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Roman domination with respect to nondegenerate graph properties: vertex and edge removal

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Abstract

For a graph property $\mathcal{P}$ and a graph G, a subset S of the vertices of G is a $\mathcal{P}$-set if the subgraph induced by S has the property $\mathcal{P}$. A $\mathcal{P}$-Roman dominating function on a graph G is a labeling $f : V(G) \to \{0, 1, 2\}$ such that every vertex with label 0 has a neighbor with label 2 and the set of all vertices with label 1 or 2 is a $\mathcal{P}$-set. The $\mathcal{P}$-Roman domination number $\gamma_{\mathcal{P}R}(G)$ of G is the minimum of $\Sigma_{v\in V(G)} f(v)$ over such functions. In this paper we present results on changing and unchanging of $\gamma_{\mathcal{P}R}(G)$ when a graph is modified by deleting an edge or a vertex. Some known results for the ordinary Roman domination number are extended and generalized to $\gamma_{\mathcal{P}R}(G)$. The $\mathcal{P}$-Roman bondage number $b_{\mathcal{P}R}(G)$ is the cardinality of a smallest set of edges whose removal from G results in a graph with $\mathcal{P}$-Roman domination number not equal to $\gamma_{\mathcal{P}R}(G)$. We obtain upper bounds in terms of (a) edge degree and maximum degree, (b) average degree and maximum degree, (c) orientable/non orientable genus and maximum degree, and (d) Euler characteristic, girth and maximum degree, for the $\mathcal{P}$-Roman bondage number of a graph on topological surfaces. We also prove that for any graph G, which admits a 2-cell embedding on a surface with non-negative Euler characteristic, either $b_{\mathcal{P}R}(G) \leq 15$ or $15 < b_{\mathcal{P}R}(G) \leq \Delta(G) - 3$.

1 Introduction

All graphs considered in this article are finite, undirected, without loops or multiple edges. We denote the vertex set and the edge set of a graph G by $V(G)$ and $E(G)$, respectively. The subgraph induced by $S \subseteq V(G)$ is denoted by $G[S]$. The complement of a graph G is denoted by $\overline{G}$. Let P_n, C_n and K_n denote the path, cycle and complete graph with n vertices, respectively. For any vertex x of a graph G, $N_G(x)$ denotes the set of all neighbors of x in G, $N_G[x] = N_G(x) \cup \{x\}$ and the degree of x is $d_G(x) = |N_G(x)|$. For a subset $A \subseteq V(G)$, let $N_G(A) = \cup_{x\in A} N_G(x)$

and $N_G[A] = N_G(A) \cup A$. For a graph G, let $x \in X \subseteq V(G)$. A vertex $y \in V(G)$ is an X-private neighbor of x if $N_G[y] \cap X = \{x\}$. The X-private neighborhood of x, denoted $pn_G[x, X]$, is the set of all X-private neighbors of x. The distance between two vertices $x, y \in V(G)$ is denoted by $d_G(x, y)$.

Let $\mathcal{I}$ denote the set of all mutually nonisomorphic graphs. A graph property is any non-empty subset of $\mathcal{I}$. We say that a graph G has the property $\mathcal{P}$ whenever there exists a graph $H \in \mathcal{P}$ which is isomorphic to G. For example, we list some graph properties:

- $\mathcal{O} = \{H \in \mathcal{I} : H$ is totally disconnected$\}$;
- $\mathcal{C} = \{H \in \mathcal{I} : H$ is connected$\}$;
- $\mathcal{T} = \{H \in \mathcal{I} : H$ is without isolates$\}$;
- $\mathcal{F} = \{H \in \mathcal{I} : H$ is a forest$\}$;
- $\mathcal{UK} = \{H \in \mathcal{I} :$ each component of H is complete$\}$;
- $\mathcal{D}^k = \{H \in \mathcal{I} : \Delta(H) \leq k\}$.

A graph property $\mathcal{P}$ is called hereditary (induced-hereditary), if from the fact that a graph G has the property $\mathcal{P}$, it follows that all subgraphs (induced subgraphs) of G also belong to $\mathcal{P}$. A property is called additive if it is closed under taking disjoint unions of graphs. A property $\mathcal{P}$ is called nondegenerate if $\mathcal{O} \subseteq \mathcal{P}$. Note that: (a) $\mathcal{O}$, $\mathcal{F}$ and $\mathcal{D}^k$ are nondegenerate, additive and hereditary properties; (b) $\mathcal{UK}$ is nondegenerate, additive, induced-hereditary and is not hereditary; (c) $\mathcal{C}$ is neither additive nor induced-hereditary nor nondegenerate; (d) $\mathcal{T}$ is additive but neither induced-hereditary nor nondegenerate. Further, an additive and induced-hereditary property is always nondegenerate. Any set $S \subseteq V(G)$ such that the subgraph $G[S]$ possesses the property $\mathcal{P}$ is called a $\mathcal{P}$-set. For a survey on this subject we refer to Borowiecki et al. [4].

A dominating set for a graph G is a subset $D \subseteq V(G)$ of vertices such that every vertex not in D is adjacent to at least one vertex in D. The minimum cardinality of a dominating set is called the domination number of G and is denoted by $\gamma(G)$. The concept of domination in graphs has many applications to several fields. Domination naturally arises in facility location problems, in problems involving finding sets of representatives, in monitoring communication or electrical networks, and in land surveying. Many variants of the basic concepts of domination have appeared in the literature. We refer to [13, 14] for a survey of the area.

The domination number with respect to the graph property $\mathcal{P}$, denoted by $\gamma_{\mathcal{P}}(G)$, is the smallest cardinality of a dominating $\mathcal{P}$-set of a graph G. Note that there may be no dominating $\mathcal{P}$-set of G at all. For example, all graphs having at least two isolated vertices are without dominating $\mathcal{P}$-sets, where $\mathcal{P} \in \{\mathcal{C}, \mathcal{T}\}$. On the other hand, if a property $\mathcal{P}$ is nondegenerate then every maximal independent set is a dominating $\mathcal{P}$-set and thus $\gamma_{\mathcal{P}}(G)$ exists. This fact will be used in the sequel, without specific reference. A dominating $\mathcal{P}$-set of G with cardinality $\gamma_{\mathcal{P}}(G)$ is called a $\gamma_{\mathcal{P}}(G)$-set. The concept of domination with respect to any graph property $\mathcal{P}$ was introduced by Goddard et al. [6] and has been studied, for example, in [15], [24], [25] and elsewhere. Note that $\gamma_{\mathcal{O}}(G)$, $\gamma_{\mathcal{C}}(G)$, $\gamma_{\mathcal{T}}(G)$, $\gamma_{\mathcal{F}}(G)$ and $\gamma_{\mathcal{D}^k}(G)$, are the well known as the independent domination number $i(G)$, the connected domination number $\gamma_c(G)$, the total domination number $\gamma_t(G)$, the acyclic domination number

$\gamma_a(G)$ and the k-dependent domination number $\gamma^k(G)$, respectively (see [13]).

A variation of domination called Roman domination was introduced by ReVelle [20, 21]. Also see ReVelle and Rosing [22] for an integer programming formulation of the problem. The concept of Roman domination can be formulated in terms of graphs. A Roman dominating function (RDF) on a graph G is a vertex labeling $f : V(G) \to \{0, 1, 2\}$ such that every vertex with label 0 has a neighbor with label 2. For an RDF f, let $V_i^f = \{v \in V(G) : f(v) = i\}$ for i = 0, 1, 2. Since this partition determines f, we can equivalently write $f = (V_0^f; V_1^f; V_2^f)$. The weight $f(V(G))$ of an RDF f on G is the value $\Sigma_{v\in V(G)} f(v)$, which equals $|V_1^f| + 2|V_2^f|$. Let $\mathcal{P}$ be nondegenerate property and let G be a graph. We define a Roman dominating function $f = (V_0^f; V_1^f; V_2^f)$ on G to be a $\mathcal{P}$-Roman dominating function, or just $\mathcal{P}$-RDF, if $V_1^f \cup V_2^f$ is a $\mathcal{P}$-set. The $\mathcal{P}$-Roman domination number $\gamma_{\mathcal{P}R}(G)$ of G is the minimum weight of a $\mathcal{P}$-RDF on G. A $\mathcal{P}$-RDF with minimum weight in a graph G will be referred to as a $\gamma_{\mathcal{P}R}$-function on G. Note that $\gamma_{\mathcal{I}R}(G)$, $\gamma_{\mathcal{O}R}(G)$, $\gamma_{\mathcal{C}R}(G)$ and $\gamma_{\mathcal{T}R}(G)$ are well known as the Roman domination number [5], the independence Roman domination number (denoted by $i_R(G)$) [1, 16], the connected Roman domination number [11] and the total Roman domination number [11].

In this paper we concentrate on $\mathcal{P}$-Roman domination when a property $\mathcal{P}$ is nondegenerate. From the above definitions we immediately obtain the following observation.

Observation 1. *Let $\mathcal{O} \subseteq \mathcal{P}_2 \subseteq \mathcal{P}_1 \subseteq \mathcal{I}$ and let G be a graph. Then*

$$\gamma_R(G) = \gamma_{\mathcal{I}R}(G) \leq \gamma_{\mathcal{P}_1R}(G) \leq \gamma_{\mathcal{P}_2R}(G) \leq \gamma_{\mathcal{O}R}(G) = i_R(G). \tag{1}$$

For convenience we omit the subscript $\mathcal{I}$.

The rest of the paper is organized as follows. Sections 2 and 3 contain known and preliminary results, respectively. It is often of interest to know how the value of a graph parameter is affected when a small change is made in a graph. In this connection, here we consider this question in the case $\gamma_{\mathcal{P}R}(G)$ when a vertex or an edge is deleted from G. In Section 4 some known results for the ordinary Roman domination number are extended and generalized to $\gamma_{\mathcal{P}R}(G)$. We also give necessary and sufficient conditions for a graph G to satisfy $\gamma_{\mathcal{P}R}(G-v) < \gamma_{\mathcal{P}R}(G)$ for each vertex v of G. In Section 5, we find all graphs G for which $\gamma_{\mathcal{P}R}(G - e) > \gamma_{\mathcal{P}R}(G)$ for every edge $e \in E(G)$. One measure of the stability of the $\mathcal{P}$-Roman domination number of a graph G under edge removal is the $\mathcal{P}$-Roman bondage number $b_{\mathcal{P}R}(G)$ which is the cardinality of a smallest set of edges whose removal from G results in a graph with $\mathcal{P}$-Roman domination number not equal to $\gamma_{\mathcal{P}R}(G)$. We obtain upper bounds in terms of (a) edge degree and maximum degree, (b) average degree and maximum degree, (c) orientable/non orientable genus and maximum degree, and (d) Euler characteristic, girth and maximum degree, for the $\mathcal{P}$-Roman bondage number of a graph. We also prove that for any graph G, which admits a 2-cell embedding on a surface with non negative Euler characteristic, either $b_{\mathcal{P}R}(G) \leq 15$ or $b_{\mathcal{P}R}(G) \leq \Delta(G) - 3$.

2 Known results

The following results are important for our investigations.

An orientable compact 2-manifold $\mathbb{S}_h$ or orientable surface $\mathbb{S}_h$ (see [23]) of genus h is obtained from the sphere by adding h handles. Correspondingly, a non-orientable compact 2-manifold $\mathbb{N}_q$ or non-orientable surface $\mathbb{N}_q$ of genus q is obtained from the sphere by adding q crosscaps. Compact 2-manifolds are called simply surfaces throughout the paper. The Euler characteristic is defined by $\chi(\mathbb{S}_h) = 2 - 2h$, $h \geq 0$, and $\chi(\mathbb{N}_q) = 2 - q$, $q \geq 1$. The Euclidean plane $\mathbb{S}_0$, the projective plane $\mathbb{N}_1$, the torus $\mathbb{S}_1$, and the Klein bottle $\mathbb{N}_2$ are all the surfaces of nonnegative Euler characteristic.

Let

$$h_1(x) = \begin{cases} 2x + 13 & \text{for } 0 \leq x \leq 3 \\ 4x + 7 & \text{for } x \geq 3 \end{cases}, \quad h_2(x) = \begin{cases} 8 & \text{for } x = 0 \\ 4x + 5 & \text{for } x \geq 1 \end{cases},$$

$$k_1(x) = \begin{cases} 2x + 11 & \text{for } 1 \leq x \leq 2 \\ 2x + 9 & \text{for } 3 \leq x \leq 5 \\ 2x + 7 & \text{for } x \geq 6. \end{cases} \quad \textit{and} \quad k_2(x) = \begin{cases} 8 & \text{for } x = 1 \\ 2x + 5 & \text{for } x \geq 2. \end{cases}$$

Theorem A (Ivančo [8])**.** *If G is a connected graph of orientable genus g and minimum degree at least 3, then G contains an edge $e = xy$ such that $d_G(x) + d_G(y) \leq h_1(g)$. Furthermore, if G does not contain 3-cycles, then G contains an edge $e = xy$ such that $d_G(x) + d_G(y) \leq h_2(g)$.*

Theorem B (Jendrol′ and Tuhársky [9])**.** *If G is a connected graph of minimum degree at least 3 on a nonorientable surface of genus $\overline{g} \geq 1$, then G contains an edge $e = xy$ such that $d_G(x)+d_G(y) \leq k_1(\overline{g})$. Furthermore, if G does not contain 3-cycles, then $d_G(x) + d_G(y) \leq k_2(\overline{g})$.*

Theorem C. *Let G be a connected graph embeddable on a surface $\mathbb{M}$ whose Euler characteristic $\chi(\mathbb{M})$ is as large as possible and let $\delta(G) \geq 5$. Then G contains an edge $e = xy$ with $d_G(x) + d_G(y) \leq 11$ if one of the following holds:*

- *(i)* (Wernicke [28] and Sanders [27], respectively) $\mathbb{M} \in \{\mathbb{S}_0, \mathbb{N}_1\}$.
- (ii) (Jendrol' and Voss [10]) $\mathbb{M} \in \{\mathbb{S}_1, \mathbb{N}_2\}$ and $\Delta(G) \geq 7$.

A path u, v, w is a path of type (i, j, k) if $d_G(u) \leq i$, $d_G(v) \leq j$, and $d_G(w) \leq k$.

Theorem D (Borodin, Ivanova, Jensen, Kostochka and Yancey [3])**.** *Let G be a planar graph with $\delta(G) \geq 3$. If no 2 adjacent vertices have degree 3 then G has a 3-path of one of the following types:*

$$(3,4,11) \quad (3,7,5) \quad (3,10,4) \quad (3,15,3) \quad (4,4,9) \quad (6,4,8) \quad (7,4,7) \quad (6,5,6).$$

Observation E (Rad and Volkmann [19])**.** *If G is a graph, then $\gamma_R(G - e) \geq \gamma_R(G)$ for any edge $e \in E(G)$.*

Theorem F (Rad and Volkmann [18])**.** *If G is a claw-free graph, then $\gamma_R(G) = i_R(G)$.*

Theorem G (Adabi, Targhi, Rad and Moradi [1]). *For any graph G of order n, $i_R(G) \leq n$. Further, the equality holds if and only if $G = mK_2 \cup \overline{K_l}$ for some non negative integers m, l with $n = 2m + l$.*

The average degree $ad(G)$ of a graph G is defined as $ad(G) = 2|E(G)|/|V(G)|$.

Theorem H. (Hartnell and Rall [7]) *For any connected nontrivial graph G, there exists a pair of vertices, say u and v, that are either adjacent or at distance 2 from each other, with the property that $d_G(u) + d_G(v) \leq 2ad(G)$.*

The girth of a graph G is the length of a shortest cycle in G; the girth of a forest is ∞.

Lemma I (Samodivkin [26]). *Let G be a connected graph embeddable on a surface $\mathbb{M}$ whose Euler characteristic χ is as large as possible and let the girth of G is $k < \infty$. Then:*

$$ad(G) \leq \frac{2k}{k-2}(1 - \frac{\chi}{|V(G)|}).$$

3 Preliminary results

Observation 2. *Let $G_1, G_2, \ldots, G_k$ be mutually vertex disjoint graphs and $G = \cup_{i=1}^k G_i$, $k \geq 2$.*

(a) If $\mathcal{P}$ is nondegenerate and additive then $\gamma_{\mathcal{P}R}(G) \leq \sum_{i=1}^k \gamma_{\mathcal{P}R}(G_i)$.

(b) If $\mathcal{P}$ is nondegenerate and induced-hereditary then $\gamma_{\mathcal{P}R}(G) \geq \sum_{i=1}^k \gamma_{\mathcal{P}R}(G_i)$.

(c) If $\mathcal{P}$ is additive and induced-hereditary then $\gamma_{\mathcal{P}R}(G) = \sum_{i=1}^k \gamma_{\mathcal{P}R}(G_i)$.

Proof. (a) Let $f_i = (V_0^{f_i}; V_1^{f_i}; V_2^{f_i})$ be a $\gamma_{\mathcal{P}R}$-function on G_i, $i = 1, 2, \ldots, k$. Since $\mathcal{P}$ is additive, $f = (\cup_{s=1}^k V_0^{f_i}; \cup_{s=1}^k V_1^{f_i}; \cup_{s=1}^k V_2^{f_i})$ is a $\mathcal{P}$-RDF on G and $\gamma_{\mathcal{P}}(G) \leq f(V(G)) = \sum_{i=1}^k f_i(V(G_i)) = \sum_{i=1}^k \gamma_{\mathcal{P}R}(G_i)$.

(b) Let f be a $\gamma_{\mathcal{P}R}(G)$-function and let $f_i = (V_0^f \cap V(G_i), V_1^f \cap V(G_i), V_2^f \cap V(G_i))$, $i = 1, 2, \ldots, k$. Since $\mathcal{P}$ is induced-hereditary, f_i is a $\mathcal{P}$-RDF on G_i. This implies $\gamma_{\mathcal{P}}(G) = f(V(G)) = \sum_{i=1}^k f(V(G_i)) = \sum_{i=1}^k f_i(V(G_i)) \geq \sum_{i=1}^k \gamma_{\mathcal{P}R}(G_i)$.

(c) Any additive and induced-hereditary property is clearly nondegenerate. It immediately follows by (a) and (b) that $\gamma_{\mathcal{P}R}(G) = \sum_{i=1}^k \gamma_{\mathcal{P}R}(G_i)$. □

By similar way we obtain:

Observation 3. *Let G_1 and $G_2 = \overline{K}_s$ ($s \geq 1$) be vertex disjoint graphs and $G = G_1 \cup G_2$.*

(a) If $\mathcal{P}$ is nondegenerate and closed under union with K_1 then $\gamma_{\mathcal{P}R}(G) \leq \gamma_{\mathcal{P}R}(G_1) + s$.

(b) If $\mathcal{P}$ is nondegenerate and induced-hereditary then $\gamma_{\mathcal{P}R}(G) \geq \gamma_{\mathcal{P}R}(G_1) + s$.

(c) If $\mathcal{P}$ is closed under union with K_1 and induced-hereditary then $\gamma_{\mathcal{P}R}(G) = \gamma_{\mathcal{P}R}(G_1) + s$.

The next lemma plays a key role in the proofs of the many of our results.

Lemma 4. *Let a property $\mathcal{P}$ be nondegenerate and induced-hereditary. Let G be a graph and $f = (V_0^f; V_1^f; V_2^f)$ a $\mathcal{P}$-RDF on G with $V_1^f \neq V(G)$. Then there is a $\mathcal{P}$-RDF $g = (V_0^g; V_1^g; V_2^g)$ on G such that $g(V(G)) \leq f(V(G))$, $V_0^f \subseteq V_0^g$, $V_1^g \subseteq V_1^f$, $V_2^f \subseteq V_2^g$ and $E(G[V_1^g \cup V_2^g]) = E(G[V_2^f])$. Furthermore:*

(i) V_2^g is a $\mathcal{P}$-set,

(ii) if f is a $\gamma_{\mathcal{P}R}$-function on G then g is a $\gamma_{\mathcal{P}R}$-function on G, and

(iii) if V_2^f is a $\mathcal{Q}$-set, where $\mathcal{Q} \subseteq \mathcal{I}$ and $\mathcal{Q}$ is closed under union with K_1 then both V_2^g and $V_1^g \cup V_2^g$ are $\mathcal{Q}$-sets and g is a $\mathcal{Q}$-RDF on G.

Proof. Since $\mathcal{P}$ is induced-hereditary, V_2^f is a $\mathcal{P}$-set. Let $h_1 = (V_0^{h_1}; V_1^{h_1}; V_2^{h_1})$, where $V_0^{h_1} = N_G(V_2^f) - V_2^f$, $V_1^{h_1} = V_1^f - N_G(V_2^f)$ and $V_2^{h_1} = V_2^f$. Hence h_1 is an RDF on G with $V_0^f \subseteq V_0^{h_1}$, $V_1^{h_1} \subseteq V_1^f$, $f(V(G)) \geq h_1(V(G))$ and no edge joins $V_1^{h_1}$ and $V_2^{h_1}$. Since $\mathcal{P}$ is induced-hereditary, h_1 is a $\mathcal{P}$-RDF on G and $V_2^{h_1}$ is a $\mathcal{P}$-set. If $V_1^{h_1}$ is empty or independent then $g = h_1$. Assume there are adjacent $u, v \in V_1^{h_1}$. Then a function $h_2 = (V_0^{h_2}; V_1^{h_2}; V_2^{h_2})$, where $V_0^{h_2} = V_0^{h_1} \cup (N_G(v) \cap V_1^{h_1})$, $V_1^{h_2} = V_1^{h_1} - N_G[v]$ and $V_2^{h_2} = V_2^{h_1} \cup \{v\}$ is a $\mathcal{P}$-RDF on G such that $V_0^{h_1} \subsetneq V_0^{h_2}$, $V_1^{h_2} \subsetneq V_1^{h_1}$, $V_2^{h_1} \subsetneq V_2^{h_2}$, $h_2(V(G)) \leq h_1(V(G))$, no edge of G joins $V_1^{h_2}$ and $V_2^{h_2}$, $|V_1^{h_2}| < |V_1^{h_1}|$ and $G[V_2^{h_2}]$ is isomorphic to $G[V_2^{h_1}] \cup K_1$. If $V_1^{h_2}$ is not independent we continue this process until we get a $\mathcal{P}$-RDF $h_k = (V_0^{h_k}; V_1^{h_k}; V_2^{h_k})$ on G, where $V_1^{h_k}$ is either empty or independent. Set $g = h_k$.

(i) Since $V_2^g \subseteq V_1^f \cup V_2^f$ and $V_1^f \cup V_2^f$ is a $\mathcal{P}$-set, V_2^g is a $\mathcal{P}$-set too ($\mathcal{P}$ is induced-hereditary).

(ii) $\gamma_{\mathcal{P}R}(G) \leq g(V(G)) \leq f(V(G)) = \gamma_{\mathcal{P}R}(G)$.

(iii) We already know that $V_2^f \subseteq V_2^g$ and $E(G[V_1^g \cup V_2^g]) = E(G[V_2^f])$. Since $\mathcal{Q}$ is closed under union with K_1, both V_2^g and $V_1^g \cup V_2^g$ are $\mathcal{Q}$-sets. □

Corollary 5. *Let $\mathcal{O} \subseteq \mathcal{P}_1 \subseteq \mathcal{P}_2 \subseteq \mathcal{I}$ and let G be a graph. Let $\mathcal{P}_1$ be closed under union with K_1 and $\mathcal{P}_2$ induced-hereditary. If there is a $\gamma_{\mathcal{P}_2R}$-function $f = (V_0^f; V_1^f; V_2^f)$ on G such that V_2^f is a $\mathcal{P}_1$-set then $\gamma_{\mathcal{P}_1R}(G) = \gamma_{\mathcal{P}_2R}(G)$.*

Proof. By Lemma 4, there is a $\gamma_{\mathcal{P}_2R}$-function $g = (V_0^g; V_1^g; V_2^g)$ on G such that $V_0^f \subseteq V_0^g$, $V_2^f \subseteq V_2^g$, $V_1^g \cup V_2^g - V_2^f$ is independent and no edge of G joins $V_1^g \cup V_2^g - V_2^f$ and V_2^f. Since $\mathcal{P}_1$ is closed under union with K_1 and V_2^f is a $\mathcal{P}_1$-set, $V_1^g \cup V_2^g$ is a $\mathcal{P}_1$-set which implies g is a $\mathcal{P}_1$-RDF. Hence $\gamma_{\mathcal{P}_2R}(G) \leq \gamma_{\mathcal{P}_1R}(G) \leq g(V(G)) = \gamma_{\mathcal{P}_2R}(G)$. □

Observation 6. *([5] when $\mathcal{P} = \mathcal{I}$) Let a property $\mathcal{P}$ be nondegenerate and induced-hereditary. Let $f = (V_0^f; V_1^f; V_2^f)$ be any $\gamma_{\mathcal{P}R}(G)$-function. Then $\Delta(G[V_1^f]) \leq 1$ and no edge of G joins V_1^f and V_2^f. If $|V_1^f|$ is a minimum then V_1^f is independent and if in addition G is isolate-free then $V_0^f \cup V_2^f$ is a vertex cover.*

Proof. Suppose $u, v, w \in V_1^f$ and $uv, vw \in E(G)$. Then $g = (V_0^f \cup \{u, w\}; V_1^f - \{u, v, w\}; V_2^f \cup \{v\})$ is an RDF on G with $g(V(G)) = f(V(G)) - 1$. Since $\mathcal{P}$ is induced-hereditary, g is a $\mathcal{P}$-RDF on G, a contradiction. Thus $\Delta(G[V_1^f]) \leq 1$. If $x \in V_1^f$ and $y \in V_2^f$ are adjacent then $h = (V_0^f \cup \{x\}; V_1^f - \{x\}; V_2^f)$ is a $\mathcal{P}$-RDF on G with $h(V(G)) = f(V(G)) - 1$, a contradiction. If $z, t \in V_1^f$ are adjacent then $l = (V_0^f \cup \{z\}; V_1^f - \{z, t\}; V_2^f \cup \{t\})$ is a $\mathcal{P}$-RDF on G with $l(V(G)) = f(V(G))$ and $|V_1^l| < |V_1^f|$. Hence if $|V_1^f|$ is a minimum then V_1^f is independent. Since the complement of an independent set of an isolate-free graph is a vertex cover, the result follows. □

Observations 1, 2, 3 and 6 will be used in the sequel without specific reference.

Proposition 7. *Let a property $\mathcal{P}$ be nondegenerate. For any graph G of order n, $1 \leq \gamma_{\mathcal{P}R}(G) \leq n$. Moreover: (a) $\gamma_{\mathcal{P}R}(G) = 1$ if and only if $G = K_1$, (b) $\gamma_{\mathcal{P}R}(G) = 2$ if and only if either $G = \overline{K}_2$ or $\Delta(G) = n - 1 \geq 1$, and (c) $\gamma_{\mathcal{P}R}(G) = n$ if and only if $\Delta(G) \leq 1$.*

Proof. (a) and (b): Obvious.

(c) If $\Delta(G) \leq 1$ then clearly $\gamma_{\mathcal{P}R}(G) = n$. If $\gamma_{\mathcal{P}R}(G) = n$ then $i_R(G) = n$ and the result follows by Theorem G. □

Observation 8. *Let a property $\mathcal{P}$ be nondegenerate. Then $\gamma_{\mathcal{P}R}(C_n) = \lceil 2n/3 \rceil$ and $\gamma_{\mathcal{P}R}(P_m) = \lceil 2m/3 \rceil$.*

Proposition 9. *Let a property $\mathcal{P}$ be nondegenerate. For any graph G, ([5] when $\mathcal{P} = \mathcal{I}$) $\gamma_{\mathcal{P}}(G) \leq \gamma_{\mathcal{P}R}(G) \leq 2\gamma_{\mathcal{P}}(G)$. Moreover, $\gamma_{\mathcal{P}}(G) = \gamma_{\mathcal{P}R}(G)$ if and only if G has no edges.*

Proof. Let $f = (V_0^f; V_1^f; V_2^f)$ be any $\gamma_{\mathcal{P}R}(G)$-function. Then $V_1^f \cup V_2^f$ is a dominating $\mathcal{P}$-set of G. Hence, $\gamma_{\mathcal{P}}(G) \leq |V_1^f| + |V_2^f| \leq |V_1^f| + 2|V_2^f| = \gamma_{\mathcal{P}R}(G)$. If $\gamma_{\mathcal{P}}(G) = \gamma_{\mathcal{P}R}(G)$ then $V_2^f = \emptyset$ which implies $V_0^f = \emptyset$. Therefore $\gamma_{\mathcal{P}}(G) = \gamma_{\mathcal{P}R}(G) = |V_1^f| = |V(G)|$. But then G has no edges. Clearly, if G has no edges then $\gamma_{\mathcal{P}}(G) = \gamma_{\mathcal{P}R}(G) = |V(G)|$.

Now, let D be a minimum dominating $\mathcal{P}$-set of G. Then $g = (V(G) - D, \emptyset, D)$ is a $\mathcal{P}$-RDF on G and $2\gamma_{\mathcal{P}}(G) = 2|D| \geq \gamma_{\mathcal{P}R}(G)$. □

We will say that a graph G is a $\mathcal{P}$-Roman graph ($\mathcal{P}$ is nondegenerate) if $\gamma_{\mathcal{P}R}(G) = 2\gamma_{\mathcal{P}}(G)$. Any nonempty n-order graph having a vertex of degree $n - 1$ is a $\mathcal{P}$-Roman graph. All Roman paths and cycles are P_{3k}, C_{3k}, P_{3k+2}, and C_{3k+2} (by Observation 8). Results on Roman graphs ($\mathcal{P} = \mathcal{I}$) may be found in [5].

4 Vertex removal

In this section we examine the effects on the $\mathcal{P}$-Roman domination number when a graph is modified by deleting a vertex. According to the effects of vertex removal on the $\mathcal{P}$-Roman domination number of a graph G, let

- $V_{\mathcal{P}R}^{+}(G) = \{v \in V(G) \mid \gamma_{\mathcal{P}R}(G - v) > \gamma_{\mathcal{P}R}(G)\}$,
- $V_{\mathcal{P}R}^{-}(G) = \{v \in V(G) \mid \gamma_{\mathcal{P}R}(G - v) < \gamma_{\mathcal{P}R}(G)\}$,

• $V^0_{\mathcal{P}R}(G) = \{v \in V(G) \mid \gamma_{\mathcal{P}R}(G - v) = \gamma_{\mathcal{P}R}(G)\}$.

Clearly $\{V^-_{\mathcal{P}R}(G), V^0_{\mathcal{P}R}(G), V^+_{\mathcal{P}R}(G)\}$ is a partition of $V(G)$.

Theorem 10. *Let a property $\mathcal{P}$ be induced-hereditary and closed under union with K_1, and let G be a graph of order at least 2. For any vertex v in a graph G, $\gamma_{\mathcal{P}R}(G) - 1 \leq \gamma_{\mathcal{P}R}(G - v)$. Moreover:*

(i) If $\gamma_{\mathcal{P}R}(G) - 1 = \gamma_{\mathcal{P}R}(G - v)$ then there is a $\gamma_{\mathcal{P}R}$-function $f = (V^f_0; V^f_1; V^f_2)$ on $G - v$ such that all vertices in V^f_1 are isolated in $(G - v)[V^f_1 \cup V^f_2]$ and one of the following holds:

(i.1) $N_G(v) \subseteq V^f_0$ and $g = (V^f_0; V^f_1 \cup \{v\}; V^f_2)$ is a $\gamma_{\mathcal{P}R}$-function on G;

(i.2) all neighbors of v but one, say w, belong to V^f_0, $f(w) = 1$, $g = (V^f_0 \cup \{w\}; V^f_1 - \{w\}; V^f_2 \cup \{v\})$ is a $\gamma_{\mathcal{P}R}$-function on G and $pn_G[v, V^g_2] = \{v, w\}$.

(ii) If there is a $\gamma_{\mathcal{P}R}$-function $f = (V^f_0; V^f_1; V^f_2)$ on G such that $f(v) = 1$ then $\gamma_{\mathcal{P}R}(G - v) = \gamma_{\mathcal{P}R}(G) - 1$.

(iii) Let $f = (V^f_0; V^f_1; V^f_2)$ be a $\gamma_{\mathcal{P}R}$-function on G such that $f(v) = 2$ and $pn_G[v, V^f_2] = \{v, u\}$. Then $\gamma_{\mathcal{P}R}(G - v) = \gamma_{\mathcal{P}R}(G) - 1$, $N_{G-v}[u] \subseteq V^f_0$, $h = (V^f_0 - \{u\}; V^f_1 \cup \{u\}; V^f_2 - \{v\})$ is a $\gamma_{\mathcal{P}R}$-function on $G - v$, and $l = (V^f_0 \cup \{v\} - \{u\}; V^f_1; V^f_2 \cup \{u\} - \{v\})$ is a $\gamma_{\mathcal{P}R}$-function on G with $pn_G[u, V^l_2] = \{u, v\}$. If $\mathcal{P}$ is closed under union with K_2 then $p = (V^f_0 - \{u\}; V^f_1 \cup \{u, v\}; V^f_2 - \{v\})$ is a $\gamma_{\mathcal{P}R}$-function on G.

Proof. Theorem 10 is true when $\gamma_{\mathcal{P}R}(G) = |V(G)|$, because of Proposition 7(c). So, let $\gamma_{\mathcal{P}R}(G - v) < \gamma_{\mathcal{P}R}(G) < |V(G)|$. We shall prove simultaneously that $\gamma_{\mathcal{P}R}(G - v) = \gamma_{\mathcal{P}R}(G) - 1$ and that (i) holds. Since $\gamma_{\mathcal{P}R}(G - v) < |V(G - v)|$, there is a $\gamma_{\mathcal{P}R}$-function on $G - v$, say t_0, with nonempty $V^{t_0}_2$. Note that no edge joins v and V^t_2 for each $\gamma_{\mathcal{P}R}$-function t on $G - v$ - otherwise $t_1 = (V^t_0 \cup \{v\}; V^t_1; V^t_2)$ is a $\mathcal{P}$-RDF on G with $t_1(V(G)) = t(V(G - v))$, a contradiction. By Lemma 4 there exists a $\gamma_{\mathcal{P}R}$-function $h = (V^h_0; V^h_1; V^h_2)$ on $G - v$ such that all vertices in V^h_1 are isolated in $(G - v)[V^h_1 \cup V^h_2]$. If all neighbors of v are in V^h_0 then since $\mathcal{P}$ is closed under union with K_1, $l = (V^h_0; V^h_1 \cup \{v\}; V^h_2)$ is a $\mathcal{P}$-RDF on G with $l(V(G)) = \gamma_{\mathcal{P}R}(G - v) + 1 \leq \gamma_{\mathcal{P}R}(G)$. Hence $\gamma_{\mathcal{P}R}(G) - 1 = \gamma_{\mathcal{P}R}(G - v)$ and (i.1) holds. Now, let $N_G(v) \cap V^h_1 = \{x_1, \ldots, x_k\}$. Since $\mathcal{P}$ is induced-hereditary and closed under union with K_1, $g = (V^h_0 \cup \{x_1, \ldots, x_k\}; V^h_1 - \{x_1, \ldots, x_k\}; V^h_2 \cup \{v\})$ is a $\mathcal{P}$-RDF on G and $\gamma_{\mathcal{P}R}(G) \leq g(V(G)) = \gamma_{\mathcal{P}R}(G - v) - k + 2 < \gamma_{\mathcal{P}R}(G) - k + 2$. Hence $k = 1$ which leads to $\gamma_{\mathcal{P}R}(G) - 1 = \gamma_{\mathcal{P}R}(G - v)$ and g is a $\gamma_{\mathcal{P}R}$-function on G. By the very definition of g it immediately follows that $pn_G[v, V^g_2] = \{v, x_1\}$. Thus (i.2) holds.

(ii) Define $h = (V^f_0; V^f_1 - \{v\}; V^f_2)$. Then h is a $\mathcal{P}$-RDF on $G - v$ with $h(V(G - v)) = \gamma_{\mathcal{P}R}(G) - 1$.

(iii) Since $v \in pn_G[v, V^f_2]$, v is isolated in $G[V^f_1 \cup V^f_2]$. Since $u \in pn_G[v, V^f_2]$, uv is the only one edge, which joins u and V^f_2. Assume M is the set of all neighbors of u which belong to V^f_1. If $M \neq \emptyset$ then $g = (V^f_0 \cup M \cup \{v\} - \{u\}; V^f_1 - M; V^f_2 \cup \{u\} - \{v\})$ is a $\mathcal{P}$-RDF on G with $g(V(G)) < f(V(G)) = \gamma_{\mathcal{P}R}(G)$ - a contradiction. Hence

$N_{G-v}[u] \subseteq V_0^f$. But then h is a $\mathcal{P}$-RDF on $G-v$ with $h(V(G-v)) = g(V(G))-1 = \gamma_{\mathcal{P}R}(G)-1 = \gamma_{\mathcal{P}R}(G-v)$ and l is a $\mathcal{P}$-RDF on G with $l(V(G-v)) = f(V(G)) = \gamma_{\mathcal{P}R}(G)$. By the definition of l and $pn_G[v, V_2^f] = \{v,u\}$ it follows $pn_G[u, V_2^l] = \{v,u\}$. If $\mathcal{P}$ is closed under union with K_2 then since $N_G(\{u,v\}) - \{u,v\} \subseteq V_0^f$ and $pn_G[v, V_2^f] = \{v,u\}$, it follows that p is a $\mathcal{P}$-RDF on G with $p(V(G)) = f(V(G)) = \gamma_{\mathcal{P}R}(G)$. □

For each nondegenerate property $\mathcal{P}$ we define the following class of graphs G:
$CV_{\mathcal{P}R}^k$: $\gamma_{\mathcal{P}R}(G-S) < \gamma_{\mathcal{P}R}(G)$ for any set $S \subsetneq V(G)$ with $|S| = k$.

Remark 11. *Let a property $\mathcal{P}$ be nondegenerate. Any n-order graph G, $n \geq 2$, with $\gamma_{\mathcal{P}R}(G) = n$ is in $CV_{\mathcal{P}R}^k$ for every k, $1 \leq k \leq n-1$ (by Proposition 7(c)). All cycles belonging to the class $CV_{\mathcal{P}R}^1$ are C_{3k+1} and C_{3k+2} (by Observation 8).*

An immediate consequence of Theorem 10 is the following characterization of the class $CV_{\mathcal{P}R}^1$.

Corollary 12. *Let a property $\mathcal{P}$ be induced-hereditary and closed under union with K_1. A graph G is in $CV_{\mathcal{P}R}^1$ if and only if for every vertex $v \in V(G)$ one of the following holds:*

(i) there is a $\gamma_{\mathcal{P}R}$-function f_v on G with $f_v(v) = 1$;

(ii) there is a $\gamma_{\mathcal{P}R}$-function h_v on G such that $h_v(v) = 2$ and $pn_G[v, V_2^{h_v}] = \{v,u\}$.

If, in addition, $\mathcal{P}$ is closed under union with K_2 then G is in $CV_{\mathcal{P}R}^1$ if and only if (i) holds for every vertex v of a graph G.

The class CV_R^1 was introduced by Rad and Volkmann [19]. Since $\mathcal{I}$ is induced-hereditary and closed under union with K_1 and with K_2, as an immediately consequence of Corollary 12 we have the following result due to Hansberg et al. [12]: G is in CV_R^1 if and only if (i) holds for every vertex v of a graph G.

Proposition 13. *([19] when $\mathcal{P} = \mathcal{I}$) Let a property $\mathcal{P}$ be induced-hereditary and closed under union with K_1, and let v be a vertex of a graph G. If $v \in V_{\mathcal{P}R}^+(G)$ then for every $\gamma_{\mathcal{P}R}$-function $f = (V_0^f; V_1^f; V_2^f)$ on G, $f(v) = 2$ and $|pn_G[v, V_2^f] \cap V_0^f| \geq 3$.*

Proof. Let $f = (V_0^f; V_1^f; V_2^f)$ be any $\gamma_{\mathcal{P}R}$-function on G. If $v \notin V_2^f$ then $f_1 = (V_0^f - \{v\}; V_1^f - \{v\}; V_2^f)$ is a $\mathcal{P}$-RDF on $G-v$ of weight at most $\gamma_{\mathcal{P}R}(G)$ - a contradiction.
Assume $|M| \leq 2$, where $M = pn_G[v, V_2^f] \cap V_0^f$. Then $f_2 = (V_0^f - M; V_1^f \cup M; V_2^f - \{v\})$ is an RDF on $G-v$ and $f_2(V(G-v)) \leq \gamma_{\mathcal{P}R}(G)$. If $V_1^{f_2} = V(G-v)$ then $\gamma_{\mathcal{P}R}(G-v) \leq |V(G-v)| = f_2(V(G-v)) \leq \gamma_{\mathcal{P}R}(G)$, a contradiction. Thus $V_2^{f_2}$ is not empty and it clearly is a $\mathcal{P}$-set. But then there is a $\mathcal{P}$-RDF f_3 on $G-v$ with $f_3(V(G-v)) \leq f_2(V(G-v))$ (by Lemma 4) - a contradiction. □

Corollary 14. *([19] when $\mathcal{P} = \mathcal{I}$) Let a property $\mathcal{P}$ be induced-hereditary and closed under union with K_1. If u and v are vertices of a graph G, $v \in V_{\mathcal{P}R}^-(G)$ and $u \in V_{\mathcal{P}R}^+(G)$ then u and v are nonadjacent.*

Proof. Proposition 13 implies $f(u) = 2$ for every $\gamma_{\mathcal{P}R}$-function f on G. By Theorem 10 it follows that all neighbors of v belong to V_0^g for some $\gamma_{\mathcal{P}R}$-function g on G. $\square$

In the case when a property $\mathcal{P}$ is induced-hereditary and closed under union with K_1, Corollary 14 allow us to give a new definition of the class $CV^1_{\mathcal{P}R}$:
$CV^1_{\mathcal{P}R}$: $\gamma_{\mathcal{P}R}(G - v) \neq \gamma_{\mathcal{P}R}(G)$ for each $v \in V(G)$.

5 Edge removal

Here we present results on changing of $\gamma_{\mathcal{P}R}(G)$ when an edge is deleted from G. When we remove an edge from a graph G, the Roman domination number with respect to the property $\mathcal{P}$ can increase or decrease. For instance, if G is a star $K_{1,p}$, $p \geq 3$, and $\{K_1, 2K_1\} \subseteq \mathcal{P} \subseteq \mathcal{I}$ then $\gamma_{\mathcal{P}R}(G) = 2$ and $\gamma_{\mathcal{P}R}(G - e) = 3$ for all $e \in E(G)$. If a graph G is obtained by three stars $K_{1,p}$ and three edges e_1, e_2, e_3 joining their centers then $\gamma_{\mathcal{F}R}(G) = 4 + p$ and $\gamma_{\mathcal{F}R}(G - e_i) = 6$, $i = 1, 2, 3$. So the edge set of G can be partitioned into

- $E^+_{\mathcal{P}R}(G) = \{e \in E(G) \mid \gamma_{\mathcal{P}R}(G - e) > \gamma_{\mathcal{P}R}(G)\}$,
- $E^-_{\mathcal{P}R}(G) = \{e \in E(G) \mid \gamma_{\mathcal{P}R}(G - e) < \gamma_{\mathcal{P}R}(G)\}$,
- $E^0_{\mathcal{P}R}(G) = \{e \in E(G) \mid \gamma_{\mathcal{P}R}(G - e) = \gamma_{\mathcal{P}R}(G)\}$.

Note that Observation E implies $E^+_R(G)$ is empty for every graph G.

Theorem 15. *Let a property $\mathcal{P}$ be hereditary and closed under union with K_1. Let $e = xy$ be an edge of a graph G.*

(i) Then $\gamma_{\mathcal{P}R}(G - e) \leq \gamma_{\mathcal{P}R}(G) + 1$.

(ii) If there is a $\gamma_{\mathcal{P}R}$-function f on $G - e$ such that $(f(x), f(y)) \neq (2, 2)$ then $\gamma_{\mathcal{P}R}(G) \leq \gamma_{\mathcal{P}R}(G - e)$.

(iii) If $(f(x), f(y)) = (2, 2)$ for some $\gamma_{\mathcal{P}R}$-function f on $G - e$ then $\gamma_{\mathcal{P}R}(G) - \min\{d_G(x), d_G(y)\} + 3 \leq \gamma_{\mathcal{P}R}(G - e)$.

If $e_1 \in E(\overline{G})$ then $\gamma_{\mathcal{P}R}(G) - 1 \leq \gamma_{\mathcal{P}R}(G + e_1)$.

Proof. (i) Let $f = (V_0^f; V_1^f; V_2^f)$ be any $\gamma_{\mathcal{P}R}$-function on G. If $\{f(x), f(y)\} \neq \{0, 2\}$ then since $\mathcal{P}$ is hereditary, f is a $\mathcal{P}$-RDF on $G - e$ which implies $\gamma_{\mathcal{P}R}(G - e) \leq f(V(G - e)) = \gamma_{\mathcal{P}R}(G)$. Let without loss of generality $f(x) = 0$ and $f(y) = 2$. Then $g = (V_0^f - \{x\}; V_1^f \cup \{x\}; V_2^f)$ is an RDF on $G - xy$, with $g(V(G - xy)) = f(V(G)) + 1 = \gamma_{\mathcal{P}R}(G) + 1$. Since $\mathcal{P}$ is hereditary, V_2^f is a $\mathcal{P}$-set. Now by Lemma 4, there is a $\mathcal{P}$-RDF h on $G - xy$ with $h(V(G - xy)) \leq g(V(G - xy)) = \gamma_{\mathcal{P}R}(G) + 1$.

By (i) it immediately follows that if $e_1 \in E(\overline{G})$ then $\gamma_{\mathcal{P}R}(G) - 1 \leq \gamma_{\mathcal{P}R}(G + e_1)$.

(ii) and (iii): Let $l = (V_0^l; V_1^l; V_2^l)$ be any $\gamma_{\mathcal{P}R}$-function on $G - xy$. Hence l is an RDF on G. If one of $l(x)$ and $l(y)$ is 0 then l is a $\mathcal{P}$-RDF on G. If $V_1^l = V(G)$ then $\gamma_{\mathcal{P}R}(G) \leq |V(G)| = |V_1^l| = \gamma_{\mathcal{P}R}(G - e)$. So let V_2^l is not empty. If $l(x) = 1$ and $l(y) \neq 0$, or visa versa then by Lemma 4 it follows that there is a $\mathcal{P}$-RDF l_1 on G with $l_1(V(G)) \leq l(V(G)) = l(V(G - xy))$. It remains the case $l(x) = l(y) = 2$. Define an RDF l_2 on G as follows: $l_2(x) = 0$, $l_2(v) = 1$ if $v \in pn_G[x, V_2^l] \cap V_0^l$ and

$l_2(v) = l(v)$ - otherwise. Hence $l_2(V(G)) = l(V(G-xy)) - 2 + |pn_{G-xy}[x, V_2^l] \cap V_1^l| \leq \gamma_{\mathcal{P}R}(G-xy) - 2 + d_{G-xy}(x) = \gamma_{\mathcal{P}R}(G-xy) - 3 + d_G(x)$. The result follows, since Lemma 4 implies the existence of a $\mathcal{P}$-RDF l_3 on G with $l_3(V(G)) \leq l_2(V(G))$. $\square$

Corollary 16. *Let a property $\mathcal{P}$ be hereditary and closed under union with K_1. Let $e = xy$ be an edge of a graph G. If $\min\{d_G(x), d_G(y)\} \leq 3$ then $\gamma_{\mathcal{P}R}(G) \leq \gamma_{\mathcal{P}R}(G-e) \leq \gamma_{\mathcal{P}R}(G)+1$. In particular, if $\Delta(G) \leq 3$ then $E(G) = E^+_{\mathcal{P}R}(G) \cup E^0_{\mathcal{P}R}(G)$.*

For every graph G and every nondegenerate property $\mathcal{P}$, we define the Roman bondage (minus Roman bondage, plus Roman bondage, respectively) number with respect to the property $\mathcal{P}$, denoted $b_{\mathcal{P}R}(G)$ ($b^-_{\mathcal{P}R}(G)$, $b^+_{\mathcal{P}R}(G)$, respectively) to be the cardinality of a smallest set of edges $U \subseteq E(G)$ such that $\gamma_{\mathcal{P}R}(G-U) \neq \gamma_{\mathcal{P}R}(G)$ ($\gamma_{\mathcal{P}R}(G-U) < \gamma_{\mathcal{P}R}(G)$, $\gamma_{\mathcal{P}R}(G-U) > \gamma_{\mathcal{P}R}(G)$, respectively). If $\gamma_{\mathcal{P}R}(G-U) \geq \gamma_{\mathcal{P}R}(G)$ ($\gamma_{\mathcal{P}R}(G-U) \leq \gamma_{\mathcal{P}R}(G)$, respectively) for all $U \subseteq E(G)$, we write $b^-_{\mathcal{P}R}(G) = \infty$ ($b^+_{\mathcal{P}R}(G) = \infty$, respectively).

Observation 17. *Let $\mathcal{P} \subseteq \mathcal{I}$ be nondegenerate and let G be a nonempty graph.*

(i) Then $b^-_{\mathcal{I}R}(G) = \infty$ and $b^+_{\mathcal{I}R}(G) = b_{\mathcal{I}R}(G) = b_R(G)$.

(ii) If $\Delta(G) = 1$ then $b^-_{\mathcal{P}R}(G) = b^+_{\mathcal{P}R}(G) = b_{\mathcal{P}R}(G) = \infty$.

(iii) If $\Delta(G) \leq 2$ then $b^-_{\mathcal{P}R}(G) = \infty$ and $b^+_{\mathcal{P}R}(G) = b^+_{\mathcal{I}R}(G) = b_R(G)$.

(iv) If $\Delta(G) \geq 2$ then $b^+_{\mathcal{P}R}(G) < \infty$ and $b_{\mathcal{P}R}(G) < \infty$.

(v) For the cycle of order n,

$$b^-_{\mathcal{P}R}(C_n) = \infty \text{ and } b^+_{\mathcal{P}R}(C_n) = b_{\mathcal{P}R}(C_n) = \begin{cases} 3 & \text{if } n \equiv 2 \ (mod\ 3), \\ 2 & \text{otherwise.} \end{cases},$$

(vi) For the path of order $n \geq 3$,

$$b^-_{\mathcal{P}R}(P_n) = \infty \text{ and } b^+_{\mathcal{P}R}(P_n) = b_{\mathcal{P}R}(P_n) = \begin{cases} 2 & \text{if } n \equiv 2 \ (mod\ 3), \\ 1 & \text{otherwise.} \end{cases},$$

Proof. (i) By Observation E, $\gamma_{\mathcal{I}R}(G-e) \geq \gamma_{\mathcal{I}R}(G)$ for every edge $e \in E(G)$.

(ii) The result immediately follows from Proposition 7(c).

(iii) If T is a graph with $\Delta(T) \leq 2$ then $\gamma_R(T) = i_R(T)$ (by Theorem F) which implies $\gamma_R(T) = \gamma_{\mathcal{P}R}(T)$. Hence $b^+_{\mathcal{P}R}(G) = b^+_{\mathcal{I}R}(G) = b_R(G)$ and $b^-_{\mathcal{P}R}(G) = b^-_{\mathcal{I}R}(G) = \infty$ (by (i)).

(iv) By Proposition 7(c), $\gamma_{\mathcal{P}R}(G) < |V(G)| = \gamma_{\mathcal{P}R}(G - E(G))$.

(v)–(vi) $b^-_{\mathcal{P}R}(C_n) = b^-_{\mathcal{P}R}(P_n) = \infty$ because of (iii). The required results for $b^+_{\mathcal{P}R}(C_n)$ and $b^+_{\mathcal{P}R}(P_n)$ provided $\mathcal{P} = \mathcal{I}$ due to Rad and Volkmann [17]. The rest follows immediately by (iii). $\square$

The star S_n of order n, $n \geq 1$, is a tree on n vertices with one vertex of degree $n-1$ and the other $n-1$ having vertex degree 1.

Theorem 18. *Let a property $\mathcal{P}$ be hereditary and closed under union with K_1. Let G be a nonempty graph of order n, $b^+_{\mathcal{P}R}(G) = k < \infty$ and the deletion of any k edges results in a graph with increased $\mathcal{P}$-Roman domination number. Then one of the following holds.*

(i) $k = 1$ and G is a nonempty forest in which each component is a star different from S_2.

(ii) $k = 2$ and $G = K_3 \cup \overline{K}_{n-3}$.

Proof. Case 1: $k = 1$. Let f be a $\gamma_{\mathcal{P}R}$-function on G. If uv is an edge of G then $\{f(u), f(v)\} = \{0, 2\}$ - otherwise f is a $\mathcal{P}$-RDF on $G - uv$, a contradiction. Assume, without loss of generality, $f(v) = 2$ and $f(u) = 0$. It immediately follows that $N_G(v) \subseteq V_0^f$. If there are vertices $w \in V(G) - \{v\}$ and $x \in N_G(v)$ which are adjacent then f is a $\mathcal{P}$-RDF on $G - xw$ - a contradiction. Thus the components of G are stars. Clearly G has no S_2 as a component. Furthermore, if S_k, $k \geq 3$, is a component of G and $e \in E(S_k)$ then obviously $\gamma_{\mathcal{P}R}(G) < \gamma_{\mathcal{P}R}(G - e)$.

Case 2: $k = 2$. Then for each edge $e \in E(G)$, $G - e$ is a forest in which each component is a star different from S_2. Since $k \neq 1$, G has exactly one component, say G_1, which has edges; moreover G_1 is not a star. First let $G_1 - e = S_r$. If $r \geq 4$ then $k = 1$, a contradiction. If $r = 3$ then $G_1 = K_3$. Since $2 = \gamma^+_{\mathcal{P}R}(K_3) = \gamma^+_{\mathcal{P}R}(P_3) < \gamma^+_{\mathcal{P}R}(K_2 \cup K_1) = 3$, the result follows.

Now let $G_1 - e = S_p \cup S_q$, where $1 \leq p \leq q$, $p \neq 2$ and $q \geq 3$. Since $b^+_{\mathcal{P}R}(P_4) = 1$, $(p, q) \neq (1, 3)$. If ($p = 1$ and $q \geq 4$) or ($p \geq 3$ and $q \geq 3$) then for any pendent edge $e_1 \in E(S_q - e)$, $G - e_1$ is neither a star nor a union of stars - a contradiction.

Case 3: $k \geq 3$. If $k = 3$ then for any edge $e \in E(G)$, $G - e = K_3 \cup \overline{K}_{n-3}$; but this is clearly impossible. Hence there are none for higher values of k. □

For each nondegenerate property $\mathcal{P}$ we define the following class of graphs G:

$CER_{\mathcal{P}R}$: $\gamma_{\mathcal{P}R}(G - e) > \gamma_{\mathcal{P}R}(G)$ for every edge $e \in E(G)$.

The following reformulation of Theorem 18(i) gives a complete characterization of the class $CER_{\mathcal{P}R}$.

Corollary 19. *Let a property $\mathcal{P}$ be hereditary and closed under union with K_1. A graph G is in the class $CER_{\mathcal{P}R}$ if and only if G is a nonempty forest in which each component is a star different from S_2.*

For any subset $U \subsetneq V(G)$, by E_U we denote the set of all edges each of which joins U and $V(G) - U$.

Theorem 20. *Let a property $\mathcal{P}$ be hereditary and closed under union with K_1. Let G be a connected graph.*

(i) If $v \in V^0_{\mathcal{P}R}(G) \cup V^+_{\mathcal{P}R}(G)$ then $\gamma_{\mathcal{P}R}(G - E_{\{v\}}) > \gamma_{\mathcal{P}R}(G)$.

(ii) If $x \in V^+_{\mathcal{P}R}(G)$ then $1 \leq \gamma_{\mathcal{P}R}(G - x) - \gamma_{\mathcal{P}R}(G) \leq d_G(x) - 2$ and for any subset $S \subseteq E_{\{x\}}$ with $|S| \geq d_G(x) - \gamma_{\mathcal{P}R}(G - v) + \gamma_{\mathcal{P}R}(G)$, $\gamma_{\mathcal{P}R}(G - S) > \gamma_{\mathcal{P}R}(G)$.

(iii) If $V^-_{\mathcal{P}R}(G) \neq V(G)$ *then* $b_{\mathcal{P}R}(G) \leq b^+_{\mathcal{P}R}(G) \leq \min\{d_G(u) - \gamma_{\mathcal{P}R}(G-u) + \gamma_{\mathcal{P}R}(G) \mid u \in V^0_{\mathcal{P}R}(G) \cup V^+_{\mathcal{P}R}(G)\} \leq \Delta(G)$.

(iv) If $b^+_{\mathcal{P}R}(G) > \Delta(G)$ *then a graph* G *is in* $CV^1_{\mathcal{P}R}$.

Proof. (i) We have $\gamma_{\mathcal{P}R}(G - E_{\{v\}}) = \gamma_{\mathcal{P}R}(G - v) + 1 > \gamma_{\mathcal{P}R}(G)$.

(ii) Assume $p = \gamma_{\mathcal{P}R}(G - x) - \gamma_{\mathcal{P}R}(G)$. Let f be any $\gamma_{\mathcal{P}R}$-function on G. Since $p > 0$, by Proposition 13 it follows that $f(x) = 2$. Consider an RDF $h = (V^f_0 - N_G(x); V^f_1 \cup (N_G(x) - V^f_2); V^f_2 - \{x\})$ on $G - x$. Since V^h_2 is a $\mathcal{P}$-set, Lemma 4 implies the existence of a $\mathcal{P}$-RDF l on $G - x$ with $l(V(G - x)) \leq h(V(G - x))$. But then $\gamma_{\mathcal{P}R}(G) + p = \gamma_{\mathcal{P}R}(G - x) \leq h(V(G - x)) \leq \gamma_{\mathcal{P}R}(G) + d_G(x) - 2$. Hence $1 \leq p \leq d_G(x) - 2$. For any set $S \subseteq E_{\{x\}}$ with $|S| \geq d_G(x) - p$ we have $\gamma_{\mathcal{P}R}(G - S) \geq \gamma_{\mathcal{P}R}(G - E_{\{x\}}) - |E_{\{x\}}| + |S| \geq (\gamma_{\mathcal{P}R}(G - x) + 1) - d_G(x) + (d_G(x) - p) = \gamma_{\mathcal{P}R}(G) + 1$, where the first inequality follows from Theorem 15.

(iii) The result follows immediately by (i) and (ii).

(iv) Immediately by (iii). □

We remark that Theorem 20(iv) shows that the class $CV^1_{\mathcal{P}R}$ will play an important role in the study of the plus bondage number with respect to property $\mathcal{P}$.

Given a graph G of order n, let $\widehat{G}$ be the graph of order $5n$ obtained from G by attaching the central vertex of a copy of P_5, to each vertex of G.

Proposition 21. *Let a property* $\mathcal{P}$ *be nondegenerate and let* G *be a graph of order* n*. Then:*

(i) $\widehat{G}$ *is a* $\mathcal{P}$*-Roman graph with* $\gamma_{\mathcal{P}}(\widehat{G}) = 2n$ *and* $\gamma_{\mathcal{P}R}(\widehat{G}) = 4n$*;*

(ii) $V(\widehat{G}) = V^-_{\mathcal{P}R} \cup V^0_{\mathcal{P}R}$ *and* $V^0_{\mathcal{P}R} = V(G)$*;*

(iii) [2] $b_R(\widehat{G}) = \delta(G) + 2$.

Proof. Clearly the set S of all support vertices of a graph $\widehat{G}$ form a $\gamma_{\mathcal{P}}(\widehat{G})$-set. Hence $\gamma_{\mathcal{P}}(\widehat{G}) = 2n$ and by Proposition 9, $\gamma_{\mathcal{P}R}(\widehat{G}) \leq 4n$. Since $\gamma_{\mathcal{P}R}(P_5) = \gamma_{\mathcal{P}R}(P_2 \cup P_2) = 4$, we have $\gamma_{\mathcal{P}R}(\widehat{G}) = 4n$. Since $f = (V(\widehat{G}) - S; \emptyset; S)$ is a $\gamma_{\mathcal{P}R}$-function on $\widehat{G}$, Proposition 13 implies $V(\widehat{G}) = V^-_{\mathcal{P}R} \cup V^0_{\mathcal{P}R}$. □

Proposition 21 shows that the bound in Theorem 20(iii) is attainable for all graphs $\widehat{G}$ when $\mathcal{P} = \mathcal{I}$.

Theorem 22. *Let a property* $\mathcal{P}$ *be hereditary and closed under union with* K_1*. Let* G *be a connected graph and* x, y, z *a path of length* 2 *in* G*. Let* H *be the graph obtained from* G *by removing the edges incident with* x, y *or* z *with exception of* yz *and all edges between* y *and* $N_G(x) \cap N_G(y)$*. Then there is a vertex* $u \in N_G(x) \cap N_G[y]$ *such that* $\gamma_{\mathcal{P}R}(H + xu) < \gamma_{\mathcal{P}R}(H)$*. In particular ([17] when* $\mathcal{P} = \mathcal{I}$*),*

$$b_{\mathcal{P}R}(G) \leq |E(G)| - |E(H)| \leq d_G(x) + d_G(y) + d_G(z) - 3 - |N_G(x) \cap N_G(y)|.$$

Proof. Let $f = (V_0^f; V_1^f; V_2^f)$ be any $\gamma_{\mathcal{P}R}$-function on H. Since x is isolated in H, $f(x) = 1$. If $f(y) = 2$ then $g = (V_0^f \cup \{x\}; V_1^f - \{x\}; V_2^f)$ is a $\mathcal{P}$-RDF on $H + xy$ of weight less than $\gamma_{\mathcal{P}R}(H)$. If $f(y) = 1$ then $f(z) = 1$ and $h = (V_0^f \cup \{x, z\}; V_1^f - \{x, y, z\}; V_2^f \cup \{y\})$ is a $\mathcal{P}$-RDF on $H+xy$ with weight less than $\gamma_{\mathcal{P}R}(H)$. Suppose $f(y) = 0$. If $f(z) = 1$ then there is $t \in N_H(y)$ with $f(t) = 2$. But then $l = (V_0^f \cup \{x\}; V_1^f - \{x\}; V_2^f)$ is a $\mathcal{P}$-RDF on $H + xt$ with weight less than $\gamma_{\mathcal{P}R}(H)$. It remains the case $f(y) = 0$ and $f(z) = 2$. Suppose T is the set of all neighbors of y in H which belong to V_1^f. As $N_H(y) \cap V_2^f = \{z\}$ then $q = (V_0^f \cup T \cup \{x, z\} - \{y\}; V_1^f - T - \{x\}; V_2^f \cup \{y\} - \{z\})$ is a $\mathcal{P}$-RDF on $H + xy$ with weight less than $\gamma_{\mathcal{P}R}(H)$.

Thus, $b_{\mathcal{P}R}(G) \leq |E(G)| - |E(H)|$ and the result follows. □

Theorem 23. *Let a property $\mathcal{P}$ be hereditary and closed under union with K_1. Let G be a planar graph with minimum degree $\delta(G) \geq 4$.*

(i) Then $b_{\mathcal{P}R}(G) \leq 15$.

(ii) Let for each path x, y, z in G if $d_G(y) = 4$ then neither $\{d_G(x), d_G(z)\} = \{6, 8\}$ nor $d_G(x) = d_G(z) = 7$. Then $b_{\mathcal{P}R}(G) \leq 14$.

Proof. The results follow by combining Theorem D and Theorem 22. □

For any edge $e = xy \in E(G)$, let $\xi(e) = d_G(x)+d_G(y)-2$ and let $\xi(G) = min\{\xi(e) : e \in E(G)\}$. The parameter $\xi(G)$ is called the minimum edge-degree of G.

Theorem 24. *Let a property $\mathcal{P}$ be hereditary and closed under union with K_1 and let G be a connected graph with $\Delta(G) \geq 2$.*

(i) Then $b_{\mathcal{P}R}(G) \leq \xi(G) + \Delta(G) - 1$.

(ii) If G is of orientable genus g and $\delta(G) \geq 3$, then $b_{\mathcal{P}R}(G) \leq h_1(g) + \Delta(G) - 3$. Furthermore, if G does not contain 3-cycles, then $b_{\mathcal{P}R}(G) \leq h_2(g) + \Delta(G) - 3$.

(iii) If G is of nonorientable genus $\overline{g}$ and $\delta(G) \geq 3$, then $b_{\mathcal{P}R}(G) \leq k_1(\overline{g})+\Delta(G)-3$. Furthermore, if G does not contain 3-cycles, then $b_{\mathcal{P}R}(G) \leq k_2(\overline{g}) + \Delta(G) - 3$.

(iv) Then $b_{\mathcal{P}R}(G) \leq 2ad(G) + \Delta(G) - 3$.

(v) Let G be embeddable on a surface $\mathbb{M}$ whose Euler characteristic χ is as large as possible. If G has order n and girth $k < \infty$ then:

$$b_{\mathcal{P}R}(G) \leq \frac{4k}{k-2}(1 - \frac{\chi}{n}) + \Delta(G) - 3.$$

Proof. (i) Since $\Delta(G) \geq 2$, there is a path x, y, z in G such that $\xi(xy) = \xi(G)$. Now, by Theorem 22 we have $b_{\mathcal{P}R}(G) \leq d_G(x) + d_G(y) + d_G(z) - 3 \leq \xi(G) + d_G(z) - 1 \leq \xi(G) + \Delta(G) - 1$.

(ii) Combining (i) and Theorem A we obtain the required.

(iii) The result follows by combining Theorem B and (i).

(iv) If G is a complete graph then clearly $b_{\mathcal{P}R}(G) \leq \Delta(G)$. Hence we may assume G has nonadjacent vertices. Theorem H implies that there are 2 vertices, say x and y, that are either adjacent or at distance 2 from each other, with the property that $d_G(x) + d_G(y) \leq 2ad(G)$. Since G is connected and $\Delta(G) \geq 2$, there is a vertex z such that xyz or xzy is a path. In either case by Theorem 22 we have $b_{\mathcal{P}R}(G) \leq d_G(x) + d_G(y) + d_G(z) - 3 \leq 2ad(G) + \Delta(G) - 3$.

(v) Lemma I and (iv) together imply the result. □

Theorem 25. *Let a property $\mathcal{P}$ be hereditary and closed under union with K_1. Let G be a connected graph 2-cell embedded on a surface with non negative Euler characteristic. Let $V_{\leq 5} = \{v \in V(G) \mid d_G(v) \leq 5\}$, $G_{\geq 6} = G - V_{\leq 5}$ and $A_k = \{u \in V(G_{\geq 6}) \mid d_{G_{\geq 6}}(u) \leq 6$ and $|N_G(u) \cap V_{\leq 5}| = k\}$. Then exactly one of the following holds:*

(i) $b_{\mathcal{P}R}(G) \leq 15$;

(ii) $A_2 = \emptyset$, $A_{\geq 3} = \cup_{i \geq 3} A_i \neq \emptyset$ *and* $15 < b_{\mathcal{P}R}(G) \leq \min\{d_G(u) \mid u \in A_{\geq 3}\} - 3 \leq \Delta(G) - 3$.

Proof. If $2 \leq \Delta(G) \leq 6$ or A_2 is not empty then Theorem 22 implies $b_{\mathcal{P}R}(G) \leq 15$. Assume now that each vertex of degree at most 6 in $G_{\geq 6}$ has no more than one neighbor in $V_{\leq 5}$. It immediately follows that $\delta(G_{\geq 6}) \geq 5$. First assume $\delta(G_{\geq 6}) = 5$. By Theorem C, there is an edge $xy \in E(G_{\geq 6})$ such that $d_{G_{\geq 6}}(x) + d_{G_{\geq 6}}(y) \leq 11$. Hence $d_G(x) + d_G(y) \leq 13$. Let without loss of generality $d_{G_{\geq 6}}(x) \leq d_{G_{\geq 6}}(y)$. Then x has exactly one neighbor in $V_{\leq 5}$, say v. By Theorem 22 applied to the path v, x, y we have $b_{\mathcal{P}R}(G) \leq 5 + 13 - 3 = 15$. Now let $\delta(G_{\geq 6}) \geq 6$. But then $G_{\geq 6}$ is a 6-regular triangulation on the torus or in the Klein bottle. If $G = G_{\geq 6}$ then Theorem 22 leads to $b_{\mathcal{P}R}(G) \leq 13$. If $G \neq G_{\geq 6}$ then G has a path x, y, z where $d_G(z) \leq 5$, and both x and y are in the same face of the triangulation. Again by Theorem 22 we obtain $b_{\mathcal{P}R}(G) \leq 7 + 7 + 5 - 3 - 2 = 14$. Assume now that $\Delta(G) \geq 7$, $A_2 = \emptyset$, $A_{\geq 3} \neq \emptyset$ and $15 < b_{\mathcal{P}R}(G)$. Let $u \in A_{\geq 3}$ and $v_1, v_2, v_3 \in N_G(u) \cap V_{\leq 5}$. Denote by E_1 the set of all edges of G which are incident to at least one of v_1, v_2 and v_3. Since $b_{\mathcal{P}R}(G) \geq 16$, $\gamma_{\mathcal{P}R}(G - E_1) = \gamma_{\mathcal{P}R}(G)$. Clearly, for any $\gamma_{\mathcal{P}R}$-function f on $G - E_1$, $f(v_1) = f(v_2) = f(v_3) = 1$. If there is a $\gamma_{\mathcal{P}R}$-function g on $G - E_1$ with $g(u) \neq 0$ then $g_1 = (V_0^g \cup \{v_1, v_2, v_3\}; V_1^g - \{u, v_1, v_2, v_3\}; V_2^g \cup \{u\})$ is a $\mathcal{P}$-RDF on $(G - E_1) \cup \{uv_1, uv_2, uv_3\}$ with weight less than $g(V(G - E_1)) = \gamma_{\mathcal{P}R}(G)$, a contradiction. Thus, for any $\gamma_{\mathcal{P}R}$-function g on $G - E_1$, $g(u) = 0$.

Let G_u be the graph obtained from G by deleting all edges incident to u with exception of uv_1, uv_2 and uv_3. If h is a $\gamma_{\mathcal{P}R}$-function on $G - E_1$ then $h_1 = ((V_0^h \cup \{v_1, v_2, v_3\}) - \{u\}; V_1^h - \{v_1, v_2, v_3\}; V_2^h \cup \{u\})$ is a $\mathcal{P}$-RDF on G_u and $\gamma_{\mathcal{P}R}(G) = h(V(G - E_1)) > h_1(V(G_u))$. □

We conclude with the following question.

Question 1. *Let a graph G admit a 2-cell embedding on a surface with non negative Euler characteristic and let a property $\mathcal{P}$ be hereditary and closed under union with K_1. Is it true that $b_{\mathcal{P}R}(G) \leq 15$?*

Note that in [2], Akbari, Khatirinejad and Qajar recently proved that $b_R(G) \leq 15$ provided G is a planar graph.

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