AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **65(1)** (2016), Pages 1–13

Critical graphs with respect to total domination and connected domination

P. Kaemawichanurat   L. Caccetta

Western Australian Centre of Excellence in Industrial Optimisation (WACEIO)
Department of Mathematics and Statistics
Curtin University, GPO Box U1987
Perth, WA 6845
Australia
pavaton@hotmail.com   L.Caccetta@exchange.curtin.edu.au

N. Ananchuen*

Department of Mathematics, Faculty of Science
Silpakorn University
Nakorn Pathom 73000
Thailand
nawarat@su.ac.th

Abstract

A graph G is said to be k-γ_t-critical if the total domination number $\gamma_t(G) = k$ and $\gamma_t(G + uv) < k$ for every $uv \notin E(G)$. A k-γ_c-critical graph G is a graph with the connected domination number $\gamma_c(G) = k$ and $\gamma_c(G + uv) < k$ for every $uv \notin E(G)$. Further, a k-tvc graph is a graph with $\gamma_t(G) = k$ and $\gamma_t(G - v) < k$ for all $v \in V(G)$, where v is not a support vertex (i.e. all neighbors of v have degree greater than one). A 2-connected graph G is said to be k-cvc if $\gamma_c(G) = k$ and $\gamma_c(G - v) < k$ for all $v \in V(G)$. In this paper, we prove that connected k-γ_t-critical graphs and k-γ_c-critical graphs are the same if and only if $3 \leq k \leq 4$. For $k \geq 5$, we concentrate on the class of connected k-γ_t-critical graphs G with $\gamma_c(G) = k$ and the class of k-γ_c-critical graphs G with $\gamma_t(G) = k$. We show that these classes intersect but they do not need to be the same. Further, we prove that 2-connected k-tvc graphs and k-cvc graphs are the same if and only if $3 \leq k \leq 4$. Similarly, for $k \geq 5$, we focus on the class of 2-connected k-tvc graphs G with $\gamma_c(G) = k$ and the class of 2-connected k-cvc graphs G with $\gamma_t(G) = k$. We finish this paper by showing that these classes do not need to be the same.

* Also at Center of Excellence Mathematics, CHE, Si Ayutthaya Rd., Bangkok 10400, Thailand

1 Introduction

Let G be a finite simple undirected graph with a vertex set $V(G)$ and an edge set $E(G)$. Denote the complement of G by $\overline{G}$. A graph H is a *subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. An *induced subgraph* $G[H]$ of a graph G is a subgraph H for which $uv \in E(H)$ if and only if $uv \in E(G)$ where $u, v \in V(H)$. The *neighborhood* $N_G(v)$ of a vertex v in G is $\{u \in V(G)|uv \in E(G)\}$. Further, the *closed neighborhood* $N_G[v]$ of a vertex v in G is $N_G(v) \cup \{v\}$. We let $N_G(S) = \cup_{v \in S} N_G(v)$ where $S \subseteq V(G)$. The *degree* of a vertex v is $|N_G(v)|$. An *end vertex* of G is a vertex of degree one and a *support vertex* of G is a vertex which is adjacent to an end vertex. A *tree* is a connected graph with no cycle. A *star* $K_{1,n}$ is a tree containing one support vertex and n end vertices.

For subsets $D, X \subseteq V(G)$, D dominates X if every vertex of X is either in D or adjacent to a vertex of D. If D dominates X, then we write $D \succ X$. Further, if $X = V(G)$, then D is a dominating set of G and we write $D \succ G$ instead of $D \succ V(G)$. A *total dominating set* of a graph G is a subset D^t of vertices of G such that every vertex of G is adjacent to some vertex of D^t. The *total domination number* $\gamma_t(G)$ of G is the minimum cardinality of a total dominating set. Note that $\gamma_t(G) \geq 2$ and every vertex in $V(G)$ is totally dominated by D^t. If D^t totally dominates G, then we write $D^t \succ_t G$. A smallest total dominating set of a graph G is called a γ_t-set of a graph G. A *connected dominating set* of a graph G is a dominating set D^c of G such that $G[D^c]$ is connected. If D^c is a connected dominating set of G, we then write $D^c \succ_c G$. The minimum cardinality of a connected dominating set of G is called the *connected domination number* of G and is denoted by $\gamma_c(G)$. A smallest connected dominating set of a graph G is called a γ_c-set of a graph G. Note that if S is a γ_c-set of G and $|S| \geq 2$, then S is also a total dominating set of G. Thus $\gamma_t(G) \leq \gamma_c(G)$ when $\gamma_c(G) \geq 2$.

A graph G is said to be *k-total domination edge critical*, or k-γ_t-critical, if $\gamma_t(G) = k$ and for every $uv \notin E(G), \gamma_t(G + uv) < k$. A graph G is said to be *k-connected domination edge critical*, or k-γ_c-critical, if $\gamma_c(G) = k$ and for every $uv \notin E(G), \gamma_c(G + uv) < k$.

In the context of vertex removal, a graph G is said to be *k-total domination vertex critical*, or k-tvc, if $\gamma_t(G) = k$ and for every vertex which is not a support vertex $v \in V(G), \gamma_t(G - v) < k$. A graph G is said to be *k-connected domination vertex critical*, or k-cvc if $\gamma_c(G) = k$ and for every vertex $v \in V(G), \gamma_c(G - v) < k$. It is easy to see that a disconnected graph cannot contain a connected dominating set. Thus, we may assume that all graphs are connected in the study on k-γ_c-critical graphs. Moreover, we assume also that all graphs are 2-connected in the study on k-cvc graphs.

The study on total domination critical graphs was started by van der Merwe et al. [9] and continued by a number of researchers (for example, Goddard et al. [4], Henning and van der Merwe [6] and van der Merwe and Loizeaux [8]).

The connected domination critical graphs was introduced by Chen et al. [3] and continued in Ananchuen [1] and Kaemawichanurat and Ananchuen [7]. Chen et al. [3] completely characterized 2-γ_c-critical graphs and gave many properties of 3-γ_c-critical graphs. Kaemawichanurat and Ananchuen [7] gave a characterization of 4-γ_c-critical graphs with cut vertices and proved that such graphs contain a perfect matching.

Chen et al. [3] showed that a graph G is 2-γ_c-critical if and only if $\overline{G} = \cup_{i=1}^{n} K_{1,n_i}$ for $n_i \geq 1$ and $n \geq 2$. Henning and van der Merwe [6] established that a graph G is 2-γ_t-critical if and only if G is a complete graph. Ananchuen [1] noted that 3-γ_c-critical graphs and 3-γ_t-critical graphs are the same. The problem that arises is whether there is a $k \geq 4$ such that the class of k-γ_c-critical graphs and the class of connected k-γ_t-critical graphs are the same.

In this paper, we show, in Section 3, that a connected graph G is 4-γ_c-critical if and only if it is 4-γ_t-critical. For $k \geq 5$, there exists a k-γ_c-critical graph which is not k-γ_t-critical. For example, Chen et al. [3] showed that C_n is an $(n-2)$-γ_c-critical graph while Goddard et al. [4] referred from Henning [5] that $\gamma_t(C_n) = \lfloor \frac{n}{2} \rfloor + \lceil \frac{n}{4} \rceil - \lfloor \frac{n}{4} \rfloor$ which is less than $n-2$ for $n \geq 7$. Clearly, C_n is not an $(n-2)$-γ_t-critical graph. We then concentrate on the class $\mathbb{G}_k$ of graphs G such that $\gamma_c(G) = \gamma_t(G) = k$ and let

$\mathbb{T}_k^e$: class of connected k-γ_t-critical graphs G with $G \in \mathbb{G}_k$ and,
$\mathbb{C}_k^e$: class of connected k-γ_c-critical graphs G with $G \in \mathbb{G}_k$.

We show that $\mathbb{T}_k^e \neq \mathbb{C}_k^e$. We finish this section by showing that $\mathbb{T}_k^e \cap \mathbb{C}_k^e \neq \emptyset$.

For vertex removal, Ananchuen et al. [2] noted that 2-connected 3-tvc graphs and 2-connected 3-cvc graphs are the same. We might ask similarly whether there is a $k \geq 4$ such that 2-connected k-cvc graphs and 2-connected k-tvc graphs are the same. Our results in Section 4 show that a 2-connected graph G is 4-cvc if and only if it is 4-tvc. Similarly, for $k \geq 5$, we focus on the class $\mathbb{G}_k$ and let

$\mathbb{T}_k^v$: class of 2-connected k-tvc graphs G with $G \in \mathbb{G}_k$ and,
$\mathbb{C}_k^v$: class of 2-connected k-cvc graphs G with $G \in \mathbb{G}_k$.

We prove that $\mathbb{T}_k^v \neq \mathbb{C}_k^v$.

2 Preliminary results

In this section, we state some results that we use in establishing our results in the next two sections. In what follows, for a pair of non-adjacent vertices u and v of G, D_{uv}^t and D_{uv}^c denote a γ_t-set of $G + uv$ and a γ_c-set of $G + uv$, respectively. Further, for a vertex v of G, D_v^t and D_v^c denote a γ_t-set of $G - v$ and a γ_c-set of $G - v$, respectively. Van der Merwe et al. [8] and [9] established fundamental properties of 4-γ_t-critical graphs described in the following propositions.

Proposition 2.1. *[8] Let G be a 4-γ_t-critical graph and let u and v be a pair of non-adjacent vertices of G. Then either*

(1) $\{u, v\} \succ G$, or

(2) for either u or v, without loss of generality, say u, $\{w, u, v\} \succ G$ for some $w \in N_G(u)$ and $w \notin N_G(v)$, or

(3) for either u or v, without loss of generality, say u, $\{x, y, u\} \succ G - v$ and $G[\{x, y, u\}]$ is connected.

Proposition 2.2. *[9] For any graph G with $\gamma_t(G) = 3$ and a γ_t-set D^t, either $G[D^t] = P_3$ or $G[D^t] = K_3$.*

Goddard et al. [4] provided some results on k-tvc graphs.

Lemma 2.3. *[4] Let G be a k-tvc graph and $v \in V(G)$. Then*

(1) $D^t_v \cap N_G[v] = \emptyset$,

(2) $|D^t_v| = k - 1$.

On connected domination critical graphs, Chen et al. [3] established the following result for k-γ_c-critical graphs.

Lemma 2.4. *[3] Let G be a k-γ_c-critical graph and let u and v be a pair of non-adjacent vertices of G. Then*

(1) $k - 2 \leq |D^c_{uv}| \leq k - 1$,

(2) $D^c_{uv} \cap \{u, v\} \neq \emptyset$.

In the concept of vertex deletion, Ananchuen et al. [2] provided some properties of k-cvc graphs as follows.

Lemma 2.5. *[2] Let G be a k-cvc graph and $v \in V(G)$. Then*

(1) $D^c_v \cap N_G[v] = \emptyset$,

(2) $|D^c_v| = k - 1$.

3 Edge critical graphs

In this section, we show that connected k-γ_t-critical graphs and k-γ_c-critical graphs are the same if and only if $3 \leq k \leq 4$. We first establish the following theorem.

Theorem 3.1. *Let G be a connected graph. Then G is a 4-γ_t-critical graph if and only if G is a 4-γ_c-critical graph.*

Proof. Suppose that G is a 4-γ_c-critical graph. Thus $\gamma_t(G) \leq \gamma_c(G) = 4$. Suppose that $\gamma_t(G) < 4$. Hence, there exists a γ_t-set D^t of G of size less than 4. Because $|D^t| < 4$, $G[D^t]$ is connected by Proposition 2.2. Therefore, D^t is a connected dominating set of G of size less than 4, a contradiction. Hence, $\gamma_t(G) = 4$.

Consider $G + uv$ for $uv \notin E(G)$. Because G is 4-γ_c-critical, there exists by Lemma 2.4(1) a γ_c-set D^c_{uv} of $G + uv$ with $|D^c_{uv}| < 4$. Clearly, D^c_{uv} is a total dominating set of $G + uv$. Thus $\gamma_t(G + uv) \leq |D^c_{uv}| = \gamma_c(G + uv) < \gamma_c(G) = \gamma_t(G)$. Hence, G is 4-γ_t-critical.

Conversely, suppose G is a 4-γ_t-critical graph. We first show that $\gamma_c(G) = 4$.

Claim : There exists a connected dominating set of size 4 of G.

Consider $G+uv$ for $uv \notin E(G)$. Let D^t_{uv} be a γ_t-set of $G+uv$. Because $|D^t_{uv}| < 4$, $(G + uv)[D^t_{uv}]$ is connected. Therefore, $D^t_{uv} \succ_c G + uv$. We distinguish 2 cases.

Case 1 : $|D^t_{uv} \cap \{u, v\}| = 1$.

By Proposition 2.1(3), $|D^t_{uv}| = 3$. We may suppose without loss of generality that $D^t_{uv} \cap \{u, v\} = \{v\}$. Since $D^t_{uv} \succ_c G + uv$ and G is connected, it follows that there exists $w \in V(G) - D^t_{uv}$ such that $wu \in E(G)$ and w must be adjacent to at least one vertex in D^t_{uv}. Because $|D^t_{uv}| = 3$, $D^t_{uv} \cup \{w\}$ is a connected dominating set of size 4 of G.

Case 2 : $|D^t_{uv} \cap \{u, v\}| = 2$.

We then distinguish 2 subcases according to Proposition 2.1(1) and (2).

Subcase 2.1 : $D^t_{uv} = \{u, v\}$.

If there is $w \in N_G(u) \cap N_G(v)$, then $\{u, v, w\}$ is a total dominating set of size 3 of G, a contradiction. Hence, $N_G(u) \cap N_G(v) = \emptyset$. Because G is connected and $\{u, v\} \succ G$, there exist x, y such that $x \in N_G(u), y \in N_G(v)$ and $xy \in E(G)$. Thus $\{u, v, x, y\}$ is a connected dominating set of size 4 of G.

Subcase 2.2 : $D^t_{uv} = \{u, v, z\}$ for some $z \in V(G)$.

Thus z is adjacent to exactly one of u or v, say v. If there is $y \in N_G(\{z, v\}) \cap N_G(u)$, then $\{u, v, y, z\}$ is a connected dominating set of size 4 of G. Suppose that $N_G(\{z, v\}) \cap N_G(u) = \emptyset$. We partition set $V(G) - \{u, v, z\}$ as $A_1 = N_G(u)$ and $A_2 = N_G(\{v, z\})$. If $v \succ A_2$, then $\{u, v\} \succ G + uv$. This contradicts the fact that $D^t_{uv} = \{u, v, z\}$ is a smallest total dominating set of $G + uv$. Hence, there is $w \in A_2$ such that $zw \in E(G)$ but $vw \notin E(G)$. Consider $G + vw$. If $|D^t_{vw} \cap \{v, w\}| = 1$, then, by similar arguments as in the proof of Case 1, G contains a connected dominating set of size 4. Thus, we now suppose $|D^t_{uv} \cap \{v, w\}| = 2$. If $D^t_{vw} = \{v, w\}$, then no vertex in D^t_{vw} dominates u because $w \in A_2$ and $A_1 \cap A_2 = \emptyset$, a contradiction. Therefore, $D^t_{vw} = \{a, v, w\}$ for some $a \in V(G)$. In fact $a \in A_1$. Thus a is adjacent to w because $A_1 \cap A_2 = \emptyset$. Since $vz, wz \in E(G)$, $\{a, v, w, z\}$ is a connected dominating set of size 4 of G and we settle our claim.

If $\gamma_c(G) < 4$, then $\gamma_t(G) \leq \gamma_c(G) < 4$, a contradiction. Hence, $\gamma_c(G) = 4$.

We finally prove the criticality by considering $G + uv$ for $uv \notin E(G)$. Because G is 4-γ_t-critical, there exists a γ_t-set D^t_{uv} of size less than 4 of $G+uv$. Since $|D^t_{uv}| < 4$, $(G + uv)[D^t_{uv}]$ is connected by Proposition 2.2. Thus $D^t_{uv} \succ_c G + uv$. Therefore, $\gamma_c(G + uv) \leq |D^t_{uv}| < 4 = \gamma_c(G)$. This completes the proof of our theorem. □

By Theorem 3.1, we have $\mathbb{T}^e_4 = \mathbb{C}^e_4$. We next show that $\mathbb{T}^e_k \neq \mathbb{C}^e_k$ for $k \geq 5$.

Theorem 3.2. *$\mathbb{T}^e_k \neq \mathbb{C}^e_k$ when $k \geq 5$.*

Proof. We prove the theorem by providing a graph $G \in \mathbb{T}^e_k/\mathbb{C}^e_k$ when $k \geq 5$. We distinguish our proof by the parity of k.

Case 1 : k is even.

Let $k = 2q$ for some positive integer $q \geq 3$. Construct the graph G from q different paths of length 2, say $P^i = x_1^i x_2^i x_3^i$ for $i = 1, \ldots, q$ and then forms a clique on $\{x_1^i | 1 \leq i \leq q\}$ (see Figure 1(a)).

We first show that $\gamma_t(G) = \gamma_c(G) = k = 2q$. Note that $\{x_1^i, x_2^i | 1 \leq i \leq q\} \succ_c G$. Hence, $\gamma_c(G) \leq 2q$. For $i = 1, .., q$, we need at least two vertices to totally dominate each of the P^i, implying that $\gamma_t(G) \geq 2q$. Therefore, $2q \leq \gamma_t(G)$. Thus $2q \leq \gamma_t(G) \leq \gamma_c(G) \leq 2q$. Hence, $\gamma_t(G) = \gamma_c(G) = 2q$.

We next consider the total domination number of $G + uv$ where $uv \notin E(G)$. If $\{u, v\} = \{x_m^i, x_p^j\}$ where $i \neq j$ and $2 \leq m, p \leq 3$, then $\{x_m^i, x_p^j\} \cup \{x_1^l, x_2^l | l \neq i, j\} \succ_t G + uv$. Hence, $\gamma_t(G + uv) \leq 2q - 2 < \gamma_t(G)$. If $\{u, v\} = \{x_1^i, x_p^j\}$ where $i \neq j$ and $p \in \{2, 3\}$, then $\{x_1^i, x_2^i, x_p^j\} \cup \{x_1^l, x_2^l | l \neq i, j\} \succ_t G + uv$. Hence, $\gamma_t(G + uv) \leq 2q - 1 < \gamma_t(G)$. Finally, if $\{u, v\} = \{x_1^i, x_3^i\}$, then $\{x_1^i\} \cup \{x_1^l, x_2^l | l \neq i\} \succ_t G + uv$. Thus $\gamma_t(G + uv) = 2q - 1 < \gamma_t(G)$. Therefore, G is k-γ_t-critical and $G \in \mathbb{T}_k^e$.

We then consider the connected domination number of $G + uv$. If $\{u, v\} = \{x_3^1, x_3^2\}$, then by Lemma 2.4(2), $D_{uv}^c \cap \{x_3^1, x_3^2\} \neq \emptyset$. Without loss of generality, we may suppose $x_3^1 \in D_{uv}^c$. Since $(G + uv)[D_{uv}^c]$ is connected, we need at least 2 vertices x_1^i, x_2^i to dominate P^i for $i \neq 1, 2$. If $x_3^2 \in D_{uv}^c$, then $x_1^2, x_2^2 \in D_{uv}^c$ or $x_1^1, x_2^1 \in D_{uv}^c$ by the connectedness of $(G+uv)[D_{uv}^c]$. Therefore $|D_{uv}^c| \geq 2q = k$. Thus G is not critical. Then $x_3^2 \notin D_{uv}^c$ and thus $x_1^1, x_2^1, x_3^1 \in D_{uv}^c$ by the connectedness of $(G + uv)[D_{uv}^c]$. Further, $x_1^2 \in D_{uv}^c$ to dominate x_2^2. Therefore, $|D_{uv}^c| \geq 2q = k$ and G is not a k-γ_c-critical graph. Thus $G \notin \mathbb{C}_k^e$.

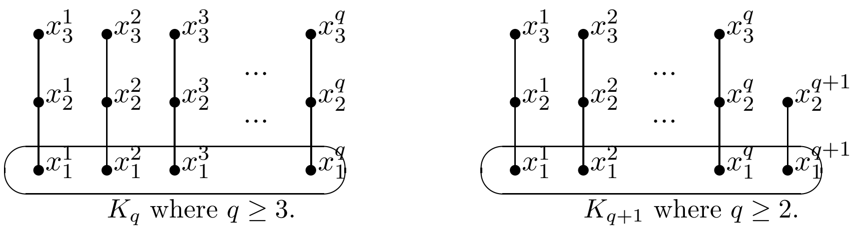

Figure 1(a) **Figure 1(b)**

Case 2 : k is odd.

Let $k = 2q + 1$ for some positive integer $q \geq 2$. Constructed the graph G from q different paths of length 2, say $P^i = x_1^i x_2^i x_3^i$ for $i = 1, \ldots, q$ and a path of length 1, say $P^{q+1} = x_1^{q+1} x_2^{q+1}$ and then forms a clique on $\{x_1^i | 1 \leq i \leq q + 1\}$ (see Figure 1(b)).

By similar arguments as in Case 1, we have $\gamma_t(G) = \gamma_c(G) = 2q + 1$. To show the criticality of $G + uv$ where $uv \notin E(G)$, we can apply similar arguments as in the proof of Case 1 when $\{u, v\} \subseteq \{x_l^i | 1 \leq i \leq q, 1 \leq l \leq 3\}$. We now suppose that $\{u, v\} \cap V(P^{q+1}) \neq \emptyset$. Because $|V(P^{q+1})| = 2$, $|\{u, v\} \cap V(P^{q+1})| = 1$. Without loss of generality, assume that $u \in V(P^{q+1})$ and $v \in V(P^j)$ for some $j \in \{1, \ldots, q\}$. If $u \in \{x_1^{q+1}, x_2^{q+1}\}$ and $v \in \{x_2^j, x_3^j\}$, then $\{u, v\} \cup \{x_1^l, x_2^l | l \neq j, q+1\} \succ_t G+uv$. Thus $\gamma_t(G + uv) \leq 2q \leq \gamma_t(G)$. Finally if $u = x_2^{q+1}$ and $v = x_1^j$, then $\{x_1^l, x_2^l | l \neq q + 1\} \succ_t$

$G+uv$. Therefore, $\gamma_t(G+uv) \le 2q < \gamma_t(G)$ and $G \in \mathbb{T}_k^e$. By considering $G + x_3^1 x_3^2$, we can show that a graph G is not a k-γ_c-critical graph by similar arguments as in Case 1.

Hence, $G \in \mathbb{T}_k^e$ but $G \notin \mathbb{C}_k^e$. Therefore, $\mathbb{T}_k^e \neq \mathbb{C}_k^e$ when $k \ge 5$. This completes the proof of our theorem. □

Chen et al. [3] characterized that a graph G is 2-γ_c-critical if and only if $\overline{G} = \cup_{i=1}^n K_{1,n_i}$ for $n_i \ge 1$ and $n \ge 2$ while Henning and van der Merwe [6] proved that a graph G is 2-γ_t-critical if and only if G is a complete graph. Thus $\mathbb{T}_2^e \neq \mathbb{C}_2^e$. Ananchuen [1] pointed out that 3-γ_t-critical graphs and 3-γ_c-critical graphs are the same. That is $\mathbb{T}_3^e = \mathbb{C}_3^e$. By Theorems 3.1 and 3.2, we have the following corollary.

Corollary 3.3. $\mathbb{T}_k^e = \mathbb{C}_k^e$ *if and only if* $3 \le k \le 4$.

Our next result shows that there exists a graph belonging to $\mathbb{T}_k^e$ and $\mathbb{C}_k^e$.

Theorem 3.4. *For* $k \ge 5$, $\mathbb{T}_k^e \cap \mathbb{C}_k^e \neq \emptyset$.

Proof. Let $G \in \mathbb{C}_k^e$. For all $uv \notin E(G)$ and a γ_c-set D_{uv}^c of $G+uv$, we have D_{uv}^c is also a total dominating set of $G+uv$. Since G is a k-γ_c-critical graph and $\gamma_t(G) = k$, it follows that $\gamma_t(G+uv) \le |D_{uv}^c| < k = \gamma_t(G)$. Therefore, $G \in \mathbb{T}_k^e$ and $\mathbb{C}_k^e \subseteq \mathbb{T}_k^e$. To prove the theorem, it suffices to establish a graph G in the class $\mathbb{C}_k^e$. We distinguish 2 cases according to the parity of k.

Case 1 : k is even.

Let $k = 2m$ for some positive integer $m \ge 3$. For $1 \le i \le k$, let K_{n_i} be a complete graph of order n_i and K_k a complete graph of order k where $V(K_k) = \{x_1, x_2, \ldots, x_k\}$. Then we join every vertex in $V(K_{n_{2i}})$ to every vertex in $V(K_{n_{2i-1}})$ for $1 \le i \le m$. Further, we join x_i to every vertex in K_{n_i} for $1 \le i \le 2m$. Finally, for $1 \le i \le m$, we join x_{2i} to every vertex in $V(K_{n_{2i-1}})$ except one vertex, say u_{2i-1}, and join x_{2i-1} to every vertex in $V(K_{n_{2i}})$ except one vertex, say u_{2i} (see Figure 2(a)).

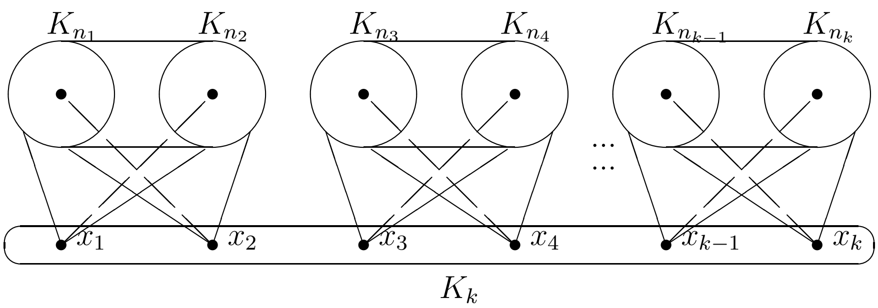

Figure 2(a)

We next show that a graph $G \in \mathbb{C}_k^e$. Clearly, $\{x_1, x_2, \ldots, x_k\} \succ_c G$. Thus $\gamma_t(G) \le \gamma_c(G) \le k$. By the construction, we need at least 2 vertices to totally dominate $K_{n_{2i}} \cup K_{n_{2i-1}}$ for $1 \le i \le m$. It follows that $\gamma_t(G) \ge k$. Hence, $k \le \gamma_t(G) \le \gamma_c(G) \le k$. Therefore, $\gamma_c(G) = \gamma_t(G) = k$.

For establishing the criticality, we consider $G+uv$ where $uv \notin E(G)$. If $\{u,v\} = \{x_{2i}, u_{2i-1}\}$, then $D^c_{uv} = \{x_i \mid i = 1,2,\ldots,k\} - \{x_{2i-1}\}$. Similarly, if $\{u,v\} = \{x_{2i-1}, u_{2i}\}$, then i $D^c_{uv} = \{x_i \mid i = 1,2,\ldots,k\} - \{x_{2i}\}$. If $\{u,v\} = \{x_{2i}, q\}$ when q is any vertex in $K_{n_{2j-1}}$ or $K_{n_{2j}}$ for $1 \le i \ne j \le m$, then $D^c_{uv} = (\{x_i \mid i = 1,2,\ldots,k\} \cup \{q\}) - \{x_{2j}, x_{2j-1}\}$. We can show that $\gamma_c(G) < k$ when $\{u,v\} = \{x_{2i-1}, q\}$ such that q is a vertex in $K_{n_{2j-1}}$ or $K_{n_{2j}}$ for $1 \le i \ne j \le m$ by a similar argument. Further, if $\{u,v\} = \{p,q\}$ when $p \in V(K_{n_{2i}})$ and $q \in V(K_{n_{2j}})$ for $1 \le i \ne j \le m$, we have $D^c_{uv} = (\{x_i | i = 1,2,\ldots,k\} \cup \{p,q\}) - \{x_{2i-1}, x_{2j}, x_{2j-1}\}$. Moreover, when $p \in V(K_{n_{2i}})$ and $q \in V(K_{n_{2j-1}})$ or $p \in V(K_{n_{2i-1}})$ and $q \in V(K_{n_{2j}})$ or $p \in V(K_{n_{2i-1}})$ and $q \in V(K_{n_{2j-1}})$ for $1 \le i \ne j \le m$, we can prove the criticality by similar arguments. Therefore, $G \in \mathbb{C}^e_k$.

Case 2 : k is odd.

Let $k = 2m+1$ for some positive integer $m \ge 2$. For $1 \le i \le k-1$, let K_{n_i} be a complete graph of order n_i, $K_{n_k} = K_1$ and K_k a complete graph of order k such that $V(K_k) = \{x_1, x_2, \ldots, x_k\}$. Then we join every vertex in $V(K_{n_{2i}})$ to every vertex in $V(K_{n_{2i-1}})$ for $1 \le i \le m$. Further, we join x_i to every vertex in K_{n_i} for $1 \le i \le 2m+1$. Finally, for $1 \le i \le m$, we join x_{2i} to every vertex in $V(K_{n_{2i-1}})$ except one vertex and x_{2i-1} to every vertex in $V(K_{n_{2i}})$ except one vertex (see Figure 2(b)). It is worth noting that, in these two constructions of Cases 1 and 2, the graphs $G \in \mathbb{T}^e_k \cap \mathbb{C}^e_k$ when $n_i = 1$ for $1 \le i \le k$ were found earlier by Henning and van der Merwe [6].

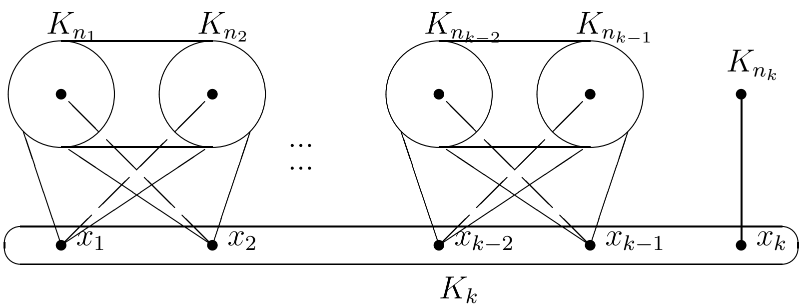

Figure 2(b)

We can show that $\gamma_c(G) = k$ by similar arguments as in Case 1. We then show the criticality of G. Let $\{a\} = V(K_{n_k})$. Consider $G+uv$ where $uv \notin E(G)$. If $\{u,v\} \subseteq \cup_{i=1}^{k-1}(V(K_{n_i}) \cup \{x_i\})$, we then establish the criticality by similar arguments as k is even. We now consider when $\{u,v\} \cap \{a, x_k\} \ne \emptyset$. If $\{u,v\} = \{x_k, p\}$ for some $p \in V(K_{n_{2i}})$ or $p \in V(K_{n_{2i-1}}), i = 1,2,\ldots,m$, then $D^c_{uv} = (\{x_i | i = 1,2,\ldots,k\} \cup \{p\}) - \{x_{2i}, x_{2i-1}\}$. If $\{u,v\} = \{a,p\}$ for some $p \in V(K_{n_{2i}})$ or $p \in V(K_{n_{2i-1}}), i = 1,2,\ldots,m$, then $D^c_{uv} = (\{x_i | i = 1,2,\ldots,k\} \cup \{p\}) - \{x_{2i-1}, x_k\}$ or $D^c_{uv} = (\{x_i | i = 1,2,\ldots,k\} \cup \{p\}) - \{x_{2i}, x_k\}$, respectively. Finally, if $\{u,v\} = \{a, x_i\}$ for $1 \le i \le k-1$, then $D^c_{uv} = \{x_i | i = 1,2,\ldots,k-1\}$. In either case, $\gamma_c(G+uv) < k$. Therefore, $G \in \mathbb{C}^e_k$ and this completes the proof of our theorem. $\square$

4 Vertex critical graphs

In this section, we show that 2-connected k-tvc graphs and k-cvc graphs are the same if and only if $3 \leq k \leq 4$. We first give the following theorem.

Theorem 4.1. *Let G be a 2-connected graph. Then G is a 4-tvc graph if and only if G is a 4-cvc graph.*

Proof. Note that for any $v \in V(G)$, v is not a support vertex and $G - v$ is connected since G is 2-connected. Let G be a 4-cvc graph. Hence, $\gamma_t(G) \leq \gamma_c(G) = 4$. If $\gamma_t(G) < 4$, then there exists a γ_t-set D^t of size less than 4 of G. Therefore, $G[D^t]$ is connected by Proposition 2.2. Thus $D^t \succ_c G$ and we have $\gamma_c(G) \leq 3$, a contradiction. Hence, $\gamma_t(G) = 4$.

We next show the criticality. For any $v \in V(G)$, $\gamma_t(G - v) \leq \gamma_c(G - v) = 3$ by Lemma 2.5(2) and because G is 4-cvc. Thus $\gamma_t(G - v) < \gamma_t(G)$ as required.

Conversely, suppose G is 4-tvc. We first show that $\gamma_c(G) = 4$. Let $v \in V(G)$. Consider $G - v$. Since G is 4-tvc, there exists a γ_t-set D^t_v of $G - v$. By Lemma 2.3(2), $|D^t_v| = 3$. By Proposition 2.2, $(G - v)[D^t_v]$ is connected. Thus $D^t_v \succ_c G - v$. By Lemma 2.3(1), there is no vertex of D^t_v adjacent to v. Since G is connected, there exists $w \in V(G) - D^t_v$ such that $vw \in E(G)$ and w is adjacent to at least one vertex of D^t_v. Thus $D^t_v \cup \{w\}$ is a γ_c-set of size 4 of G. We now have $\gamma_c(G) \leq 4$. Suppose there exists D^c which is a γ_c-set of size less than 4. Since $G[D^c]$ is connected, there is no isolated vertex in $G[D^c]$. Thus $D^c \succ_t G$. Therefore, $\gamma_t(G) \leq |D^c| < 4 = \gamma_t(G)$, a contradiction. Thus $\gamma_c(G) = 4$. In the proof of criticality, since $|D^t_v| = 3$, $(G-v)[D^t_v]$ is connected. Hence, D^t_v is a connected dominating set of $G-v$. Therefore, $\gamma_c(G - v) \leq |D^t_v| = 3 < 4 = \gamma_c(G)$ and this completes the proof of our theorem. □

Recall that

$\mathbb{T}^v_k$: class of 2-connected k-tvc graphs G with $G \in \mathbb{G}_k$ and,

$\mathbb{C}^v_k$: class of 2-connected k-cvc graphs G with $G \in \mathbb{G}_k$.

By Theorem 4.1, we have $\mathbb{T}^v_4 = \mathbb{C}^v_4$. However, we next show that $\mathbb{T}^v_k$ and $\mathbb{C}^v_k$ when $k \geq 5$ are different.

Theorem 4.2. *$\mathbb{T}^v_k \neq \mathbb{C}^v_k$ when $k \geq 5$.*

Proof. We prove this theorem by giving a construction of a graph G such that $G \in \mathbb{T}^v_k$ but $G \notin \mathbb{C}^v_k$ when $k \geq 5$. We distinguish 2 cases according to the parity of k.

Case 1 : k is even.

Let $k = 2m + 2$ where $m \geq 2$. Let $P^i = a^i_1 a^i_2 a^i_3 a^i_4$ for $1 \leq i \leq m$. Let $V(G) = \cup^m_{i=1} V(P^i) \cup \{x, y\}$ and $E(G) = \{xy\} \cup \{xa^i_1 | 1 \leq i \leq m\} \cup \{ya^i_4 | 1 \leq i \leq m\}$ (see Figure 3(a)).

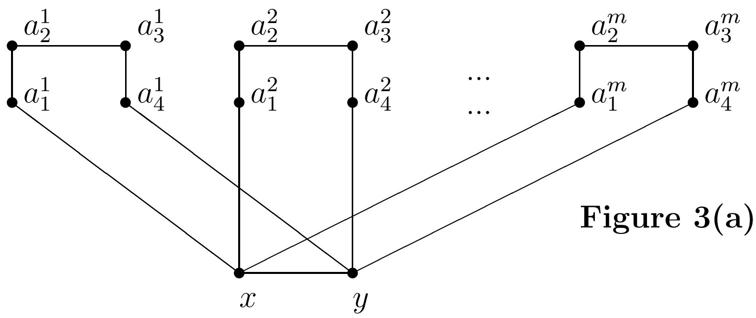

Figure 3(a)

Clearly, $\{x, y\} \cup \{a_1^i, a_4^i | 1 \leq i \leq m\} \succ_c G$. Thus $\gamma_c(G) \leq 2m+2$. Since a γ_c-set of G is also a γ_t-set of G, $\gamma_t(G) \leq \gamma_c(G) \leq 2m+2$. To show that $\gamma_t(G) = \gamma_c(G) = 2m+2$, we need only show that $2m + 2 \leq \gamma_t(G)$. Let D^t be a γ_t-set of G. We next establish the following claim.

Claim 1 : For $1 \leq i \leq m$, $|D^t \cap V(P^i)| \geq 2$.

Suppose first that $a_2^i \in D^t$. Thus $a_3^i \in D^t$ or $a_1^i \in D^t$. It follows that $a_3^i, a_2^i \in D^t$ or $a_1^i, a_2^i \in D^t$. We then suppose that $a_2^i \notin D^t$. If $a_3^i \in D^t$, then $a_4^i \in D^t$. Finally, consider when $a_3^i \notin D^t$. Thus $a_1^i, a_4^i \in D^t$ to dominate a_2^i, a_3^i and we settle Claim 1.

Suppose first that $\{x, y\} \subseteq D^t$. By Claim 1, $|D^t| \geq 2m + 2$.

We next suppose that $|\{x, y\} \cap D^t| = 1$. Without loss of generality, assume that $\{x, y\} \cap D^t = \{x\}$. Since $x \in D^t$, x is adjacent to some vertex in D^t. Thus $a_1^i \in D^t$ for some $i \in \{1, \ldots, m\}$. Without loss of generality, $a_1^1 \in D^t$. We first suppose that $a_4^1 \notin D^t$. Since $D^t \succ_t a_4^1$ and $y \notin D^t$, $a_3^1 \in D^t$. Because $a_3^1 \in D^t$ and $a_4^1 \notin D^t$, it follows that $a_2^1 \in D^t$. Hence, $\{x, a_1^1, a_2^1, a_3^1\} \subseteq D^t$. By Claim 1, $|D^t \cap V(P^i)| \geq 2$ for $2 \leq i \leq m$. Therefore, $|D^t| \geq 2(m - 1) + 4 = 2m + 2$. We then consider when $a_4^1 \in D^t$. Since $y \notin D^t$, $a_3^1 \in D^t$. Hence, $\{x, a_1^1, a_4^1, a_3^1\} \subseteq D^t$. Similarly, $|D^t| \geq 2(m - 1) + 4 = 2m + 2$.

We finally suppose that $\{x, y\} \cap D^t = \emptyset$. Since $D^t \succ_t \{x, y\}$, $a_1^i, a_4^j \in D^t$ for some $i, j \in \{1, \ldots, m\}$. Suppose first that $i = j$. With out loss of generality, $i = j = 1$. Since $x, y \notin D^t$, $a_1^1, a_4^1 \in D^t$ and $a_1^1 a_4^1 \notin E(G)$, it follows that $a_2^1, a_3^1 \in D^t$ and thus $\{a_1^1, a_2^1, a_3^1, a_4^1\} \subseteq D^t$. By Claim 1, $|V(P^i) \cap D^t| \geq 2$ for $2 \leq i \leq m$. Thus $|D^t| \geq 2(m - 1) + 4 = 2m + 2$. We now consider $j \neq i$. Without loss of generality, let $i = 1, j = 2$. Since $\{x, y\} \cap D^t = \emptyset$ and $a_1^1, a_4^2 \in D^t$, it follows that we need at least 3 vertices in $D^t \cap V(P^l)$ to totally dominate P^l for $l \in \{1, 2\}$. Therefore, by Claim 1, $|D^t| \geq 2(m - 2) + 3 + 3 = 2m + 2$.

Hence, $2m + 2 \leq \gamma_t(G) \leq \gamma_c(G) \leq 2m + 2$ and we have that $\gamma_t(G) = \gamma_c(G) = 2m + 2$. We next establish the total domination criticality. Consider $G - v$ where $v \in V(G)$. We have to show that $|D_v^t| = 2m + 1$. Suppose first that $v = a_1^i$. Thus $D_v^t = \{a_3^i, a_4^i, y\} \cup \{a_2^j, a_3^j | 1 \leq i \neq j \leq m\}$ and $|D^t| = 2(m-1)+3 = 2m+1$. We then suppose that $v = a_2^i$. Hence, $D_v^t = \{x, y, a_4^i\} \cup \{a_2^j, a_3^j | 1 \leq j \neq i \leq m\}$ and $|D_v^t| = 2(m - 1) + 3 = 2m + 1$. When $v = x$, we have $D_v^t = \{a_2^1, a_3^1, a_4^1\} \cup \{a_2^i, a_3^i | 2 \leq i \leq m\}$

and $|D_v^t| = 2(m-1)+3 = 2m+1$. We can prove the criticality when $v = a_4^i, v = a_3^i$ and $v = y$ where $i \in \{1,\ldots,m\}$ by the same arguments as when $v = a_1^i, v = a_2^i$ and $v = x$, respectively. Hence, $G \in \mathbb{T}_k^v$. The graph G is not a k-cvc because when we consider $G - x$, by Lemma 2.5(1), $y \notin D_x^c$ and it follows that $(G-x)[D_x^c]$ is not connected. Therefore, $G \notin \mathbb{C}_k^v$.

Case 2 : k is odd.

Let $k = 2m+1$ when $m \geq 2$. Let $P^i = a_1^i a_2^i a_3^i a_4^i$ for $2 \leq i \leq m$ and $P^1 = a_1^1 a_2^1 a_3^1$. Let $V(G) = \cup_{i=1}^m V(P^i) \cup \{x,y\}$ and $E(G) = \{xy, a_3^1 y\} \cup \{xa_1^i | 1 \leq i \leq m\} \cup \{ya_4^i | 2 \leq i \leq m\}$ (see Figure 3(b)).

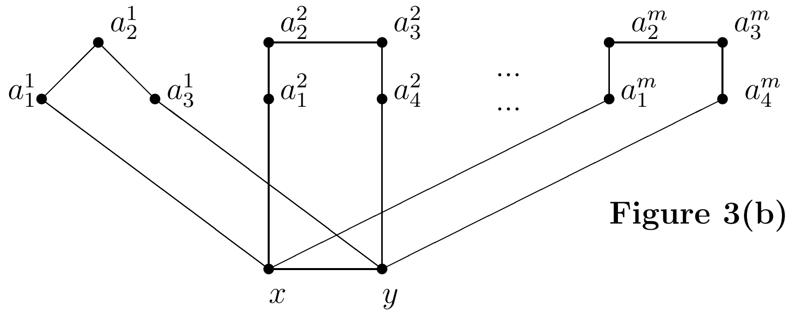

Figure 3(b)

We see that $\{x,y,a_1^1\} \cup \{a_1^i, a_4^i | 2 \leq i \leq m\} \succ_c G$. Thus $\gamma_c(G) \leq 2(m-1)+3 = 2m+1$. To show that $\gamma_t(G) = \gamma_c(G) = 2m+1$, we need only show that $\gamma_t(G) \geq 2m+1$. Let D^t be a γ_t-set of G. We also establish the following claim.

Claim 2 : For $2 \leq i \leq m$, $|D^t \cap V(P^i)| \geq 2$.

By applying the same arguments as in the proof of Claim 1, $|D^t \cap V(P^i)| \geq 2$ for all i such that $|V(P^i)| = 4$.

We first suppose that $\{x,y\} \subseteq D^t$. To dominate $a_2^1, a_1^1 \in D^t$ or $a_3^1 \in D^t$. Hence, $\{a_1^1, x, y\} \subseteq D^t$ or $\{a_3^1, x, y\} \subseteq D^t$. By Claim 2, $|D^t \cap V(P^i)| \geq 2$ for $2 \leq i \leq m$. Thus $|D^t| \geq 2(m-1)+3 = 2m+1$.

Suppose $|\{x,y\} \cap D^t| = 1$. Without loss of generality, assume that $\{x,y\} \cap D^t = \{x\}$. Since $x \in D^t$ and $y \notin D^t$, it follows that $a_1^i \in D^t$ for some $i \in \{1,\ldots,m\}$. We first suppose that $i > 1$, without loss of generality $i = 2$. Thus $a_1^2 \in D^t$. Since $y \notin D^t$ and $D^t \succ_t P^1$, it follows that $|D^t \cap V(P^1)| \geq 2$. Because $D^t \succ_t a_4^2$, $\{x, a_1^2, a_2^2, a_3^2\} \subseteq D^t$ when $a_4^2 \notin D^t$ and $\{x, a_1^2, a_3^2, a_4^2\} \subseteq D^t$ when $a_4^2 \in D^t$. Hence, by Claim 2, $\gamma_t(G) = |D^t| \geq 2(m-2)+2+4 = 2m+2 > 2m+1 = \gamma_c(G)$, a contradiction. Therefore, $i = 1$. Since $y \notin D^t$, $D^t \succ_t a_3^1$ and $a_1^1 a_3^1 \notin E(G)$, it follows that $|D^t \cap V(P^1)| \geq 2$. By Claim 2, $|D^t \cap V(P^j)| \geq 2$ for $j \in \{2,\ldots,m\}$. Hence, $|D^t| \geq 2(m-1)+2+1 = 2m+1$.

Suppose $\{x,y\} \cap D^t = \emptyset$. To totally dominate $\{x,y\}$, $\{a_1^i, a_3^1\} \subseteq D^t$ or $\{a_1^i, a_4^j\} \subseteq D^t$ for some $1 \leq i \leq m, 2 \leq j \leq m$.

We first consider the case when $\{a_1^i, a_4^j\} \subseteq D^t$ for some $1 \leq i \leq m, 2 \leq j \leq m$. Since $x, y \notin D^t$, $|D^t \cap V(P^1)| \geq 2$. We first suppose that $i > 1$. If $i \neq j$, then

$|D^t \cap V(P^i)| = |D^t \cap V(P^j)| = 3$ to dominate a_4^i and a_1^j because $x, y \notin D^t$. By Claim 2, $\gamma_t(G) = |D^t| \geq 2(m-3)+3+3+2 = 2m+2 > 2m+1 = \gamma_c(G)$, a contradiction. Hence, $i = j$. Since $a_1^i, a_4^i \in D^t, x, y \notin D^t$ and $a_1^i a_4^i \notin E(G)$, it follows that $a_2^i, a_3^i \in D^t$. Thus, by Claim 2, $\gamma_t(G) = |D^t| \geq 2(m-2) + 2 + 4 = 2m + 2 > 2m + 1 = \gamma_c(G)$, again a contradiction. Hence, $i = 1$. Therefore, $\{a_1^1, a_2^1\} \subseteq D^t$ and $\{a_2^j, a_3^j, a_4^j\} \subseteq D^t$ to totally dominate a_1^j. Thus $|D^t| \geq 2(m-2) + 2 + 3 = 2m + 1$.

We now consider when $\{a_1^i, a_3^1\} \subseteq D^t$ for some $1 \leq i \leq m$. If $i = 1$, then $D^t \cap V(P^1) = \{a_1^1, a_2^1, a_3^1\}$ because $a_1^1 a_3^1 \notin E(G)$. Thus, by Claim 2, $|D^t| \geq 2(m-1) + 3 = 2m + 1$. If $i > 1$, without loss of generality let $i = 2$, then $a_2^1 \in D^t$ because $a_3^1 \in D^t$ and $y \notin D^t$. Since $a_1^2 \in D^t$ and $x, y \notin D^t$, it follows that $|D^t \cap V(P^2)| = 3$ to totally dominate a_4^2. By Claim 2, $|D^t| \geq 2(m-2) + 2 + 3 = 2m + 1$. Hence, $2m + 1 \leq \gamma_t(G) \leq \gamma_c(G) \leq 2m + 1$. Therefore, $\gamma_t(G) = \gamma_c(G) = 2m + 1$.

We finally establish the criticality of a graph G. Consider $G - v$ where $v \in V(G)$. We have to show that $|D_v^t| = 2m$. Suppose first that $v = x$, then $D_v^t = \{a_2^i, a_3^i | 2 \leq i \leq m\} \cup \{a_2^1, a_3^1\}$ and $|D_v^t| = 2(m-1)+2 = 2m$. Similarly, $|D_y^t| = 2m$. We then suppose $v = a_1^1$. Thus $D_v^t = \{a_2^i, a_3^i | 2 \leq i \leq m\} \cup \{a_3^1, y\}$ and $|D_v^t| = 2(m-1) + 2 = 2m$. We also show that $|D_{a_3^1}^t| = 2m$ by a similar argument as $v = a_1^1$. If $v = a_2^1$, then $D_v^t = \{a_2^i, a_3^i | 2 \leq i \leq m\} \cup \{x, y\}$ and $|D_v^t| = 2(m-1) + 2 = 2m$. If $v = a_1^i$ for $2 \leq i \leq m$, then $D_v^t = \{a_2^j, a_3^j | 2 \leq j \neq i \leq m\} \cup \{a_3^i, a_4^i\} \cup \{a_1^1, a_2^1\}$. It follows that $|D_v^t| = 2(m-2)+2+2 = 2m$. Further, if $v = a_4^i$ for $2 \leq i \leq m$, then $D_v^t = \{a_2^j, a_3^j | 2 \leq j \neq i \leq m\} \cup \{a_1^i, a_2^i\} \cup \{a_3^1, a_2^1\}$. It follows that $|D_v^t| = 2(m-2) + 2 + 2 = 2m$. If $v = a_2^i$ for $2 \leq i \leq m$, then $D_v^t = \{a_2^j, a_3^j | 2 \leq j \neq i \leq m\} \cup \{a_1^1, a_4^i, x, y\}$. It follows that $|D_v^t| = 2(m-2) + 4 = 2m$. Finally, if $v = a_3^i$ for $2 \leq i \leq m$, then $D_v^t = \{a_2^j, a_3^j | 2 \leq j \neq i \leq m\} \cup \{a_1^1, a_1^i, x, y\}$. It also follows that $|D_v^t| = 2(m-2) + 4 = 2m$. Hence, $G \in \mathbb{T}_k^v$.

We can show that G is not a k-cvc graph by the same arguments as in Case 1. Hence, $G \notin \mathbb{C}_k^v$ and this completes the proof of our theorem. □

Goddard et al. [4] mentioned that K_2 is a 2-tvc graph while Ananchuen et al. [2] claimed that a 2-cvc graph is K_{2n} delete a perfect matching where $n \geq 2$. Thus $\mathbb{T}_2^v \neq \mathbb{C}_2^v$. Ananchuen et al. [2] also pointed out that 2-connected 3-tvc graphs and 2-connected 3-cvc graphs are the same. Therefore, $\mathbb{T}_3^v = \mathbb{C}_3^v$. By Theorems 4.1 and 4.2, we can conclude the following corollary.

Corollary 4.3. $\mathbb{T}_k^v = \mathbb{C}_k^v$ *if and only if* $3 \leq k \leq 4$.

Acknowledgements

P. Kaemawichanurat was supported by Development and Promotion of Science and Technology i Talents Project (DPST), Thailand. This work was done while N. Ananchuen was a visitor at Curtin University.

References

[1] N. Ananchuen, On domination critical graphs with cut vertices having connected domination number 3, *Int. Math. Forum* 2 (2007), 3041–3052.

[2] W. Ananchuen, N. Ananchuen and M. D. Plummer, Vertex criticality for connected domination, *Utilitas Math.* 86 (2011), 45–64.

[3] X. G. Chen, L. Sun and D. X. Ma, Connected domination critical graphs, *Appl. Math. Lett.* 17 (2004), 503–507.

[4] W. Goddard, T. W. Haynes, M. A. Henning and L. C. van der Merwe, The diameter of total domination vertex critical graphs, *Discrete Math.* 286 (2004), 255–261.

[5] M. A. Henning, Graphs with large total domination number, *J. Graph Theory* 35 (2000), 21–45.

[6] M. A. Henning and L. C. van der Merwe, Properties of total domination edge-critical graphs, *Discrete Appl. Math.* 158 (2010), 147–153.

[7] P. Kaemawichanurat and N. Ananchuen, On 4-γ_c-critical graphs with cut vertices, *Utilitas Math.* 82 (2010), 253–268.

[8] L. C. van der Merwe and M. Loizeaux, 4_t-critical graphs with maximum diameter, *J. Combin. Math. Combin. Comput.* 60 (2007), 65–80.

[9] L. C. van der Merwe, C. M. Mynhardt and T. W. Haynes, Total domination edge critical graphs, *Utilitas Math.* 54 (1998), 229–240.

(Received 13 Feb 2015; revised 24 Mar 2016)