1 > n \ge \mu(w)$, for any $w \in N_H(u)$. Therefore, the mapping ρ', given by

$$\rho'(x) = \begin{cases} \rho(x) & \text{if } x \in N_{G \cup_h H}(u) - \{h(u)\}, \\ z & \text{if } x = h(u), \end{cases}$$

(or $\rho' = \rho$, when $h(u) = v$) is an injective mapping from $N_{G \cup_h H}(u) - \{v\}$ to $N_{G \cup_h H}(v) - \{u\}$ such that $\mu(\rho'(x)) = \mu(\rho(x)) \ge \mu(x)$ for each $x \in N_{G \cup_h H}(u) - \{v, h(u)\}$, and by (a3), $\mu(\rho'(h(u))) = \mu(z) = n + \pi(z) \ge n + \pi(h(u)) = \mu(h(u))$.

If $\mu(u) > n$ then u and v are vertices of G. Moreover, $\pi(v) - \pi(u) = \mu(v) - \mu(u) = 1$. Thus, there is an injection σ from $N_G(u) - \{v\}$ to $N_G(v) - \{u\}$ such that $\pi(\sigma(x)) \ge \pi(x)$ for each $x \in N_G(u) - \{v\}$. By (a2), there is an injective mapping ϱ from $\{w : h(w) = u\}$ to $\{w : h(w) = v\} \cup N_G(v) - \big(\{u\} \cup \{\sigma(x) : x \in N_G(u) - \{v\}\}\big)$. According to (a1), $\mu(\varrho(x)) \ge \mu(x)$ for every $x \in \{w : h(w) = u\}$. Since $N_{G \cup_h H}(u) = N_G(u) \cup \{w : h(w) = u\}$, the mapping σ', given by

$$\sigma'(x) = \begin{cases} \sigma(x) & \text{if } x \in N_G(u) - \{v\}, \\ \varrho(x) & \text{if } x \in \{w : h(w) = u\}, \end{cases}$$

is an injective mapping from $N_{G\cup_h H}(u)-\{v\}$ to $N_{G\cup_h H}(v)-\{u\}$ such that $\mu(\sigma'(x)) = \mu(\sigma(x)) = n+\pi(\sigma(x)) \geq n+\pi(x) = \mu(x)$ for each $x \in N_G(u)-\{v\}$ and $\mu(\sigma'(x)) = \mu(\varrho(x)) \geq \mu(x)$ for each $x \in \{w : h(w) = u\}$. □

Let k be a positive integer and let G_i, for $i \in [0,k]$, be a graph. Let $n_0 = 1$ and $n_{i+1} = n_i \cdot |V(G_i)|$, for $i \in [0,k-1]$. Denote by $\odot(G_0, G_1, \ldots, G_k)$ the graph satisfying:

(b1) $\bigcup_{i=0}^{k} n_i G_i$ is its spanning subgraph;

(b2) its subgraph induced by $V(n_i G_i \cup n_{i+1} G_{i+1})$ is $n_i G_i \odot G_{i+1}$, $i \in [0,k-1]$;

(b3) any its edge belongs to some induced subgraph considered in (b2).

Theorem 4. *Let k be a positive integer and let G_i be an overlaid graph for each $i \in [0,k]$. Suppose that the following conditions are satisfied:*

(c1) $\Delta(G_0) = |V(G_0)| - 1$;

(c2) *G_i is a d_i-regular graph for each $i \in [1,k]$;*

(c3) *$d_0 = \delta(G_0) > d_1$ when $|V(G_0)| > 1$;*

(c4) *$d_i \geq d_{i+1}$ for each $i \in [1,k-1]$;*

(c5) *$d_i + |V(G_{i+1})| \leq d_{i-1} + |V(G_i)|$ for each $i \in [1,k-1]$.*

Then $\odot(G_0, G_1, \ldots, G_k)$ is an overlaid graph.

Proof. Let $n_0 = 1$ and $n_{i+1} = n_i \cdot |V(G_i)|$, for $i \in [0,k-1]$. For $j \in [1,k]$, the regular graph G_j is overlaid and, by Corollary 3, $n_j G_j$ is also an overlaid graph. Therefore, there is an overlaying ν_j of $n_j G_j$. Moreover, according to proof of Theorem 1, we can assume that the values of vertices of rth copy of G_j belong to $[(r-1)|V(G_j)| + 1, r|V(G_j)|]$, $r \in [1, n_j]$. This means that $\nu_j^{-1}(t)$ is a vertex of $\lceil t/|V(G_j)| \rceil$th copy of G_j, for each $t \in [1, n_j |V(G_j)|]$.

If $G_0 = K_1$ then $\odot(G_0, G_1)$ is isomorphic to $G_1 \oplus K_1$. Therefore, by Theorem 2, the graph $\odot(G_0, G_1)$ is overlaid.

Similarly, if $|V(G_0)| > 1$ then G_0 is overlaid and there is an overlaying π_0 of G_0. Let h_1 be a mapping from $n_1 G_1$ to G_0 given by

$$h_1\big(\nu_1^{-1}(t)\big) = \pi_0^{-1}(\lceil t/|V(G_1)| \rceil), \quad t \in [1, n_1 |V(G_1)|].$$

Suppose that u and v are vertices of $n_1 G_1$ such that $\nu_1(u) < \nu_1(v)$. Let $\nu_1(u) = r$, $\nu_1(v) = s$. Then $r < s$ and

$$\begin{aligned} \pi_0\big(h_1(u)\big) &= \pi_0\big(h_1(\nu_1^{-1}(r))\big) = \pi_0\big(\pi_0^{-1}(\lceil r/|V(G_1)| \rceil)\big) \\ &= \lceil r/|V(G_1)| \rceil \leq \lceil s/|V(G_1)| \rceil \\ &= \pi_0\big(\pi_0^{-1}(\lceil s/|V(G_1)| \rceil)\big) = \pi_0\big(h_1(\nu_1^{-1}(s))\big) = \pi_0\big(h_1(v)\big). \end{aligned}$$

Suppose that u and v are vertices of G_0 such that $\pi_0(u) < \pi_0(v)$. Then, by Lemma 1, $\deg_{G_0}(u) \le \deg_{G_0}(v)$ and

$$\begin{aligned}\deg_{G_0}(u) + |\{w : h_1(w) = u\}| &= \deg_{G_0}(u) + |V(G_1)|\\ &\le \deg_{G_0}(v) + |V(G_1)| = \deg_{G_0}(v) + |\{w : h_1(w) = v\}|.\end{aligned}$$

Set $z = \pi_0^{-1}(|V(G_0)|)$. According to Lemma 1 and (c1), we have

$$\deg_{G_0}(z) = \Delta(G_0) = |V(G_0)| - 1.$$

Thus, $z \in N_{G_0}\big(\pi_0^{-1}(1)\big)$ and $\pi_0(h_1(\nu_1^{-1}(|V(n_1G_1)|))) \le \pi_0(z)$.

Moreover, by (c3), $\Delta(n_1G_1) = d_1 < \delta(G_0)$. Therefore, the assumptions of Lemma 4 hold, and so the graph $G_0 \cup_{h_1} n_1G_1$ is overlaid. As $G_0 \cup_{h_1} n_1G_1$ is isomorphic to $\odot(G_0, G_1)$, the graph $\odot(G_0, G_1)$ is overlaid.

Note that in both cases (see proofs of Lemmas 2 and 4) there is an overlaying π_1 of $\odot(G_0, G_1)$ such that $\pi_1^{-1}(t)$ is a vertex of $\lceil t/|V(G_1)|\rceil$th copy of G_1, for $t \in [1, n_1|V(G_1)|]$, and $\pi_1^{-1}(t)$ is a vertex of G_0, for $t > n_1|V(G_1)|$.

Now suppose that there is an overlaying π_p of $O_p = \odot(G_0, G_1, \dots, G_p)$, $p \in [1, k-1]$, such that $\pi_p^{-1}(t)$ is a vertex of $\lceil t/|V(G_p)|\rceil$th copy of G_p, for $t \in [1, n_m|V(G_p)|]$, and $\pi_p^{-1}(t)$ is a vertex of $\odot(G_0, \dots, G_{p-1})$, for $t > n_p|V(G_p)|$. Define the mapping $h_{p+1} : n_{p+1}G_{p+1} \to O_p$ by

$$h_{p+1}\big(\nu_{p+1}^{-1}(t)\big) = \pi_p^{-1}(\lceil t/|V(G_{p+1})|\rceil), \quad t \in [1, n_{p+1}|V(G_{p+1})|].$$

In the same manner as above we can show that if $\nu_{p+1}(u) < \nu_{p+1}(v)$, $u, v \in n_{p+1}G_{p+1}$, then $\pi_p\big(h_{p+1}(u)\big) \le \pi_p\big(h_{p+1}(v)\big)$.

If $u \in V(n_pG_p)$ then $\deg_{O_p}(u) + |\{w : h_{p+1}(w) = u\}| = 1 + d_p + |V(G_{p+1})|$. If $v \in V(\odot(G_0, \dots, G_{p-1}))$ then

$$\deg_{O_p}(v) + |\{w : h_{p+1}(w) = v\}| = \deg_{O_p}(v) \ge 1 + d_{p-1} + |V(G_p)|.$$

Therefore, according to (c5),

$$\deg_{O_p}(u) + |\{w : h_{p+1}(w) = u\}| \le \deg_{O_p}(v) + |\{w : h_{p+1}(w) = v\}|,$$

for $u, v \in V(O_p)$ such that $\pi_p(u) < \pi_p(v)$.

Let $z = h_p(\nu_p^{-1}(1))$. Then $z \in \odot(G_0, \dots, G_{p-1})$ and $\pi_p(z) > n_p|V(G_p)|$. As $\nu_p^{-1}(1) = \pi_p^{-1}(1)$, z belongs to $N_{O_p}(\pi_p^{-1}(1))$ and

$$\begin{aligned}\pi_p(h_{p+1}(\nu_{p+1}^{-1}(|V(n_{p+1}G_{p+1})|))) &= \pi_p(\pi_p^{-1}(n_{p+1}))\\ &= n_{p+1} = n_p|V(G_p)| < \pi_p(z).\end{aligned}$$

By (c4), $\Delta(n_{p+1}G_{p+1}) = d_{p+1} < 1 + d_p = \delta(O_p)$. Therefore, the assumptions of Lemma 4 hold, and so the graph $O_p \cup_{h_{p+1}} n_{p+1}G_{p+1}$ is overlaid. As $O_p \cup_{h_{p+1}} n_{p+1}G_{p+1}$ is isomorphic to $\odot(G_0, G_1, \dots, G_{p+1})$, the graph $\odot(G_0, G_1, \dots, G_{p+1})$ is overlaid. Moreover, by proof of Lemma 4, there is an overlaying π_{p+1} of $\odot(G_0, G_1, \dots, G_{p+1})$ such that $\pi_{p+1}^{-1}(t)$ is a vertex of $\lceil t/|V(G_{p+1})|\rceil$th copy of G_{p+1}, for $t \in [1, n_{p+1}|V(G_{p+1})|]$, and $\pi_{p+1}^{-1}(t)$ is a vertex of O_p, for $t > n_{p+1}|V(G_{p+1})|$.

Therefore, by induction, $\odot(G_0, G_1, \dots, G_k)$ is an overlaid graph. □

A tree in which every vertex that is not a leaf has the degree d is called *d-regular* tree. If G_0 is a tree and $G_1, \dots, G_k$ are totally disconnected graphs then the graph $\odot(G_0, G_1, \dots, G_k)$ is a tree. Thus, by Theorem 4, we immediately have

Corollary 6. *Let r be a positive integer. Then the $(r+1)$-regular trees $\odot(K_1, D_{r+1}, D_r, \dots, D_r)$ and $\odot(K_2, D_r, \dots, D_r)$ are overlaid graphs.*

Let P_n denote a path on n vertices. Clearly, $P_1 = K_1$, $P_2 = K_2$ and $P_3 = K_{2,1}$ are overlaid. The path on $n \geq 4$ vertices is a 2-regular tree $\odot(K_2, D_1, \dots, D_1)$ when n is even, and $\odot(K_1, D_2, D_1, \dots, D_1)$ when n is odd. Thus, we get

Corollary 7. *The path P_n is an overlaid graph for each $n \geq 1$.*

Corollary 8. *The cycle C_n is an overlaid graph for each $n \geq 3$.*

Proof. By Observation 1, the cycle $C_3 = K_3$ is an overlaid graph.

The path P_m, $m \geq 3$, is overlaid. Thus, there is an overlaying π of P_m. By Lemma 1, $\pi^{-1}(1)$ and $\pi^{-1}(2)$ are vertices of degree 1, and $\pi^{-1}(3)$ is a vertex of degree 2. Therefore, by Lemma 3, $P_m(M\triangleright)$ is an overlaid graph when $M = \{u \in V(P_m) : \pi(u) \in [1,2]\}$. Clearly, the graph $P_m(M\triangleright)$ is isomorphic to C_{m+1}. $\square$

Combining Theorem 2 and Corollary 7 (Corollary 8) we get

Corollary 9. *The fan $P_n \oplus K_1$ and the wheel $C_n \oplus K_1$ are overlaid graphs.*

Theorem 5. *For any graph G there is an overlaid graph which contains an induced subgraph isomorphic to G.*

Proof. If G is totally disconnected then, by Observation 2, it is overlaid.

If $G = G' \cup D_n$ and there is an overlaid graph H' containing an induced subgraph isomorphic to G' then, by Theorem 1, the disjoint union $H' \cup D_n$ is also an overlaid graph. Clearly, $H' \cup D_n$ contains an induced subgraphs isomorphic to G. Therefore, next we can assume that $\delta(G) \geq 1$.

Set $p = |V(G)|$, $q = |E(G)|$ and $k = p + 2q + 1$. Denote the vertices of G by $v_1, v_2, \dots, v_p$ in such a way that $\deg_G(v_i) \leq \deg_G(v_j)$ holds for every $i < j$. For $i \in [1,p]$, let $s_i = \sum_{t=1}^{i} \deg_G(v_t)$. Now consider the graph H satisfying:

(i) $V(H) = \{v_i : i \in [1,k]\}$;

(ii) the subgraph of H induced by $\{v_i : i \in [1,p]\}$ is G;

(iii) the subgraph of H induced by $\{v_i : i \in [p+2,k]\}$ is K_{2q};

(iv) $E(H) = \bigcup_{i=1}^{p} \{v_{i+1}v_j : j \in [k+1-s_i, k]\} \cup E(G) \cup E(K_{2q})$.

Define the bijection $\pi : V(H) \to [1,k]$ by $\pi(v_i) = i$ and distinguish the following cases.

Let $i = 1$. Then $N_H(v_1) = N_G(v_1)$ and $N_H(v_2) = N_G(v_2) \cup \{v_j : j \in [k+1-\deg_G(v_1), k]\}$. As $|N_H(v_1)| = \deg_G(v_1) = |[k+1-\deg_G(v_1), k]|$, there is an injective mapping σ from $N_H(v_1) - \{v_2\}$ to $\{v_j : j \in [k+1-\deg_G(v_1), k]\}$. Clearly, $\pi(\sigma(x)) > p \geq \pi(x)$ for every $x \in N_H(v_1) - \{v_2\}$.

Let $i \in [2, p-1]$. Then $N_H(v_i) = N_G(v_i) \cup \{v_j : j \in [k+1-s_{i-1}, k]\}$ and $N_H(v_{i+1}) = N_G(v_{i+1}) \cup \{v_j : j \in [k+1-s_i, k]\}$. As $|N_G(v_i)| = \deg_G(v_i) = |[k+1-s_i, k-s_{i-1}]|$, there is an injective mapping ρ from $N_G(v_i) - \{v_{i+1}\}$ to $\{v_j : j \in [k+1-s_i, k-s_{i-1}]\}$. Clearly, the mapping σ, given by

$$\sigma(x) = \begin{cases} \rho(x) & \text{if } x \in N_G(v_i) - \{v_{i+1}\}, \\ x & \text{if } x \in \{v_j : j \in [k+1-s_{i-1}, k]\}, \end{cases}$$

is an injective mapping from $N_H(v_i) - \{v_{i+1}\}$ to $N_H(v_{i+1}) - \{v_i\}$ such that $\pi(\sigma(x)) = \pi(\rho(x)) > p \geq \pi(x)$ for each $x \in N_G(v_i) - \{v_{i+1}\}$ and $\pi(\sigma(x)) = \pi(x)$ for $x \in \{v_j : j \in [k+1-s_{i-1}, k]\}$.

Let $i = p$. Then $N_H(v_p) = N_G(v_p) \cup \{v_j : j \in [k+1-s_{p-1}, k]\}$ and $N_H(v_{p+1}) = \{v_j : j \in [k+1-s_p, k]\}$. As $|N_G(v_p)| = |[k+1-s_p, k-s_{p-1}]|$, there is an injection ρ from $N_G(v_p)$ to $\{v_j : j \in [k+1-s_p, k-s_{p-1}]\}$. Clearly, the mapping σ, given by

$$\sigma(x) = \begin{cases} \rho(x) & \text{if } x \in N_G(v_p), \\ x & \text{if } x \in \{v_j : j \in [k+1-s_{p-1}, k]\}, \end{cases}$$

is an injective mapping from $N_H(v_p) - \{v_{p+1}\}$ to $N_H(v_{p+1}) - \{v_p\}$ such that $\pi(\sigma(x)) = \pi(\rho(x)) > p > \pi(x)$ for each $x \in N_G(v_p)$ and $\pi(\sigma(x)) = \pi(x)$ for $x \in \{v_j : j \in [k+1-s_{p-1}, k]\}$.

Let $i \in [p+1, k-1]$. Then $\{v_j : j \in [p+1, k]\} \subseteq N_H(v_i) \cup \{v_i\} \subseteq N_H(v_{i+1}) \cup \{v_{i+1}\}$. Therefore, the injection σ from $N_H(v_i) - \{v_{i+1}\}$ to $N_H(v_{i+1}) - \{v_i\}$, given by $\sigma(x) = x$, satisfies $\pi(\sigma(x)) = \pi(x)$.

Thus, π is an overlaying of H and G is an induced subgraph of the overlaid graph H. □

Combining Theorem 5 and Proposition 1 we get

Corollary 10. *For any graph G there is a sharp ordered super TAT graph which contains an induced subgraph isomorphic to G.*

4 Conclusion

In the paper we present overlaid graphs. By Proposition 1, these graphs are sharp ordered super TAT. So, we present a large class of sharp ordered super TAT graphs. Moreover, two conjectures are stated in [1] (namely, 1: *every graph $G \oplus K_1$ is TAT* and 2: *every complete graph is TAT*). Corollary 1 confirms Conjecture 2. Conjecture 1 is still open, however Corollary 10 is a weak version of this conjecture.

Hartsfield and Ringel [4] conjectured that every connected graph except P_2 admits a vertex-antimagic edge labeling. We believe that the following analogy of this conjecture is true.

Conjecture. *Every graph is TAT.*

Acknowledgements

This work was supported by Slovak VEGA Grant 1/0652/12 and by Slovak VEGA Grant 1/0368/16.

References

[1] M. Bača, M. Miller, O. Phanalasy, J. Ryan, A. Semaničová-Feňovčíková and A. A. Sillasen, Totally antimagic total graphs, *Australas. J. Combin.* **61** (2015), 42–56.

[2] G. Exoo, A. C. H. Ling, J. P. McSorley, N. C. K. Phillips and W. D. Wallis, Totally magic graphs, *Discrete Math.* **254** (2002), 103–113.

[3] J. A. Gallian, A dynamic survey of graph labeling, *Electron. J. Combin.* (2015), #DS6.

[4] N. Hartsfield and G. Ringel, *Pearls in Graph Theory*, Academic Press, Boston - San Diego - New York - London, 1990.

[5] M. Miller, O. Phanalasy and J. Ryan, All graphs have antimagic total labelings, *Electron. Notes Discrete Math.* **38** (2011), 645–650.

[6] W. D. Wallis, *Magic Graphs*, Birkhäuser, Boston, 2001.

(Received 25 Aug 2015)