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Coloring k-trees with forbidden monochrome and rainbow triangles

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Abstract

An$(\mathscr{H}, H)$-*good* coloring is the coloring of the edges of a (hyper)graph $\mathscr{H}$ such that no subgraph $H \subseteq \mathscr{H}$ is monochrome or rainbow. Similarly, we define an $(\mathscr{H}, H)$-*proper* coloring to be the coloring of the vertices of $\mathscr{H}$ with forbidden monochromatic and rainbow copies of H. An $(\mathscr{H}, K_t)$-good coloring is also known as a mixed Ramsey coloring when $\mathscr{H} = K_n$ is a complete graph, and an $(\mathscr{H}, \overline{K_t})$-proper coloring is a mixed hypergraph coloring of a t-uniform hypergraph $\mathscr{H}$. We highlight these two related theories by finding the number of (T_k^n, K_3)-good and proper colorings for some k-trees, T_k^n with $k \geq 2$. Further, a partition of an edge/vertex set into i nonempty classes is called *feasible* if it is induced by a good/proper coloring using i colors. If r_i is the number of feasible partitions for $1 \leq i \leq n$, then the vector $(r_1, \ldots, r_n)$ is called the *chromatic spectrum*. We investigate and compare the exact values in the chromatic spectrum for some 2-trees, given (T_2^n, K_3)-good versus (T_2^n, K_3)-proper colorings. In particular, we find that when T_2^n is a fan, r_2 follows a Fibonacci recurrence.

1 Preliminaries

It is customary to define a *hypergraph* $\mathscr{H}$ to be the ordered pair $(X, \mathcal{E})$, where X is a finite set of vertices with *order* $|X| = n$ and $\mathcal{E}$ is a collection of nonempty subsets of X, called *(hyper)edges*. $\mathscr{H}$ is said to be *linear* (otherwise it is *nonlinear*) if $E_1 \cap E_2$ is either empty or a singleton, for any pair of hyperedges. The number of vertices contained in E of $\mathcal{E}$, denoted $|E|$, is the *size* of E. When $|E| = r$, $\mathscr{H}$ is said to be r*-uniform* and a 2-uniform hypergraph $\mathscr{H} = G$ is a *graph*. For more basic definitions of graphs and hypergraphs, we recommend [17].

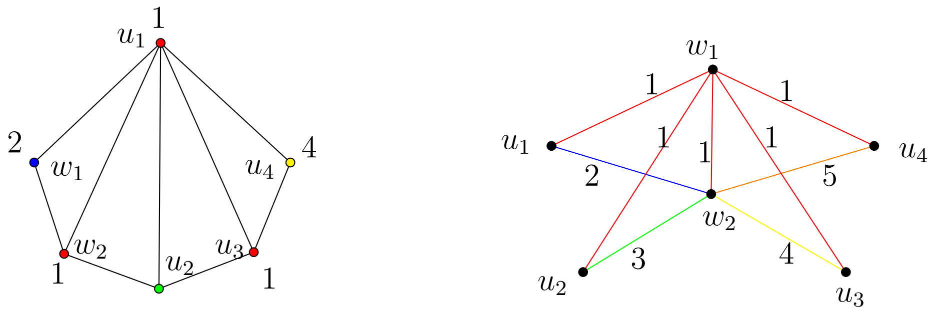
Consider the mapping $c : A \to \{1, 2, \ldots, \lambda\}$ being a λ-coloring of the elements of A. A subset $B \subseteq A$ is said to be *monochrome* if all of its elements share the same color and B is *rainbow* if all of its elements have distinct colors. Let H be a subgraph of a graph G. An edge coloring of G is called $(G; H)$*-good* if it admits no monochromatic copy of H and no rainbow copy of H. Likewise, a $(G; H)$*-proper* coloring is the coloring of the vertices of G such that no copy of H is monochrome or rainbow. Figure 1(A) is an example of a $(G; K_3)$-proper coloring while Figure 1(B) shows a $(G; K_3)$-good coloring.

Axenovich et al. [2] have referred to (K_n, K_3)-good coloring as mixed-Ramsey coloring, a hybrid of classical Ramsey and anti-Ramsey colorings [2, 8, 14] and the minumum and maximum numbers of colors used in a (K_n, K_3)-good coloring have been the subject of extensive research in [2, 3], for instance. Further, in mixed hypergraph colorings [16], a hypergraph $\mathscr{H}$ that admits an $(\mathscr{H}; H)$-proper coloring is called a *bihypergraph* when $H = \overline{K_t}$, the complement of a complete graph on $t \geq 3$ vertices. We note here that, mixed hypergraphs are often used to encode partitioning constraints, and recently bihypergraphs have appeared in communication models for cyber security [11]. Although this paper focuses on graphs, it is worth noting that the results concern some linear and nonlinear bihypergraphs as well.

A partition of an edge/vertex set into i nonempty classes is called *feasible* if it is induced by a good/proper coloring using i colors. If r_i is the number of feasible partitions for each $1 \leq i \leq n$, then the vector $(r_1, \ldots, r_n)$ is called the *chromatic spectrum*. The chromatic spectrum of mixed hypergraphs has been well studied by several researchers such as Kràl and Tuza [5, 6, 12, 13]. Here, we found the values in the chromatic spectrum for any $(G; H)$-good or $(G; H)$-proper colorings when G is some non-isomorphic 2-trees, which are triangulated graphs, and H is a triangle. A comparative analysis of these values is presented in our effort to establish some bounds. In the process, we found that when G is a fan, r_2 follows a shifted Fibonacci recurrence. If we denote the falling factorial by $\lambda^{\underline{i}} = \lambda(\lambda-1)(\lambda-2)\ldots(\lambda-i+1)$, then the *(chromatic) polynomial* $P(G; H, \lambda) = P(G; H) = \sum_{i=1}^{n} r_i \lambda^{\underline{i}}$, counts the number of colorings given some constraint on H, using at most λ colors. This polynomial is well known in the case of vertex colorings of graphs with a forbidden monochrome subgraph $H \in \{K_2, \overline{K_t}\}$ [4, 7, 15]. In this paper, we also presented this polynomial for k-trees with forbidden monochrome or rainbow K_t for all $t \geq 3$. Here, the Stirling number of the second kind is denoted by $\left\{ {n \atop k} \right\}$; it counts the number of partitions of a set of n elements into k nonempty subsets. See Table 4 for some of its values. These notations and other combinatorial identities can be found in [10]. In Appendix, we present some arrays of the values of the parameters involved in this article; the zero entries are omitted in each table.

2 Chromatic polynomial of some k-trees

As a generalization of a tree, a *k-tree* is a graph which arises from a k-clique by 0 or more iterations of adding n new vertices, each joined to a k-clique in the old graph; This process generates several non-isomorphic k-trees. Figure 1 shows two non-isomorphic 2-trees on 6 vertices. K-trees, when $k \geq 2$, are shown to be useful in constructing reliable network in [9]. Here, we denote by T_k^n, a k-tree on $n+k$ vertices which is obtained from a k-clique S, by repeatedly adding n new vertices and making them adjacent to all the vertices of S. When $k = 2$, this particular 2-tree is also known as an $(n+1)$-bridge $\theta(1, 2, \ldots, 2)$. See Figure 1(B) when $n = 4$.



(A) Fan graph, F^4 (B) 5-bridge graph, T_2^4

Figure 1: Two non-isomorphic 2-trees with a unique $(F^4; K_3)$-proper 4-coloring and a unique $(T_2^4; K_3)$-good 5-coloring

Theorem 2.1. *Suppose T_k^n is a k-tree on $n+2$ vertices. The number of its $(T_k^n; K_{k+1})$-good colorings is*

$$P(T_k^n; K_{k+1}) = \lambda(\lambda^k - 1)^n + \lambda^{\underline{\binom{k}{2}}}(\lambda^k - (\lambda - \binom{k}{2})^{\underline{k}})^n + (\lambda^{\binom{k}{2}} - \lambda^{\underline{\binom{k}{2}}} - \lambda)\lambda^{nk}.$$

Proof. Given any coloring of T_k^n, one of the following is true:

(i) S is monochromatic, giving λ colorings. For each such coloring, there are $\lambda^k - 1$ ways to color the remaining k edges, that arise from each of the n vertices added, giving the first term.

(ii) S is rainbow, giving $\lambda^{\underline{|S|}}$ colorings. For each such coloring, there are $\lambda^k - (\lambda - |S|)^{\underline{k}}$ ways to color the remaining k edges of each of the $n(k+1)$ cliques, giving the second term.

(iii) S is neither monochromatic nor rainbow, giving $\lambda^{|S|} - \lambda^{\underline{|S|}} - \lambda$ colorings. For each such coloring, there are λ^k ways to color the remaining edges of each added vertex, giving the last term. The result follows from the fact that $|S| = \binom{k}{2}$. ☐

Using a similar argument as in the proof of Theorem 2.1 when $|S| = k$, gives

Theorem 2.2. *Suppose T_k^n is a k-tree on $n+2$ vertices. The number of its $(T_k^n; K_{k+1})$-proper colorings is given by*

$$P(T_k^n; K_{k+1}) = \lambda(\lambda-1)^n + \lambda^{\underline{k}} k^n + (\lambda^k - \lambda^{\underline{k}} - \lambda)\lambda^n.$$

Remark 1: Following the argument for the proof of Theorem 2.1, we can deduce that the number of $(G; K_3)$-good colorings for any 2-tree G is the same; this gives rise to equivalent chromatic spectral values. However, in the next section, we show that this is not the case for $(G; K_3)$-proper colorings, when G is some 2-tree.

3 Chromatic spectra of (monochrome and rainbow)-triangle free 2-trees

Here, we find and compare the values in the chromatic spectrum of some 2-trees. The next proposition is instrumental in expressing several formulas in the previous section into a falling factorial form, giving the chromatic spectral values.

Proposition 3.1. *The equality* $\lambda(\lambda-1)^n = \sum_{k=2}^{n+1} \Big[\sum_{s=0}^{n-k+1} (-1)^s \binom{n}{s} \left\{ {n+1-s \atop k} \right\} \Big] \lambda^{\underline{k}}$ *holds for all $n \geq 1$.*

Proof. Clearly,

$$\begin{aligned}
\lambda(\lambda-1)^n &= \sum_{i=0}^{n} (-1)^i \binom{n}{i} \lambda^{n+1-i} \\
&= \sum_{i=0}^{n} (-1)^i \binom{n}{i} [\sum_{k=1}^{n+1-i} \left\{ {n+1-i \atop k} \right\} \lambda^{\underline{k}}] \\
&= \sum_{k=1}^{n+1} \Big[\sum_{s=0}^{n-k+1} (-1)^s \binom{n}{s} \left\{ {n+1-s \atop k} \right\} \Big] \lambda^{\underline{k}} \\
&= \sum_{s=0}^{n} (-1)^s \binom{n}{s} \left\{ {n+1-s \atop 1} \right\} \lambda^{\underline{1}} \qquad (1) \\
&+ \sum_{k=2}^{n+1} \Big[\sum_{s=0}^{n-k+1} (-1)^s \binom{n}{s} \left\{ {n+1-s \atop k} \right\} \Big] \lambda^{\underline{k}}. \qquad (2)
\end{aligned}$$

The result follows from the fact that (1) is equal to zero. □

Corollary 3.1. *If G is a 2-tree on $n+2$ vertices then the chromatic spectrum of its $(G; K_3)$-good coloring is $(r_2, \ldots, r_k, \ldots, r_{n+1})$,*
where $r_k = 3^n\Big(\sum_{i=0}^{n-k+1}(-1)^i\binom{n}{i}\left\{{n+1-i \atop k}\right\}\Big)$, $k = 2, \ldots, n+1$.

Proof. The result follows from Theorem 2.1 when $k = 2$, Remark 1 and Proposition 3.1. □

Here is the analogous result in a $(G; K_3)$-proper coloring when G is the specific 2-tree T_2^n which we described in the previous section.

Corollary 3.2. *If $G = \theta(1, 2, \ldots, 2)$, a 2-tree on $n+2$ vertices, then the chromatic spectrum of its $(G; K_3)$-proper coloring is $(r'_2, \ldots, r'_k, \ldots, r'_{n+1})$, where*

$$r'_k = \begin{cases} \sum_{i=0}^{n-k+1}(-1)^i\binom{n}{i}\left\{{n+1-i \atop k}\right\} & \text{if } k \geq 3 \\ 2^n + 1 & \text{otherwise.} \end{cases}$$

Proof. Since $G = T_2^n$, a 2-tree, it follows from Theorem 2.2 (when $k = 2$) that $P(G; K_3) = P(T_2^n; K_3) = \lambda(\lambda-1)^n + 2^n\lambda^{\underline{2}}$, to which we then apply Proposition 3.1. Also, observe that from (2) when $k = 2$,

$$\sum_{i=0}^{n-1}(-1)^i\binom{n}{i}\left\{{n+1-i \atop 2}\right\} = 1,$$

given the second statement. □

Now, we take a closer look at another well-known 2-tree. Construct a graph G as follows: start with a triangle $\{w_1, w_2, u_1\}$, and iteratively add $n-1$ new vertices, such that each additional vertex u_i is adjacent to the pair $\{u_1, u_{i-1}\}$, for $i = 3, \ldots, n+1$, and u_2 is adjacent to the pair $\{u_1, w_2\}$. G is often called a *fan* on and we denote it by F^n; see Figure 1(A) for an example when $n = 4$. Further, from the construction, it is clear that F^n is also a 2-tree. Here, we color the vertices of F^n, and recursively count the number of $(F^n; K_3)$-proper colorings. To help illustrate this recursion, we present the next example.

Example 3.1. *Chromatic spectrum of an $(F^4; K_3)$-proper coloring.*

Consider the fan F^4, obtained by iteratively adding $n = 4$ vertices to a base edge $\{w_1, w_2\}$ as shown in Figure 1(A). When $n = 1$, it is clear that there are exactly $2\lambda^{\underline{2}} + \lambda^{\underline{2}}$ ways to color the vertices of the triangle $\{w_1, w_2, u_1\}$ so that it is neither monochrome nor rainbow. The first and second terms count the cases when (a) $c(u_1) \neq c(w_2)$ and (b) $c(u_1) = c(w_2)$, respectively. When $n = 2$, from (a) it follows that for each such colorings, there are exactly two ways to color u_2; either $c(u_2) = c(u_1) \neq c(w_2)$ or $c(u_2) = c(w_2) \neq c(u_1)$. Likewise from (b), there are $\lambda - 1$ ways to color u_2 such that $c(u_2) \neq c(u_1) = c(w_2)$. Together, we have

$$P(F^2; K_3) = 2(2\lambda^{\underline{2}}) + (\lambda-1)\lambda^{\underline{2}} = \lambda^{\underline{2}}[2 + (\lambda - 1)] + 2\lambda^{\underline{2}}. \tag{3}$$

As the terms in the last expression of (3) are arranged so that the first term counts the case when $c(u_1) \neq c(u_2)$ and the last term counts the case when $c(u_1) = c(u_2)$, we can apply once again the same argument to the newly added vertex u_3. Thus, we have

$$P(F^3; K_3) = 2[\lambda^{\underline{2}}(2+(\lambda-1))] + (\lambda-1)[2\lambda^{\underline{2}}] = \lambda^{\underline{2}}[2+3(\lambda-1)] + \lambda^{\underline{2}}[2+(\lambda-1)]. \tag{4}$$

Similarly, by adding u_4 to F^3, we obtain from (4),

$$P(F^4; K_3) = \lambda^{\underline{2}}[2 + 5(\lambda-1) + (\lambda-1)^2] + \lambda^{\underline{2}}[2 + 3(\lambda-1)], \tag{5}$$

after rearranging the expression so that the first and last terms count the cases when $c(u_1) \neq c(u_4)$ and $c(u_1) = c(u_4)$, respectively. Hence,

$$P(F^4; K_3) = 4\lambda(\lambda-1) + 8\lambda(\lambda-1)^2 + 1\lambda(\lambda-1)^3. \tag{6}$$

Now apply Proposition 3.1 to each term of (6) to obtain

$$\begin{aligned}
P(F^4; K_3) &= 4[\binom{1}{0}\genfrac\{\}{0pt}{}{2}{2}]\lambda^{\underline{2}} + 8[\binom{2}{0}\genfrac\{\}{0pt}{}{3}{2} - \binom{2}{1}\genfrac\{\}{0pt}{}{2}{2}]\lambda^{\underline{2}} \\
&+ 1[\binom{3}{0}\genfrac\{\}{0pt}{}{4}{2} - \binom{3}{1}\genfrac\{\}{0pt}{}{3}{2} + \binom{3}{2}\genfrac\{\}{0pt}{}{2}{2}]\lambda^{\underline{2}} \\
&+ 8[\binom{2}{0}\genfrac\{\}{0pt}{}{3}{3}]\lambda^{\underline{3}} + 1[\binom{3}{0}\genfrac\{\}{0pt}{}{4}{3} - \binom{3}{1}\genfrac\{\}{0pt}{}{3}{3}]\lambda^{\underline{3}} + 1[\binom{4}{0}\genfrac\{\}{0pt}{}{4}{4}]\lambda^{\underline{4}} \\
&= [4 + 8(3-2) + 1(7 - 3\cdot 3 + 3)]\lambda^{\underline{2}} + [8 + 1(6-3)]\lambda^{\underline{3}} + 1[\lambda^{\underline{4}}] \\
&= 13\lambda^{\underline{2}} + 11\lambda^{\underline{3}} + 1\lambda^{\underline{4}}. \qquad (7)
\end{aligned}$$

Thus, the chromatic spectrum of any $(F^4; K_3)$-proper coloring is $(13, 11, 1)$.

To support a general recursion presented in the next theorem, we let $a_{4,0} = 2$, $a_{4,1} = 5$, $a_{4,2} = 1$, $a_{4,3} = 2$ and $a_{4,4} = 3$; Table 2 shows the values of each $a_{i,j}$ (when $n = 4$). With these coefficients we obtain directly from (5):

$$\begin{aligned}
P(F^4; K_3) &= \lambda^{\underline{2}}[a_{4,0}(\lambda-1)^0 + a_{4,1}(\lambda-1)^1 + a_{4,2}(\lambda-1)^2] \\
&+ \lambda^{\underline{2}}[a_{4,3}(\lambda-1)^0 + a_{4,4}(\lambda-1)^1] \\
&= \phi(4,0)\lambda(\lambda-1)^1 + \phi(4,1)\lambda(\lambda-1)^2 + \phi(4,2)\lambda(\lambda-1)^3, \qquad (8)
\end{aligned}$$

where

$$\phi(4,r) = \begin{cases} a_{4,r} + a_{4,3+r} & \text{if } r < 2 \\ a_{4,2} & \text{otherwise.} \end{cases}$$

We note that (8) follows from Theorem 3.1, when $n = 4$. Now, Proposition 3.1 gives

$$\begin{aligned} P(F^4; K_3) &= [\phi(4,0)(1) + \phi(4,1)(3-2) + \phi(4,2)(7 - 3\cdot 3 + 3)]\lambda^{\underline{2}} \\ &+ [\phi(4,1)(1) + \phi(4,2)(6-3)]\lambda^{\underline{3}} + [\phi(4,2)]\lambda^{\underline{4}} \\ &= [\phi(4,0)+\phi(4,1)+\phi(4,2)]\lambda^{\underline{2}} + [\phi(4,1)+3\phi(4,2)]\lambda^{\underline{3}} + \phi(4,2)\lambda^{\underline{4}}. \quad (9) \end{aligned}$$

Again, observe that (9) follows from (13) when $n = 4$. The values of $\phi(n,r)$ when $n = 11$ are recorded in Table 3, with $0 \le r \le \lfloor \frac{n}{2} \rfloor$. Thus, since $\phi(4,0) = 4$, $\phi(4,1) = 8$, $\phi(4,2) = 1$, we have

$$P(F^4; K_3) = 13\lambda^{\underline{2}} + 11\lambda^{\underline{3}} + 1\lambda^{\underline{4}}.$$

Table 1 in the Appendix shows some of the chromatic spectral values given a $(T_2^n; K_3)$-good coloring, a $(T_2^n; K_3)$-proper coloring and an $(F^n; K_3)$-proper coloring when $n = 1, \ldots, 6$. These values can be derived from Corollary 3.1, Corollary 3.2, and Corollary 3.3 respectively, for each coloring condition.

Theorem 3.1. *The number of* $(F^n; K_3)$*-proper colorings is*

$$P(F^n; K_3) = \sum_{0 \le r \le \lfloor \frac{n}{2} \rfloor} \phi(n,r)\lambda(\lambda-1)^{r+1}, \text{ where}$$

$$\phi(n,r) = \begin{cases} a_{n,r} + a_{n,\lceil \frac{n+1}{2} \rceil + r} & \text{if } r < \frac{n}{2} \\ a_{n,\frac{n}{2}} & \text{otherwise} \end{cases}$$

and the values of $a_{i,j}$ *satisfy, for* $0 \le j \le i \le n$,

(i) $a_{i,0} = 2$ *and* $a_{1,1} = 1$

(ii) for all even $i \ge 2$, $a_{i,j} = \begin{cases} a_{i-1,j} + a_{i-1,j+\lfloor \frac{i-1}{2} \rfloor} & ; 1 \le j \le \lceil \frac{i-1}{2} \rceil \\ 1 & ; j = \frac{i}{2} \\ a_{i-1,j-\lceil \frac{i+1}{2} \rceil} & ; \lceil \frac{i+1}{2} \rceil \le j \le i \end{cases}$

(iii) for all odd $i \ge 3$, $a_{i,j} = \begin{cases} a_{i-1,j} + a_{i-1,j+\lfloor \frac{i-1}{2} \rfloor} & ; 1 \le j \le \frac{i-1}{2} \\ a_{i-1,j-\lceil \frac{i}{2} \rceil} & ; \lceil \frac{i}{2} \rceil \le j \le i \end{cases}$

Proof. When $n = 1$, it follows that $P(F^1; K_3) = \phi(1,0)\lambda(\lambda-1)^1 = [a_{1,0} + a_{1,1}]\lambda(\lambda-1)^1 = 3\lambda(\lambda-1)$, since $a_{1,0} = 2$ and $a_{1,1} = 1$ by condition (i). For $n \ge 2$, at each iteration, we separate the cases when $c(u_1) \ne c(u_k)$ from when $c(u_1) = c(u_k)$. Further, we rearrange the terms of the resulting expression of $P(F^k; K_3)$ so that the first counts the colorings $c(u_1) \neq c(u_k)$, and the last counts the colorings $c(u_1) = c(u_k)$ for $k = 1, \ldots, n$. Hence, for $n \geq 1$,

$$\begin{aligned} P(F^n; K_3) &= \lambda^{\underline{2}}\Big(\sum_{1 \leq k \leq \lceil \frac{n+1}{2} \rceil} a_{n,k-1}(\lambda - 1)^{k-1} \Big) \\ &\qquad + \lambda^{\underline{2}}\Big(\sum_{1+\lceil \frac{n+1}{2} \rceil \leq k \leq n} a_{n,k-1}(\lambda - 1)^{k - \lceil \frac{n+1}{2} \rceil - 1} \Big) \\ &= \sum_{1 \leq k \leq \lceil \frac{n+1}{2} \rceil} [a_{n,k-1} + a_{n,\lceil \frac{n+1}{2} \rceil + k - 1}]\lambda(\lambda - 1)^{k+1}, \end{aligned} \tag{10}$$

where the coefficients $a_{i,j}$ are obtained recursively from items (i)–(iii). By letting $a_{i,j} = 0$ when $i < j$, it follows that

$$P(F^n; K_3) = \sum_{0 \leq r \leq \lfloor \frac{n}{2} \rfloor} \phi(n, r)\lambda(\lambda - 1)^{r+1}, \tag{11}$$

where $\phi(n, r) = \begin{cases} a_{n,r} + a_{n,\lceil \frac{n+1}{2} \rceil + r} & \text{if } r < \frac{n}{2} \\ a_{n,\frac{n}{2}} & \text{otherwise.} \end{cases}$ □

Observation 1: The previous result can be reinterpreted as follows: Let $a_{0,0} = 2$ and define an $(n + 1) \times (n + 1)$ matrix A whose entries are the coefficients $a_{i,j}$ for $0 \leq i, j \leq n$. It follows that (10) is equivalent to the equation $P = \lambda A \cdot B$, where

$$P = \begin{bmatrix} P(F^0; K_3) + \lambda(\lambda - 2) \\ P(F^1; K_3) \\ \vdots \\ P(F^n; K_3) \end{bmatrix}, \ A = \begin{bmatrix} a_{0,0} & & & \\ a_{1,0} & a_{1,1} & & \\ \vdots & \vdots & \ddots & \\ a_{n,0} & a_{n,1} & \ldots & a_{n,n} \end{bmatrix}, \ B = \left[B^1 | B^2 \right],^T$$

$$\text{with } B^1 = \left[(\lambda - 1)^1 \ldots (\lambda - 1)^{\lceil \frac{n+1}{2} \rceil} \right] \text{ and } B^2 = \left[(\lambda - 1)^1 \ldots (\lambda - 1)^{\lfloor \frac{n+1}{2} \rfloor} \right].$$

When $n = 10$, we present the entries of the lower triangular matrix A in Table 2 to help in the verification of the formula. The matrix A has several interesting properties, some of which we discuss in the next observation. For now, it is easy to see that its determinant is

$$\det(A) = \prod_{i=0}^{n} a_{i,i} = 2(\lceil \frac{n+1}{2} \rceil)!,$$

and its characteristic polynomial is given by

$$(-1)^{n+1}(x - 1)^{\lceil \frac{n}{2} \rceil}(x - 2)^2(x - 3) \ldots (x - \lceil \frac{n+1}{2} \rceil).$$

Corollary 3.3. *The values in the chromatic spectrum of any $(F^n; K_3)$-proper coloring are given by* $r''_k = \sum_{k-2 \le r \le \lfloor \frac{n}{2} \rfloor} \phi(n,r)\Big(\sum_{0 \le i \le r-k+2} (-1)^i \binom{r+1}{i} \left\{ {r+2-i \atop k} \right\} \Big)$, *for each* $k = 2, \ldots, \lceil \frac{n+1}{2} \rceil + 1$, *with*

$$\phi(n,r) = \begin{cases} a_{n,r} + a_{n,\lceil \frac{n+1}{2} \rceil + r} & \text{if } r < \lfloor \frac{n}{2} \rfloor \\ a_{n,\lfloor \frac{n}{2} \rfloor} & \text{otherwise} \end{cases}$$

Proof. For each $r = 0, \ldots, \lfloor \frac{n}{2} \rfloor$, we apply Proposition 3.1 to $P(F^n; K_3)$, giving

$$\begin{aligned} P(F^n; K_3) &= \phi(n,0)[(-1)^0 \binom{1}{0} \left\{ {2 \atop 2} \right\}]\lambda^{\underline{2}} \\ &+ \phi(n,1)[(-1)^0 \binom{2}{0} \left\{ {3 \atop 2} \right\} + (-1)^1 \binom{2}{1} \left\{ {2 \atop 2} \right\}]\lambda^{\underline{2}} \\ &\qquad + \phi(n,1)[(-1)^0 \binom{2}{0} \left\{ {3 \atop 3} \right\}]\lambda^{\underline{3}} \\ &+ \phi(n,2)[(-1)^0 \binom{3}{0} \left\{ {4 \atop 2} \right\} + (-1)^1 \binom{3}{1} \left\{ {3 \atop 2} \right\} + (-1)^2 \binom{3}{2} \left\{ {2 \atop 2} \right\}]\lambda^{\underline{2}} \\ &+ \phi(n,2)[(-1)^0 \binom{3}{0} \left\{ {4 \atop 3} \right\} + (-1)^1 \binom{3}{1} \left\{ {4 \atop 4} \right\}]\lambda^{\underline{3}} \\ &+ \phi(n,3)[(-1)^0 \binom{3}{0} \left\{ {4 \atop 4} \right\}]\lambda^{\underline{4}} \\ &\vdots \\ &+ \phi(n, \lfloor \frac{n}{2} \rfloor)[(-1)^0 \binom{\lceil \frac{n+1}{2} \rceil}{0} \left\{ {\lceil \frac{n+1}{2} \rceil + 1 \atop \lceil \frac{n+1}{2} \rceil + 1} \right\}]\lambda^{\underline{\lceil \frac{n+1}{2} \rceil + 1}}. \end{aligned} \tag{12}$$

Therefore,

$$P(F^n; K_3) = \sum_{k=2}^{\lceil \frac{n+1}{2} \rceil + 1} \Big(\sum_{r=k-2}^{\lfloor \frac{n}{2} \rfloor} \phi(n,r) \Big[\sum_{0 \le i \le r-k+2} (-1)^i \binom{r+1}{i} \left\{ {r+2-i \atop k} \right\} \Big] \Big) \lambda^{\underline{k}}, \tag{13}$$

giving the result. □

Observation 2: When $k = \lceil \frac{n+1}{2} \rceil + 1$, the last term of (13) is

$$\phi(n, \lfloor \tfrac{n}{2} \rfloor) = \begin{cases} 1 & \text{if n is even} \\ 3 + \frac{n-1}{2} & \text{otherwise} \end{cases}$$

Also, it is worth noting that when $k = 2$,

$$\sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} \phi(n,r) \Big[\sum_{0 \le i \le r} (-1)^i \binom{r+1}{i} \left\{ {r+2-i \atop 2} \right\} \Big] = \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} \phi(n,r);$$

this proceeds from the simple fact that $\sum_{i=0}^{n}(-1)^i\binom{n}{i}\left\{{n+1-i \atop 2}\right\} = 1$, for all n. Further, observe that if we define $b_i = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \phi(n,j)$ for each $i \leq n$, it follows that $b_i = \sum_j a_{i,j}$ and the sequence $\{b_n\}$ satisfies the shifted Fibonacci recurrence given by $b_0 = 2$, $b_1 = 3$ and $b_n = b_{n-1} + b_{n-2}$, for $n \geq 2$. From this observation, we determine the generating function in the next proposition.

Proposition 3.2. *The number of partitions of the $n+2$ vertices of a fan into 2 nonempty classes such that no triangle is monochrome or rainbow is given by*

$$b_n = \frac{1}{\sqrt{5}}[(2+\sqrt{5})\alpha^n - (2-\sqrt{5})\beta^n], \text{ where } \alpha = \frac{1+\sqrt{5}}{2} \text{ and } \beta = \frac{1-\sqrt{5}}{2}.$$

Proof. Let $b(x) = \sum_{n=0}^{\infty} b_n x^n$ such that $b_0 = 2$, $b_1 = 3$ and $b_n = b_{n-1} + b_{n-2}$. It follows that

$$\begin{aligned}
b(x) &= 2 + 3x + \sum_{n=2}^{\infty} b_n x^n \\
&= 2 + 3x + x\sum_{k=1}^{\infty} b_k x^k + x^2 \sum_{k=0}^{\infty} b_k x^k \\
&= 2 + 3x + x(\sum_{k=0}^{\infty} b_k x^k - 2) + x^2 \sum_{k=0}^{\infty} b_k x^k \\
&= 2 + x + xb(x) + x^2 b(x).
\end{aligned}$$

This implies that $b(x) = \frac{2+x}{1-x-x^2} = -\frac{2+x}{(x+\alpha)(x+\beta)}$, with $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$. Using a partial fraction decomposition, and subsequently the power series, we obtain

$$\begin{aligned}
b(x) &= \frac{1}{\sqrt{5}}\Big[\frac{\beta-2}{x+\beta} - \frac{\alpha-2}{x+\alpha}\Big] \\
&= \frac{1}{\sqrt{5}}\Big[\frac{\beta-2}{\beta}(\sum_{n=0}^{\infty}\alpha^n x^n) - \frac{\alpha-2}{\alpha}(\sum_{n=0}^{\infty}\beta^n x^n)\Big] \\
&= \sum_{n=0}^{\infty}\frac{1}{\sqrt{5}}\Big[\frac{\beta-2}{\beta}\alpha^n - \frac{\alpha-2}{\alpha}\beta^n\Big]x^n,
\end{aligned}$$

giving that $b_n = \frac{1}{\sqrt{5}}\Big[\frac{\beta-2}{\beta}\alpha^n - \frac{\alpha-2}{\alpha}\beta^n\Big]$. The result follows, after a simplification. $\square$

In summary, the extreme chromatic spectral values given the aforementioned colorings are clear; the lower values are, $r_2 = 3^n$, $r'_2 = 2^n + 1$, $r''_2 = b(x)$ where $b(x) = \frac{1}{\sqrt{5}}\Big[\frac{\beta-2}{\beta}\alpha^n - \frac{\alpha-2}{\alpha}\beta^n\Big]$. Also, for all $n > 1$, the upper values are also shown to be $r_{n+1} = 3^n$, $r'_{n+1} = 1$, and $r''_{\lceil\frac{n+1}{2}\rceil+1} = \begin{cases} 1 & \text{if } n \text{ is even,} \\ 3 + \frac{n-1}{2} & \text{otherwise.} \end{cases}$

4 Conclusion and future work

To the best of our knowledge, the problem of finding the exact chromatic spectral values in a $(K_n; K_t)$-good coloring remains open for all $t \geq 3$ and larger values of n; this particular problem has greatly inspired this research. When G is a 2-tree, the findings in Corollaries 3.1, 3.2, and 3.3 suggest the existence of some constant $c < 1$, such that $r^*_k = cr_k$ where r^*_k and r_k are the corresponding values in the chromatic spectra of a $(G; K_3)$-proper and a $(G; K_3)$-good coloring, respectively. For instance, $c = (\frac{1}{3})^n$ when G is an $(n+1)$-bridge. Further work is needed to determine whether the values in the chromatic spectrum of a $(G; H)$-good coloring remain upper bounds for their counterparts in a $(G; H)$-proper coloring, given any other graph G and some subgraph H.

Also, the original definition of a $(G; H)$-proper coloring can be extended to include more than one subgraph. For instance, a $(G; H_1, \ldots, H_m)$-proper coloring can be defined as the coloring of the vertices of G such that no copy of (distinct) subgraphs H_i is monochrome or rainbow, for $i = 1, \ldots, m$. As such, when $G = \mathscr{H}$ and $H_i = \overline{K}_{t_i}$, $\mathscr{H}$ is a non-uniform bihypergraph with hyperedges of sizes $t_i \geq 3$. Some related results concerning non-uniform bihypergraphs can be found in [1]. As a step in this direction for graphs, we propose the next lemma. This lemma shows that the chromatic spectral values of any $(F^n; K_3, H)$-proper coloring are identical when $H \in \{P^*_n, K_{1,n-1}, C_n, \theta(1, n_1, n_2)\}$, where P^*_n, $K_{1,n-1}$ and C_n denote respectively, an n-path that includes a fixed vertex (apex) u_1, an n-cycle, and an n-star.

Corollary 4.1. *Suppose G is a fan on $n \geq 4$ vertices. Any $(G; K_3)$-proper coloring is a $(G; H)$-proper coloring where $H \in \{P^*_s, K_{1,t}, C_r, \theta(1, n_1, n_2)\}$ with $s \geq 4$, $\lfloor\frac{n-1}{2}\rfloor \leq t \leq n-1$, $r \geq 3$, and $2 \leq n_1 \leq n_2$ such that $n_1 + n_2 \leq n$.*

Proof. Suppose G is a fan on $n \geq 4$ vertices which we can construct as follow: start with a triangle, say (u_1, u_2, u_3), and iteratively add $n-3$ new vertices such that each additional vertex u_i is adjacent to the pair (u_1, u_{i-1}), for $i = 4, \ldots, n$. Assume there is a $(G; K_3)$-proper coloring.

(i) Observe that for $s \geq 4$, every path $P^*_s \subseteq G$ contains the subgraph $u_1u_iu_{i+1}$ for some i $(2 \leq i \leq n-2)$. If some 3-path (that includes u_1) is monochrome/rainbow then the triangle (u_1, u_i, u_{i+1}) is monochrome/rainbow, violating our $(G; K_3)$-proper coloring assumption. Hence G admits a $(G; P^*_s)$-proper coloring for all $s \geq 4$.

(ii) By letting the vertices of $K_{1,t} \subseteq G$ be all the vertices of G, it follows that $t \leq n-1$. Now, consider the coloring such $c(u_1) = c(u_{2k})$ and $c(u_1) \neq c(u_{2k+1})$

for $k = 1, \ldots, \lceil \frac{n-1}{2} \rceil$. Clearly, such coloring does not violate our original coloring assumption. The lower bound of t is satisfied by letting the vertices of $K_{1,t}$ be $\{u_1, u_2\} \cup \{u_{2k+1} : k = 1, \ldots, \lceil \frac{n-1}{2} \rceil\}$, which guarantees a $(G; K_{1,t})$-proper coloring for all $t \geq \lceil \frac{n-1}{2} \rceil$.

(iii) For $r \geq 4$, since every cycle $C_r \subseteq G$ includes the apex u_1, there exists an $s \leq r$ such that $P_s^* \subseteq C_r$, with $4 \leq s \leq r \leq n$. From (i), we can conclude that there is a $(G; C_r)$-proper coloring. The case when $r = 3$ is trivial.

(iv) Likewise, since $\theta(1, n_1, n_2)$ contains $C_{1+q} \subseteq G$ with $q \in \{n_1, n_2\}$, the result follows from (iii) that, for all $2 \leq n_1 \leq n_2$ such that $n_1 + n_2 \leq n$, there is a $(G; \theta(1, n_1, n_2))$-proper coloring. □

In conclusion, it is worth noting that future work can address the coloring of the vertices/edges of a graph with either forbidden monochrome subgraphs or forbidden rainbow subgraphs (but not both). As a step in this direction, we present a simple case when coloring the elements of an n-set such that no t-subset is rainbow.

Corollary 4.2. *The chromatic spectral values in the colorings of the vertices of a complete graph K_n such that no K_t is rainbow are given by $r_k = \left\{ {n \atop k} \right\}$, for $k = 1, \ldots, t-1$.*

Note that these values also correspond to the chromatic spectral values of any complete t-uniform cohypergraph of order n; cohypergraphs are hypergraphs whose hyperedges are forbidden to be rainbow given any proper (vertex) coloring [16].

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Appendix

	$(T_2^n;K_3)$-good	$(T_2^n;K_3)$-proper	$(F^n;K_3)$-proper
$n=1$	**(3)**	**(3)**	**(3)**
$n=2$	3^2**(1,1)**	**(5,1)**	**(5,1)**
$n=3$	3^3**(1,3,1)**	**(9,3,1)**	**(8,4)**
$n=4$	3^4**(1,7,6,1)**	**(17,7,6,1)**	**(13,11,1)**
$n=5$	3^5**(1,15,25,10,1)**	**(33,15,25,10,1)**	**(27,17,5)**
$n=6$	3^6**(1,31,90,65,15,1)**	**(65,31,90,65,15,1)**	**(37,62,7,1)**

Table 1: chromatic spectral values of some $(G;K_3)$-good colorings and some $(G;K_3)$-proper colorings for $n \leq 6$

$n\backslash j$	0	1	2	3	4	5	6	7	8	9	10	11
0	**2**											
1	**2**	**1**										
2	**2**	**1**	**2**									
3	**2**	**3**	**2**	**1**								
4	**2**	**5**	**1**	**2**	**3**							
5	**2**	**7**	**4**	**2**	**5**	**1**						
6	**2**	**9**	**9**	**1**	**2**	**7**	**4**					
7	**2**	**11**	**16**	**5**	**2**	**9**	**9**	**1**				
8	**2**	**13**	**25**	**14**	**1**	**2**	**11**	**16**	**5**			
9	**2**	**15**	**36**	**30**	**6**	**2**	**13**	**25**	**14**	**1**		
10	**2**	**17**	**49**	**55**	**20**	**1**	**2**	**15**	**36**	**30**	**6**	
11	**2**	**19**	**64**	**91**	**50**	**7**	**2**	**17**	**49**	**55**	**20**	**1**

Table 2: Table of values of $a_{i,j}$, which are the entries of the matrix A when $n = 11$

$n\backslash r$	0	1	2	3	4	5
0	**2**					
1	**3**					
2	**4**	**1**				
3	**4**	**4**				
4	**4**	**8**	**1**			
5	**4**	**12**	**5**			
6	**4**	**16**	**13**	**1**		
7	**4**	**20**	**25**	**6**		
8	**4**	**24**	**41**	**19**	**1**	
9	**4**	**28**	**61**	**44**	**7**	
10	**4**	**32**	**85**	**85**	**26**	**1**
11	**4**	**36**	**113**	**146**	**70**	**8**

Table 3: Table of values of $\phi(n,r)$ when $n = 11$

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10	11
0	**1**											
1	**0**	**1**										
2	**0**	**1**	**1**									
3	**0**	**1**	**3**	**1**								
4	**0**	**1**	**7**	**6**	**1**							
5	**0**	**1**	**15**	**25**	**10**	**1**						
6	**0**	**1**	**31**	**90**	**65**	**15**	**1**					
7	**0**	**1**	**63**	**301**	**350**	**140**	**21**	**1**				
8	**0**	**1**	**127**	**966**	**1701**	**1050**	**266**	**28**	**1**			
9	**0**	**1**	**255**	**3025**	**7770**	**6951**	**2646**	**462**	**36**	**1**		
10	**0**	**1**	**511**	**9330**	**34105**	**42525**	**22827**	**5880**	**750**	**45**	**1**	
11	**0**	**1**	**1023**	**2850**	**145750**	**246730**	**179487**	**63987**	**11880**	**1155**	**55**	**1**

Table 4: Table of values of $\left\{ n \atop k \right\}$ when $n = 11$

References

[1] J. Allagan and D. Slutzky, Chromatic Polynomials of Mixed Hypergraphs, *Australas. J. Combin.* **58(1)** (2014), 197–213.

[2] M. Axenovich and J. Choi, A note on monotonicity of mixed Ramsey numbers, *Discrete Math.* **311** (2011), 2020–2023.

[3] M. Axenovich and P. Iverson, Edge-colorings avoiding rainbow and monochromatic subgraphs, *Discrete Math.* **308** (2008), no. 20, 4710–4723.

[4] M. Borowiecki and E. Łazuka, Chromatic polynomials of hypergraphs, *Discuss. Math., Graph Theory* **20 (2)** (2000), 293–301.

[5] C. Bujtàs and Zs. Tuza, Uniform mixed hypergraphs: the possible numbers of colors, *Graphs Combin.* **24** (2008), 1–12.

[6] K. Diao, K. Wang and P. Zhao, The chromatic spectrum of 3-uniform bi-hypergraphs, *Discrete Math.* **3(11)** (2011), 650–656.

[7] F.M. Dong, K.M. Koh and K.L. Teo, *Chromatic polynomials and chromaticity of graphs*, Singapore: World Scientific Publishing. xxvii, 2005.

[8] P. Erdős, M. Simonovits and V.T. Sòs, Anti-Ramsey theorems, Infinite and finite sets (Colloq., Keszthely, 1973; dedicated to P. Erdös on his 60th birthday), Vol. II, pp. 633–643. *Colloq. Math. Soc. Janos Bolyai*, Vol. 10, North-Holland, Amsterdam, 1975.

[9] A.M. Farley, Networks Immune to Isolated Failures, *Networks* **11** (1981), 225–268.

[10] R.L. Graham, D.E. Knuth and O. Patashnik, *Concrete Mathematics*, Addison–Wesley Publishing, 1994, pp. 264–265.

[11] A. Jaffe, T. Moscibroda and S. Sen, On the price of equivocation in Byzantine agreement, *Proc. 31st Principles of Distributed Computing (PODC)*, 2012.

[12] T. Jiang, D. Mubayi, Zs. Tuza, V. Voloshin and D. West, The Chromatic Spectrum of Mixed Hypergraphs, *Graphs Combin.* **18** (2002), 309–318.

[13] D. Kràl, On Feasible Sets of Mixed Hypergraphs, *Electr. J. Combin.* **11 (1)** (2004), R19.

[14] F.P. Ramsey, On a Problem of Formal Logic, *Proc. London Math. Soc.* **30** (1930), 264–286.

[15] I. Tomescu, Chromatic coefficients of linear uniform hypergraphs, *J. Combin. Theory Ser. B* **72** (1998), 229–235.

[16] V.I. Voloshin, *Coloring Mixed Hypergraphs: Theory, Algorithms and Applications*, Fields Institute Monographs, 17. Amer. Math. Soc., Providence, RI, 2002. xiv+181 pp.

[17] D. West, *Introduction to Graph Theory*, Prentice Hall, 2001.

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