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The relaxed square property

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Abstract

Graph products are characterized by the existence of non-trivial equivalence relations on the edge set of a graph that satisfy a so-called square property. We investigate here a generalization, termed *RSP-relations*. The class of graphs with non-trivial RSP-relations in particular includes graph bundles. Furthermore, RSP-relations are intimately related with covering graph constructions. For $K_{2,3}$-free graphs finest RSP-relations can be computed in polynomial-time. In general, however, they are not unique and their number may even grow exponentially. They behave well for graph products, however, in sense that a finest RSP-relation can be obtained easily from finest RSP-relations on the prime factors.

1 Introduction

Modern proofs of prime factor decomposition (PFD) theorems for the Cartesian graph product rely on characterizations of the product relation σ on the edge set of the given graph [17]. The key property of σ is that connected components of the subgraphs induced by the classes of σ are precisely the layers, i.e., $(e, f) \in \sigma$ if and only if the edges e and f belong to copies of the same (Cartesian) prime factor [10, 22]. Classical results in the theory of graph products establish that σ can be derived from other, easily computable, relations on the edge set:

$$\sigma = \mathfrak{C}(\delta) = (\theta \cup \tau)^*,$$

where $\mathfrak{C}(\delta)$ denotes the convex closure of the so-called δ-relation and $(\theta \cup \tau)^*$ is the transitive closure of two different relations known as the Djoković-Winkler relation θ and relation τ [10, 17].

Of particular interest for us is the relation δ. An equivalence relation R is said to have the *square property* if (i) any pair of adjacent edges which belong to distinct equivalence classes span a unique chordless square and (ii) the opposite edges of any chordless square belong to the same equivalence class. The importance of δ stems from the fact that it is the unique, finest relation on $E(G)$ with the square property.

An equivalence relation has the *unique square property* if any two adjacent edges e and f from distinct equivalence classes span a unique chordless square with opposite edges in the same equivalence class. The slight modification, in fact a mild generalization, of the relation δ turned out to play a fundamental role for the characterization of graph bundles [24] and forms the basis of efficient algorithms to recognize Cartesian graph bundles [16, 23]. Graph bundles [21], the combinatorial analog of the topological notion of a fiber bundle [14], are a common generalization of both Cartesian products [10] and covering graphs [1].

The key distinction of the unique square property is that, in contrast to the square property, opposite edges do not have to be in the same equivalence class for all chordless squares. Any such relation that is in addition weakly 2-convex yields the structural properties of a graph bundle [24]. Moreover, every Cartesian graph bundle over a triangle-free simple base can be characterized by the relation δ^*, which satisfies the unique square property [16]. In a recent attempt to better understand the structure of equivalence relations on the edge set of a graph G that satisfy the unique square property, we uncovered a surprising connection to equitable partitions on the vertex set of G [13] and a Cartesian factorization of certain quotient graphs that was previously observed in the context of quantum walks on graphs [2]. It was shown that for any equivalence class φ of a relation R with unique square property the connected components of the graph $G_{\overline{\varphi}} = (V(G), E(G) \setminus \varphi)$ form a natural *equitable* partition $\mathcal{P}^R_{\overline{\varphi}}$ of the vertex set of G. Moreover, the so-called common refinement $\mathcal{P}^R$ of this partitions $\mathcal{P}^R_{\overline{\varphi}}$ yields again an equitable partition of $V(G)$ and the quotient $G/\mathcal{P}^R$ has then a product representation as $G/\mathcal{P}^R \cong \Box_{\varphi \sqsubseteq R} G_{\varphi}/\mathcal{P}^R_{\overline{\varphi}}$.

In [20], it was shown that a further relaxation of the unique square property to the relaxed square property still retains the product decomposition of these quotient graphs. The connected components of $G_\varphi = (V(G), \varphi)$ have a natural interpretation as fibers, while the graph $G_{\overline{\varphi}}/\mathcal{P}^R_{\varphi}$ can be seen as base graph. Such a decomposition is a graph bundle if and only if edges in G linking distinct connected components of G_φ induce an isomorphism between them. Thus, graphs with this type of relations on the edge set, which we call *RSP-relations* for short, are a natural generalization of graph bundles.

In this contribution we will examine RSP-relations more systematically. First we show that, as in the case of the unique square property, there is no uniquely determined finest RSP-relation for given graphs in general. Even more, the number of such finest relations on a graph can grow exponentially. However, we will see that the finest RSP-relations R are "bounded" by relations δ_0 and δ_1 so that $\delta_1^* \subseteq R \subseteq \delta_0^*$. We explain how (finest) RSP-relations can be determined in certain graph products, given the RSP-relations in the factors. The main difficulty in determining finest RSP-relations derive from $K_{2,3}$ as induced subgraphs. We provide a polynomial-time algorithm for $K_{2,3}$-free graphs and give a recipe how finest RSP-relations can be constructed in complete and complete bipartite graphs. Finally, we examine the close connection of covering graphs and RSP-relations.

2 Preliminaries

Notation. In the following we consider finite, connected, undirected, simple graphs unless stated otherwise. A graph G has vertex set $V = V(G)$ and edge set $E = E(G)$. An *isomorphism* $f : G \to H$ between two graphs G, H is a bijective mapping $f : V(G) \to V(H)$ such that for all $u, v \in V(G)$ holds $[f(u), f(v)] \in E(H)$ if and only if $[u, v] \in E(G)$. We say G and H are *isomorphic*, in symbols $G \cong H$, if there exists an isomorphism between them. A graph H is a subgraph of G, $H \subseteq G$, if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A subgraph H is an *induced subgraph* of G if $x, y \in V(H)$ and $[x, y] \in E(G)$ implies $[x, y] \in E(H)$. H is called *spanning subgraph* if $V(H) = V(G)$. If none of the subgraphs H of G is isomorphic to a graph K, we say that G is *K-free*. A subgraph $H = (\{a, b, c, d\}, \{[a, b], [b, c], [c, d], [a, d]\})$ is called *square*, will often be denoted by $a-b-c-d$ and we say that $[a, b]$ and $[c, d]$, resp.,

$[b, c]$ and $[a, d]$ are *opposite* edges. The *complete* graph on n vertices is denoted by K_n and the *complete bipartite* graph on $n+m$ vertices by $K_{m,n}$. $\mathcal{L}K_1$ is the one-vertex graph with a loop.

We will consider equivalence relations R on E and denote equivalence classes of R by Greek letters, $\varphi \subseteq E$. We will furthermore write $\varphi \sqsubseteq R$ to indicate that φ is an equivalence class of R. The complement $\overline{\varphi}$ of an R-class φ is defined as $\overline{\varphi} := E \setminus \varphi$. For an equivalence class $\varphi \sqsubseteq R$, an edge e is called φ-edge if $e \in \varphi$. The subgraph G_φ has vertex set $V(G)$ and edge set φ. The connected component of G_φ containing vertex $x \in V(G)$ is called *φ-layer through* x, denoted by G_φ^x. Analogously, the subgraphs $G_{\overline{\varphi}}$ and $G_{\overline{\varphi}}^x$ are defined. Two φ-layers G_φ^x, G_φ^y are said to be *adjacent*, if there exists an edge $[x', y'] \in \overline{\varphi}$ with $x' \in V(G_\varphi^x)$ and $y' \in V(G_\varphi^y)$.

An equivalence relation Q is *finer* than a relation R while the relation R is *coarser* than Q if $(e, f) \in Q$ implies $(e, f) \in R$, i.e, $Q \subseteq R$. In other words, for each class ϑ of R there is a collection $\{\chi | \chi \subseteq \vartheta\}$ of Q-classes, whose union equals ϑ. Equivalently, for all $\varphi \sqsubseteq Q$ and $\psi \sqsubseteq R$ we have either $\varphi \subseteq \psi$ or $\varphi \cap \psi = \emptyset$. If R is not an equivalence relation, then we will denote with R^* the finest equivalence relation that contains R. Moreover, an equivalence relation R is *non-trivial* if it has at least two equivalence classes.

For a given partition $\mathcal{P} = \{V_1, \dots, V_l\}$ of $V(G)$ of a graph G, the *quotient graph* $G/\mathcal{P}$ has as its vertex set $\mathcal{P}$ and there is an edge $[A, B]$ for $A, B \in \mathcal{P}$ if and only if there are vertices $a \in A$ and $b \in B$ such that $[a, b] \in E(G)$. A partition $\mathcal{P}$ of the vertex set $V(G)$ of a graph G is *equitable* if, for all (not necessarily distinct) classes $A, B \in \mathcal{P}$, every vertex $x \in A$ has the same number $m_{AB} := |N_G(x) \cap B|$ of neighbors in B.

Graph Cover and Homomorphisms. A *homomorphism* $f : G \to H$ between two graphs G and H is a mapping $f : V(G) \to V(H)$ such that $f(u)$ and $f(v)$ are adjacent in H whenever u and v are adjacent in G. A homomorphism $f : G \to H$ is called *locally surjective* if $f(N_G(u)) = N_H(f(u))$ for all vertices $u \in V(G)$, i.e., if $f_{|N_G(u)} : N_G(u) \to N_H(f(u))$ is a surjection. We use here the obvious notation $N_G(v)$ for the open neighborhood of v in the graph G. Analogously, f is called *locally bijective* if for all vertices $u \in V(G)$ we have $f(N_G(u)) = N_H(f(u))$ and $|f(N_G(u))| = |N_H(f(u))|$, i.e., $f_{|N_G(u)} : N_G(u) \to N_H(f(u))$ is a bijection. Notice, a locally surjective homomorphism $f : G \to H$ is already globally surjective if H is connected. If there exists a locally surjective homomorphism $f : G \to H$, we call G a *quasi-cover* of H. Locally surjective homomorphisms are also known as role colorings [4]. A locally bijective homomorphism is called a *covering map.* G is a *(graph) cover* or *covering graph* of H if there exists a covering map from G to H, in which case we say that G *covers* H. $|V(H)|$ is then a multiple of $|V(G)|$, i.e., $|V(H)| = k|V(G)|$. H is referred to as *k-fold cover* of G. Moreover, every covering map $f : H \to G$ satisfies $|f^{-1}(u)| = k$ for all $u \in V(G)$ [6]. For more detailed information about locally constrained homomorphisms and graph cover we refer to [6, 7].

Graph Products. There are three associative and commutative standard graph products, the *Cartesian product* $G \,\square\, H$, the *strong product* $G \boxtimes H$, and the *direct product* $G \times H$, see [10].

All products have as vertex set the Cartesian set product $V(G) \times V(H)$. Two vertices (g_1, h_1), (g_2, h_2) are adjacent in $G \boxtimes H$ if (i) $[g_1, g_2] \in E(G)$ and $h_1 = h_2$, or (ii) $[h_1, h_2] \in E(G_2)$ and $g_1 = g_2$, or (iii) $[g_1, g_2] \in E(G)$ and $[h_1, h_2] \in E(G_2)$. Two vertices (g_1, h_1), (g_2, h_2) are adjacent in $G \square H$ if they satisfy only (i) or (ii), while these two vertices are adjacent in $G \times H$ if they satisfy only (iii).

Every finite connected graph G has a decomposition $G = \square_{i=1}^n G_i$, resp., $G = \boxtimes_{i=1}^n G_i$ into prime factors that is unique up to isomorphism and the order of the factors [22]. For the direct product an analogous result holds for non-bipartite connected graphs.

The mapping $p_i : V(\square_{i=1}^n G_i) \to V(G_i)$ defined by $p_i(v) = v_i$ for $v = (v_1, v_2, \ldots, v_n)$ is called *projection* on the i-th factor of G. By $p_i(W) = \{p_i(w) \mid w \in W\}$ the set of projections of vertices contained in $W \subseteq V(G)$ is denoted. An equivalence relation R on the edge set $E(G)$ of a Cartesian product $G = \square_{i=1}^n G_i$ of (not necessarily prime) graphs G_i is a *product relation* if $(e, f) \in R$ if and only if there exists a $j \in \{1, \ldots, n\}$ such that $|p_j(e)| = |p_j(f)| = 2$. The G_i-layer G_i^w of G is then the induced subgraph with vertex set $V(G_i^w) = \{v \in V(G) \mid p_j(v) = w_j, \text{ for all } j \neq i\}$. It is isomorphic to G_i.

Given two graphs G and H, a map $p : G \to H$ is called a *graph map* if p maps adjacent vertices of G to adjacent or identical vertices in H and edges of G to edges or vertices of H. A graph G is a *(Cartesian) graph bundle* if there are two graphs F, the fiber, and B, the base graph, and a graph map $p : G \to B$ such that: For each vertex $v \in V(B)$, $p^{-1}(v) \cong F$ and for each edge $e \in E(B)$ we have $p^{-1}(e) \cong K_2 \square F$.

3 RSP-Relations: Definition and Basic Properties

As mentioned in the introduction, relations that have the *square property* play a fundamental role for the $\square$-PFD of graphs. In particular, the relation δ is the unique, finest relation on $E(G)$ with the square property. For such relations two adjacent edges of different classes span exactly one chordless square and this square has opposite edges in the same equivalence classes. A mild generalization of the latter kind of relations are relations that have the *unique square property*. Here two adjacent edges e and f of different classes might span more than one square, however, there must be exactly one chordless square spanned by e and f with opposite edges in the same equivalence classes. As it turned out, a further generalization of such relations plays an important role for the characterization of certain properties of hypergraphs [20]. Here, we examine this generalization in realm of undirected graph in a systematic manner.

Definition 3.1. Let R be an equivalence relation on the edge set $E(G)$ of a connected graph G. We say R has the *relaxed square property* if any two adjacent edges e, f of G that belong to distinct equivalence classes of R span a square with opposite edges in the same equivalence class of R.

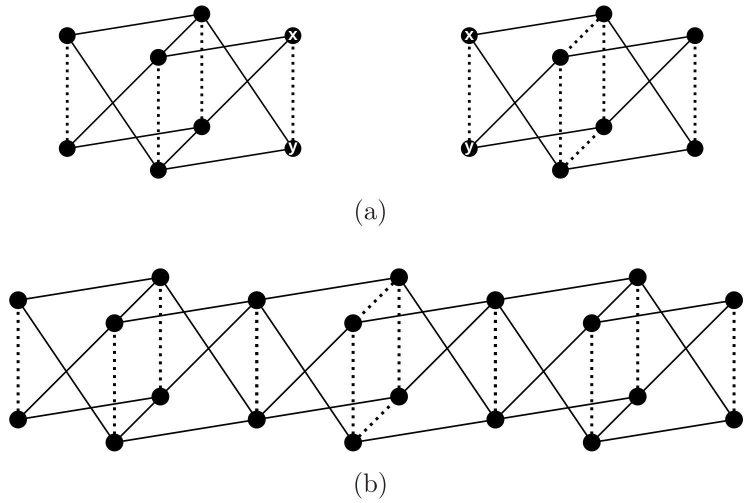

Figure 1: In Fig. (a) two isomorphic graphs with two non-equivalent finest RSP-relations are shown. Each RSP-relation has two equivalence classes, highlighted by dashed and solid edges. By stepwisely identifying the vertices marked with x and y, resp., one obtains a chain of graphs G, see Fig. (b). For each subgraph that is a copy of the graph above, a finest RSP-relation can be determined independently of the remaining parts of the graph G. Hence, with an increasing number of vertices of such chains G the number of finest RSP-relations is growing exponentially.

An equivalence relation R on $E(G)$ with the relaxed square property will be called an *RSP-relation* for short. In contrast to the more familiar (unique) square property, we do not require there that squares spanned by adjacent edges that belong to different equivalence classes are unique or chordless.

The following basic result was shown in [20] for hypergraphs and equivalence relations with the "relaxed grid property", of which graphs and RSP-relations are a special case.

Lemma 3.2 ([20])**.** *Let R be an RSP-relation on E of a connected graph $G = (V, E)$. Then each vertex of G is incident to at least one edge of each R-class and thus, the number of R-classes is bounded by the minimum degree of G. Moreover, if S is a coarser equivalence relation, $R \subseteq S$, then S is also an RSP-relation.*

For later reference we record the following technical result:

Lemma 3.3. *Let R be an RSP-relation on the edge set E of a connected graph $G = (V, E)$ and φ be an equivalence class of R. Moreover, let S be the equivalence relation on the edge set $E \setminus \varphi$ of the spanning subgraph $G' = (V, E \setminus \varphi)$ of G that retains all equivalence classes of R different from φ, i.e., $\psi \sqsubseteq S$ if and only if $\psi \sqsubseteq R$ and $\psi \neq \varphi$. Then S is an RSP-relation.*

Proof: Let e, f be adjacent edges in $E(G')$ such that $(e, f) \notin S$, say $e \in \psi, f \in \psi'$, $\varphi \neq \psi, \psi' \sqsubseteq S \subseteq R$. By construction, $e, f \in E(G)$ and $(e, f) \notin R$. Thus, there exists a

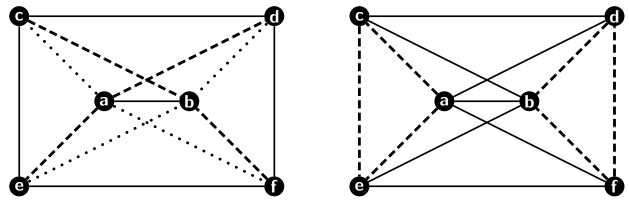

Figure 2: The two panels show two distinct finest RSP-relations R and S on a graph with different number of equivalence classes, see Example 1.

square with edges e, f, e', f' such that e, e' and f, f' are opposite edges and $e' \in \psi$ as well as $f' \in \psi'$. Hence, $e', f' \in E(G')$ and thus the assertion follows. $\Box$

The RSP-relation S on the spanning subgraph, as defined in Lemma 3.3, need not be a finest RSP-relation, although R might be a finest one. Consider the right graph in Figure 2. If S consists only of the class $\overline{\varphi}$ that is highlighted by the solid edges, then the spanning subgraph $H = (V(G), E(G) \setminus \varphi)$ is the Cartesian graph product of a path on three vertices and an edge. The finest RSP-relation on $E(H)$ is thus the product relation σ w.r.t. the unique $\Box$-PFD of H with two equivalence classes.

As the examples in Figures 1, 2 and 3 show, there is no unique finest RSP-relation for a given graph G and finest RSP-relations need not to have the same number of equivalence classes. Even more, the number of such finest relations on a graph can grow exponentially as the example in Figure 1 shows.

Example 1. There are graphs $G = (V, E)$ with distinct finest RSP-relations that even have a different number of equivalence classes. Consider the graph in Figure 2. We leave it to the reader to verify that the relations, whose equivalence classes are indicated by different line styles, indeed satisfy the relaxed square property. The RSP-relation on the left graph has three and on the right graph two equivalence classes. It remains to show, that both RSP-relations are finest ones.

Left Graph: For all equivalence classes there is a vertex that is incident to exactly one edge of each class. Lemma 3.2 implies that R is a finest RSP-relation.

Right Graph: Assume the relation is not finest. The equivalence class indicated by the dashed edges cannot be subdivided further since this would lead to vertices that are not met by each of the two or more subclasses, thus contradicting Lemma 3.2. The equivalence class depicted by solid edges is isomorphic to a Cartesian product $P_3 \Box K_2$. Using Lemma 3.3, the only possible split would be the product relation on this subgraph, i.e., with classes $\psi_1 = \{[a, b], [c, d], [e, f]\}$ and $\psi_2 = \{[a, d], [a, f], [b, c], [b, e]\}$. But then there is no square with opposite edges in the same equivalence classes spanned by the edges $[b, c]$ and $[c, e]$, again a contradiction.

We next discuss the relationship of (finest) RSP-relations with relations of the edge set that play a role in the theory of product graphs and graph bundles.

Definition 3.4 ([5])**.** Two edges $e = \{x, z\}$ and $f = \{z, y\}$ are in the *relation* τ, $e\tau f$ if z is the unique common neighbor of x and y.

In other words, two edges are in relation τ if they are adjacent and there is no square containing both of them. Obviously, τ is symmetric. Its reflexive and transitive closure, i.e. the smallest equivalence relation containing τ, will be denoted by τ^*. By definition, $\tau^* \subseteq R$ for any RSP-relation R.

Definition 3.5. Two edges $e, f \in E(G)$ are in the *relation* δ_0, $e\delta_0 f$, if one of the following conditions is satisfied:

(i) e and f are opposite edges of a square.

(ii) e and f are adjacent and there is no square containing e and f, i.e. $(e, f) \in \tau$.

(iii) $e = f$.

The relation δ_0 is reflexive and symmetric. Its transitive closure, denoted with δ_0^*, is therefore an equivalence relation.

Proposition 3.6. *Let G be a connected $K_{2,3}$-free graph and R an equivalence relation on $E(G)$. Then R has the relaxed square property if and only if $\delta_0 \subseteq R$.*

PROOF: It is easy to see, that δ_0^* has the relaxed square property and moreover, that any equivalence relation containing δ_0 has the relaxed square property.

Let R be an RSP-relation on the edge set of a connected $K_{2,3}$-free graph G. Notice, if G contains no $K_{2,3}$ then any pair of adjacent edges of G spans at most one square. Let e, f be two edges in G such that $(e, f) \in \delta_0$. We have to show that this implies $(e, f) \in R$. If $e = f$, then $(e, f) \in R$ is trivially fulfilled since R is an equivalence relation. If e and f are not adjacent, they have to be opposite edges of a square. Let g be an edge of this square that is adjacent to both edges e and f. If e and g are not in relation R, by the relaxed square property, they span some square with opposite edges in the same equivalence class. Since G contains no $K_{2,3}$, this square is unique, thus $(e, f) \in R$. Assume now, $(e, g) \in R$. If e and f are not in the same equivalence class of R, we can conclude that also f and g are in distinct equivalence classes, since R is an equivalence relation. Thus, by the relaxed square property, f and g span a square with opposite edges in the same equivalence class and as G is $K_{2,3}$-free, this square has to be unique, which implies $(e, f) \in R$, a contradiction. Now let e and f be two adjacent edges and suppose for contraposition $(e, f) \notin R$. Hence, e and f have to span a square. Thus, condition (ii) in the definition of δ_0 is not satisfied, hence, $(e, f) \notin \delta_0$. In summary, we can conclude $\delta_0 \subseteq R$. □

Proposition 3.6 implies that there is a uniquely determined finest RSP-relation, namely the relation δ_0^* if G is $K_{2,3}$-free. However, if G is not $K_{2,3}$-free, there is no uniquely determined finest RSP-relation, see Fig. 1, 2 and 3. Moreover, the quotient graphs that are induced by these relations (see [13, 20]) need not be isomorphic.

By construction, δ_0 places all edges of a $K_{2,3}$-subgraph in the same equivalence class. In many graphs this leads to an RSP-relation which is not finest. On the other hand, the

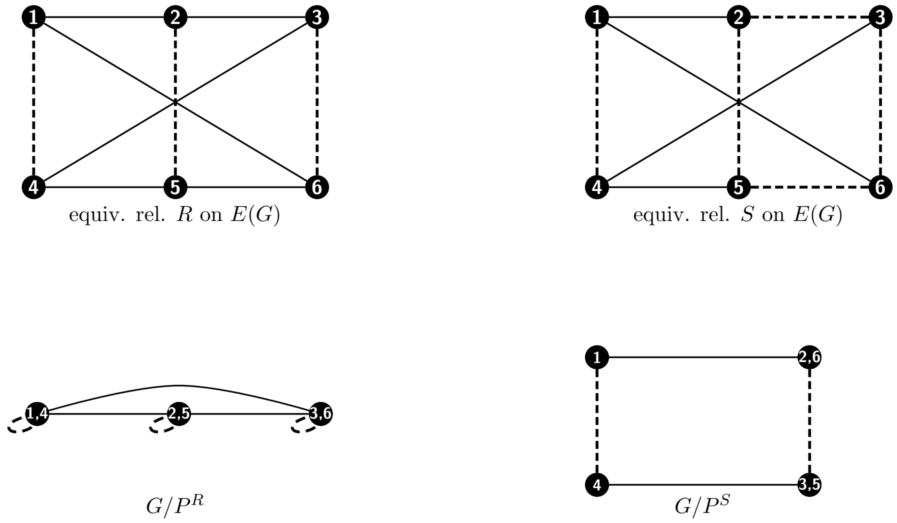

Figure 3: Two distinct RSP-relations R and S on the edge set of the same graph G and the quotient graphs induced by these relations (below). Their coarsest common refinement, i.e., the coarsest equivalence relation T with $T \subseteq R$ and $T \subseteq S$ does not have the relaxed square property. Moreover, the quotient graphs induced by these relations are not isomorphic.

opposite edges of a square that is not contained in a $K_{2,3}$ must always be in the same equivalence class. This motivates us to introduce the following

Definition 3.7. Two edges $e, f \in E(G)$ are in the *relation* δ_1, $e\delta_1 f$, if one of the following conditions is satisfied:

(i) e and f are opposite edges of a square that is not contained in any $K_{2,3}$ subgraph of G.

(ii) e and f are adjacent and there is no square containing e and f, i.e. $(e, f) \in \tau$.

(iii) $e = f$.

If G is $K_{2,3}$-free then it is easy to verify that $\delta_0 = \delta_1$. Proposition 3.6 implies that δ_1^* is contained in any RSP-relation and therefore, that it is a uniquely determined finest RSP-relation on $K_{2,3}$-free graphs. We can summarize this discussion of the properties of finest RSP-relations as follows:

Theorem 3.8. *Let G be an arbitrary graph and R be a finest RSP-relation on $E(G)$. Then the following holds:*

$$\delta_1^* \subseteq R \subseteq \delta_0^*.$$

Moreover, if G is $K_{2,3}$-free, then $\delta_1^* = R = \delta_0^*$.

Theorem 3.8 suggests that $K_{2,3}$-subgraphs are to blame for complications in understanding RSP-relations. It will therefore be useful to consider a subclass of RSP-relations that

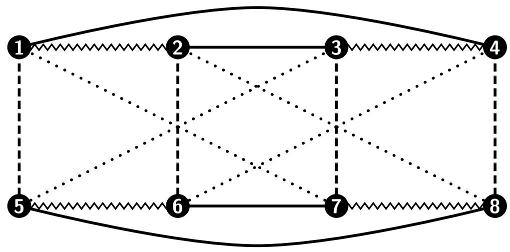

Figure 4: The well-behaved RSP-relation R on the edge set $E(G)$ of the "diagonalized cube" G has the four equivalence classes $\varphi_1, \varphi_2, \varphi_3$ and φ_4 depicted by solid, zigzag, dotted and dashed edges, respectively. In addition, R satisfies the unique square property. The relation R' with classes φ_3, φ_4 and $\psi_1 = \varphi_1 \cup \varphi_2$, however, is not well-behaved, because the $K_{2,3}$-subgraph with partition $\{1, 6\}$ and $\{2, 4, 5\}$ has a forbidden coloring. Note, R' has the unique square property.

are "well-behaved" on $K_{2,3}$-subgraphs. They will turn out to play a crucial role to establish the connection of RSP-relations, (quasi-)covers, and equitable partitions. We fix the notation for $K_{2,3}$ so that $\{x, y\}, \{a, b, c\}$ is the canonical partition of the vertex set. We say that graph $K_{2,3}$ has a *forbidden coloring* if the edges $[a, x]$, $[x, c]$, and $[y, b]$ are in one equivalence class φ and the other edges are in the union $\overline{\varphi}$ of the classes different from φ.

Definition 3.9. An RSP-relation is *well-behaved* (on G) if G does not contain a subgraph isomorphic to a $K_{2,3}$ with a forbidden coloring.

For a graph G and an RSP-relation R consisting of only two equivalence classes we can strengthen this definition. It is easy to verify that in this case the two statements are equivalent:

(i) R is well-behaved

(ii) for each pair of adjacent edges $[a, b]$, $[a, c]$ which are not in relation R there exists a unique (not necessarily chordless) square $a - b - d - c$ with opposite edges in the same classes, i.e., $([a, b], [c, d]), ([a, c], [b, d]) \in R$.

In the general case (i) implies (ii). To see this, note that if there are adjacent edges that span more than one square, say SQ_1 and SQ_2, with opposite edges in the same classes, then there is a $K_{2,3}$ with forbidden coloring that consists of the squares SQ_1 and SQ_2. Hence, R cannot be well-behaved. The converse is not true in general, as shown in Fig. 4. by the non-well-behaved RSP-relation R' that nevertheless has the property (ii).

To obtain well-behaved RSP-relations R on G one can simply use δ_0 and coarsenings of it. That is, *any* equivalence relation R with $\delta_0 \subseteq R$ is well-behaved. In this case, all edges of any $K_{2,3}$-subgraph are in the same equivalence class. However, coarsenings of arbitrary well-behaved RSP-relation R need not be well-behaved, see Fig. 4.

Furthermore, if R is not well-behaved, this is equivalent to the existence of squares with two adjacent edges in same class $\varphi \sqsubseteq R$ and others in class(es) different from φ, see Figure 5

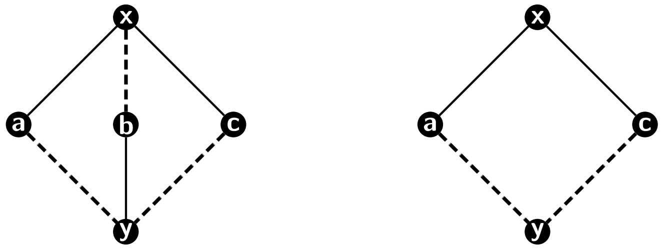

Figure 5: Forbidden coloring of a (sub)graph isomorphic to $K_{2,3}$ based on the classes φ and $\overline{\varphi}$ of a (non-well-behaved) RSP-relation. The class $\overline{\varphi}$ might consist of more than one equivalence class. The existence of a forbidden coloring is equivalent to the existence of squares spanned by edges in same equivalence class with opposite edges in different equivalence classes. Such a square contained in the (sub)graph $K_{2,3}$ is shown on the right.

and the next explanations. It is easy to verify that any $K_{2,3}$(-subgraph) with a forbidden coloring contains such a square. By way of example, consider the square $a-x-c-y$ in Figure 5. Conversely, let R be an RSP-relation on $E(G)$ and suppose that G contains a square $a-x-c-y$ with $([a,x],[c,x]) \in R$ and $([a,x],[c,y]),([a,y],[c,x]) \notin R$. By the relaxed square property, $[a,x]$ and $[c,y]$ span a square, say $a-x-b-y$ with opposite edges in the same equivalence class. Hence, there is a complete bipartite graph $K_{2,3}$ with partition $\{x,y\}$ and $\{a,b,c\}$ of $V(K_{2,3})$ and forbidden coloring.

Let us now turn to the computational aspects of RSP-relations. It is an easy task to determine finest relations that have the square property in polynomial time, see [11, 12]. In contrast, it seems to be hard in general to determine one or all finest RSP-relations. We conjecture that the corresponding decision problem is NP- or GI-hard for general graphs. For definitions of NP- and GI-hard see [8, 18].

On the other hand, an efficient polynomial-time solution exists for $K_{2,3}$-free graphs since δ_0 can be constructed efficiently, e.g., by listing all squares [3]. Algorithm 1 serves as a heuristic to find a finest RSP-relation for general graphs. The basic idea is to start from the lower bound $R=\delta_1^*$ and to unite equivalence classes of R stepwisely until an RSP-relation is obtained.

Proposition 3.10. *Let $G=(V,E)$ be a given graph with maximum degree Δ. Algorithm 1 computes an RSP-relation R on E in $O(|V||E|^2\Delta^4)$ time. If G is $K_{2,3}$-free, then Algorithm 1 computes a finest RSP-relation on E.*

PROOF: Clearly, δ_1^* must be contained in every RSP-relation R. The set Q contains all adjacent candidate edges (e,f), where we have to ensure that they span a square with opposite edges in the same equivalence class. Since we already computed $\tau \subseteq \delta_1$, we can conclude that if e and f are contained in Q, then they span some square. Thus, in Line 10 we check whether there are opposite edges e_i of e and f_i of f in one of those squares spanned by e and f with $(e,e_i),(f,f_i) \in R_j^*$, i.e., e_i and e, resp., f_i and f are in the same equivalence class. If so, we can safely remove (e,f) from Q. If not, we will construct a

Algorithm 1 `Compute RSP-Relation`

1: **INPUT:** A connected graph $G = (V, E)$
2: Compute $R_0 = \delta_1^*$;
3: $Q \leftarrow \{(e, f) \mid e, f \in E, e \cap f \neq \emptyset\} \setminus R_0$;
4: $j \leftarrow 0$;
5: {Note, edges e and f with $(e, f) \in Q$ are adjacent, span a square and are necessarily distinct}
6: **while** $Q \neq \emptyset$ **do**
7: Take an arbitrary pair $(e, f) \in Q$ with $e \cap f \neq \emptyset$;
8: Let $sq_1, \dots, sq_k$ be the squares spanned by e and f;
9: Find the opposite edges e_i of e and f_i of f in sq_i;
10: **if** there is a square sq_i with $(e, e_i) \in R_j^*$ and $(f, f_i) \in R_j^*$ **then**
11:   $Q \leftarrow Q \setminus \{(e, f), (f, e)\}$;
12: **else**
13:   take an arbitrary square, say sq_1 {with edge set $E_0 = (e, f, e_1, f_1)$};
14:   $R_{j+1} \leftarrow R_j^* \cup \{(e, e_1), (e_1, e), (f_1, f), (f, f_1)\}$;
15:   compute R_{j+1}^*;
16:   $Q \leftarrow Q \setminus R_{j+1}^*$;
17:   $j \leftarrow j + 1$;
18: **end if**
19: **end while**
20: $R \leftarrow R_j^*$
21: **OUTPUT:** An RSP-relation R on E;

square spanned by e and f with opposite edges in the same class and the pair (e, f) will be removed from Q in the next run of the while-loop (Line 11). To be more precise, we take one of those squares spanned by e and f and add (e, e_i) and (f, f_i) to R_j resulting in R_{j+1}. Hence, e and f now span a square with opposite edges in the same class. We then compute the transitive closure R_{j+1}^*. This might result in new pairs $(a, b) \in R_{j+1}^*$ of adjacent edges, which can safely be removed from Q since they are in the same equivalence class, and thus need not span a square with opposite edges in the same class. Hence, we compute $Q \leftarrow Q \setminus R_{j+1}^*$. When Q is empty all adjacent pairs (which span at least one square) are added in a way that at least one square has opposite edges in the same equivalence class. Thus, R satisfies the relaxed square property. Note, if G is $K_{2,3}$-free, then all pairs (e, f) of adjacent edges e and f already span a square with opposite edges in the same class, due to δ_1. Hence, all such pairs (e, f) will be removed from Q, without adding any new pair to R_0^*. In this case we obtain $R = \delta_1^*$.

In order to determine the time complexity we first consider the relation δ_1. Note that there are at most $O(|E|\Delta^2)$ squares in a graph, that can be listed efficiently in $O(|E|\Delta)$ time, see Chiba and Nishizeki [3]. For the computation of δ_1, we first have to check for each square $a - b - c - d$ whether it is contained in a $K_{2,3}$ subgraph or not. Thus, we need to verify whether a and c have a common neighbor $x \notin \{b, d\}$, and, if b and d have a common neighbor $x \notin \{a, c\}$, respectively. If none of the cases occur, i.e., the square is not part of a $K_{2,3}$ subgraph, then we put the pairs $([a, b], [c, d])$ and $([a, d], [b, c])$ to δ_1. This task can be done in $O(\Delta^2)$ time for each square, resulting in an overall time complexity of $O(|E|\Delta^4)$. The relation $\tau \subseteq \delta_1$ can then be computed in $O(|V||E|)$ time [10, Prop. 23.5] and the transitive closure δ_1^* in $O(|E|^2)$ time, [10, Prop. 18.2]. Thus, we end in time complexity $O(|E|^2\Delta^4)$ for the computation of δ_1^*. Finally, we have to check for the at most $|V|\Delta^2$ pairs of adjacent edges whether they already span a square with opposite edges in the same class or not and compute the transitive closure R_{j+1}^* if necessary. Since there are at most $|E|\Delta^2$ squares, $|E| \leq |V|\Delta$, and the transitive closure can be computed in $O(|E|^2)$ time, the latter task can be done in $O(|V||E|^2\Delta^3)$ time. $\square$

As the following example shows, the order in which the squares are examined does matter in the general case, hence Alg. 1 does not produce a finest RSP-relation in general.

Example 2. Consider the complete graph $K_5 = (V, E)$ with vertex set $V = \mathbb{Z}_5$ and natural edge set. After the init step we have $R_0 = \{(e, e) \mid e \in E\}$ and hence, Q contains all pairs of adjacent edges. To obtain a finest RSP-relation, we could start with the pair $([0, 1][1, 4]) \in Q$ that span the square $0 - 1 - 4 - 3$ get as classes $\varphi_1 = \{[0, 1], [3, 4]\}$ and $\varphi_2 = \{[1, 4], [0, 3]\}$ of R_1^*. Continuing with $([0, 1][1, 2]) \in Q$ and the square $0-1-2-3$, we obtain the classes $\varphi_1 \cup \{[2, 3]\}$ and $\varphi_2 \cup \{[1, 2]\}$ of R_2^*. Next, take $([0, 1][0, 4]) \in Q$ and the square $0-1-2-4$, followed by the pair $([0, 1][0, 2]) \in Q$ and the square $0-1-4-2$, resulting in the classes $\varphi_1 = \{[0, 1][2, 3], [3, 4], [2, 4]\}$ and $\varphi_2 = \{[0, 2], [0, 3], [0, 4], [1, 2], [1, 4]\}$ for R_4^*. Finally, take $([0, 1][1, 3]) \in Q$ and the square $0 - 1 - 3 - 4$ to obtain the classes φ_1 and $\varphi_2 \cup \{[1, 3]\}$ for a valid finest RSP-relation, see Example 3 for further details. Note, the computed RSP-relation is not well-behaved.

However, if we start with the pair $([0, 1][0, 4]) \in Q$ and square $0 - 1 - 3 - 4$, followed by $([1, 2][1, 3]) \in Q$ and $1 - 2 - 4 - 3$, then $([1, 4][3, 4]) \in Q$ and $1 - 2 - 3 - 4$, next $([0, 1][0, 3]) \in Q$ and $0 - 1 - 2 - 3$ and finally $([0, 2][2, 3]) \in Q$ and $0 - 2 - 3 - 4$, the resulting RSP-relation has only one equivalence class.

4 RSP-Relations and Graph Products

Graph products are intimately related with the square property. It seems natural, therefore to ask whether finest RSP-relations can be found more easily in products. We use the symbol $\circledast$ for one of the three graph products defined in Section 2.

Definition 4.1. For $\circledast \in \{\square, \boxtimes, \times\}$ let $G = \circledast_{i\in I} G_i$. For each $i \in I$ let R_i be an equivalence relation on $E(G_i)$. Furthermore, define for $e \in E(G)$ the set $I_e := \{i \in I \mid p_i(e) \in E(G_i)\}$. We define an equivalence relation $\circledast_{i\in I} R_i$ on $E(G)$ as follows: $(e, f) \in \circledast_{i\in I} R_i$ if and only if $I_e = I_f$ and $(p_i(e), p_i(f)) \in R_i$, for all $i \in I_e$.

If $\circledast = \square$ then $|I_e| = 1$ for all $e \in E(G)$, and if $\circledast = \times$ then $I_e = I$ for all $e \in E(G)$.

Lemma 4.2. *For $\circledast \in \{\Box, \boxtimes, \times\}$ let $G = \circledast_{i\in I} G_i$. For each $i \in I$ let R_i be an equivalence relation on $E(G_i)$. Then $R := \circledast_{i\in I} R_i$ is an RSP-relation if and only if R_i is an RSP-relation for all $i \in I$.*

PROOF: First suppose R_i has the relaxed square property for all $i \in I$. We have to show that R has the relaxed square property. Therefore, let $e = [x, y], f = [x, z] \in E(G)$ such that $(e, f) \notin R$. We need to show that there exists a vertex $w \in V(G)$ such that $e' = [w, z] \in E(G)$, $f' = [w, y] \in E(G)$ and $(e, e') \in R$ as well as $(f, f') \in R$.

Let $I_0 := \{i \in I \mid (p_i(e), p_i(f)) \in R_i\}$. Notice, that $I_0 \subseteq I_e \cap I_f$. Moreover, we have $(p_j(e), p_j(f)) \notin R_j$ for all $j \in (I_e \cap I_f) \setminus I_0 =: I^*$. Since R_i has the relaxed square property for all $i \in I$, for all $j \in I^*$ there exists a vertex $w_j \in V(G_j)$ such that $(p_j(e), [p_j(z), w_j]) \in R_j$ as well as $(p_j(f), [p_j(y), w_j]) \in R_j$.

Let $w \in V(G)$ such that

$$\begin{aligned} p_i(w) &= p_i(x), && \text{for all } i \in I_0 \text{ and} \\ p_i(w) &= w_i, && \text{for all } i \in I^* \text{ and} \\ p_i(w) &= p_i(z), && \text{for all } i \in I \setminus I_e \text{ and} \\ p_i(w) &= p_i(y), && \text{for all } i \in I \setminus I_f. \end{aligned}$$

Since $I = I_0 \mathbin{\dot\cup} I^* \mathbin{\dot\cup} (I \setminus (I_e \cap I_f))$, $I \setminus (I_e \cap I_f) = I \setminus I_e \cup I \setminus I_f$ and $p_i(z) = p_i(x) = p_i(y)$ for all $i \in I \setminus I_e \cap I \setminus I_f$, this vertex exists in $V(G)$ and is well defined.

We now have to verify that w has the desired properties. More precisely, we have to verify the following statements:

(i) $p_i(w) = p_i(z)$ for all $i \in I \setminus I_e$,

(ii) $p_i(w) = p_i(y)$ for all $i \in I \setminus I_f$,

(iii) $e'_i := [p_i(z), p_i(w)] \in E(G_i)$ and $(p_i(e), e'_i) \in R_i$ for all $i \in I_e$,

(iv) $f'_i := [p_i(y), p_i(w)] \in E(G_i)$ and $(p_i(f), f'_i) \in R_i$ for all $i \in I_f$.

Assertions (i) and (ii) are trivially fulfilled by construction. To prove assertion (iii), note that $I_e = I_0 \dot\cup I^* \dot\cup (I_e \setminus I_f)$. From $p_i(w) = p_i(x)$ for all $i \in I_0$, we conclude $e'_i = [p_i(z), p_i(x)] = p_i(f) \in E(G_i)$, and moreover, by construction of I_0 and since R_i is an equivalence relation, we have $(p_i(e), e'_i) \in R_i$ for all $i \in I_0$. By the choice of w, it follows that $e'_i \in E(G_i)$ and $(p_i(e), e'_i) \in R_i$ for all $i \in I^*$. Finally, we have $e'_i = [p_i(z), p_i(y)] = [p_i(x), p_i(y)] = p_i(e) \in E(G_i)$ for all $i \in I_e \setminus I_f$ and since R_i is an equivalence relation, $(e'_i, p_i(e)) \in R_i$. Thus, $e' = [w, z] \in E(G)$ and $(e, e') \in R$.

Assertion (iv), which implies $f' = [w, y] \in E(G)$ and $(f, f') \in R$, can be shown analogously.

Now suppose R is an RSP-relation. We have to show that for all $i \in I$, R_i has the relaxed square property. Therefore, let $i \in I$ and $e_i = [x_i, y_i], f_i = [x_i, z_i]$ be two adjacent edges in G_i such that $(e_i, f_i) \notin R_i$. We need to show, that there exists some vertex $w_i \in V(G_i)$ such that $e'_i := [w_i, z_i], f'_i := [w_i, y_i]$ are edges in G_i with $(e_i, e'_i) \in R_i$ and $(f_i, f'_i) \in R_i$. By definition of $\circledast$, there exist edges $e = [x, y], f = [x, z] \in E(G)$, $p_i(x) = x_i, p_i(y) = y_i, p_i(z) = z_i$, with $p_i(e) = e_i$ and $p_i(f) = f_i$, that are adjacent. It holds that $i \in I_e \cap I_f$

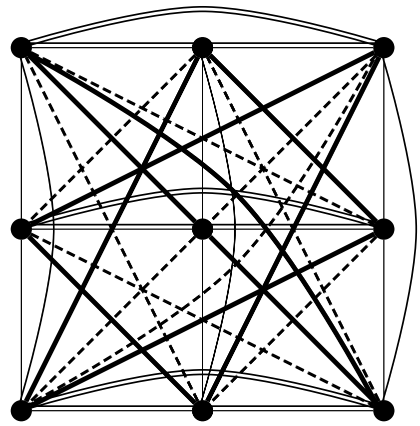

Figure 6: Refinement of product of relations of K_9 w.r.t. $K_9 \cong K_3 \boxtimes K_3$

and by definition of R, $(e, f) \notin R$. Since R has the relaxed square property, there exists some vertex $w \in V(G)$ such that $e' := [w, z], f' := [w, y]$ are edges in G with $(e, e') \in R$ and $(f, f') \in R$. That is, by definition of R, $I_e = I_{e'}$ and $(p_j(e), p_j(e')) \in R_j$ for all $j \in I_e$ as well as $I_f = I_{f'}$ and $(p_j(f), p_j(f')) \in R_j$ for all $j \in I_f$. Thus, we have in particular $(e_i, p_i(e')), (f_i, p_i(f')) \in R_i$ and $z_i \neq p_i(w) \neq y_i$. Moreover, $p_i(w) \neq x_i$, since otherwise $p_i(e') = [p_i(w), p_i(z)] = [x_i, z_i] = f_i$ and therefore $(f_i, e_i) = (p_i(e'), p_i(e)) \in R_i$ must hold, a contradiction. Hence, with $w_i := p_i(w)$ the assertion follows. $\square$

For $\circledast \in \{\times, \boxtimes\}$, the relation $R = \circledast_{i \in I} R_i$ need not be the finest RSP-relation on $E(G) = E(\circledast_{i \in I} G_i)$ although R_i is a finest RSP-relation on $E(G_i)$ for all $i \in I$. See Fig. 6 for an example: Shown is the complete graph K_9 with a finest RSP-relation consisting of four equivalence classes depicted by solid, double, dashed and thick lines. Joining the two classes with dashed and thick edges to one class, one gets a coarser relation $R_1 \boxtimes R_2$, w.r.t. $K_9 \cong K_3 \boxtimes K_3$ where R_i denotes the trivial relation on $E(K_3)$. This together with Lemma 3.3 implies that also $R_1 \times R_2$ is not a finest RSP-relation on $E(K_3 \times K_3)$.

However, this does not hold for the Cartesian product $\Box$. Moreover, we have:

Lemma 4.3. *Let $G = \Box_{i \in I} G_i$ be a connected and simple graph. Then R is a finest RSP-relation on $E(G)$ if and only if $R = \Box_{i \in I} R_i$ where each R_i is a finest RSP-relation on $E(G_i)$.*

Proof: First, observe the following: Let R' be an arbitrary RSP-relation on G and $[x, y], [y, z] \in E(G)$ adjacent edges that lie in the same layer of G, i.e. $p_j([x, y]) \in E(G_j)$ and $p_j([y, z]) \in E(G_j)$ for some $j \in I$. Moreover, let $[x, y]$ and $[y, z]$ be in different equivalence classes of R'. Since R' is an RSP-relation, they lie on a four cycle $x - y - z - w$ with opposite edges in the same equivalence class. By the definition of the Cartesian product w is also in the same layer as x, y, z, that is $w \in V(G_j^y)$. This shows that R' limited to subgraph G_j^y is also an RSP-relation.

Let now $[x, y], [w, z] \in E(G)$ be such edges that lie on a four cycle $x - y - z - w$ with $j \in I$ such that $p_j([x, y]) = p_j([w, z]) \in E(G_j)$. Assume that $[x, y]$ and $[w, z]$ do not lie in the same equivalence class of R'. Then at least one of the pairs $[x, y], [x, w]$ or $[w, z], [x, w]$ does not lie in the same equivalence class of R'. Without loss of generality let $[x, y]$ and $[x, w]$ lie in different equivalence classes of R'. By the definition of the Cartesian product, $x - y - z - w$ is the only four cycle that contains $[x, y]$ and $[x, w]$. Since R' is an RSP-relation, $[x, y]$ and $[w, z]$ lie in the same equivalence class. By connectedness of G, all layers are connected. Therefore, all edges $\{[a, b] \in E(G) : p_j([a, b]) = p_j([x, y])\}$ are in the same equivalence class.

Assume now that R is a finest RSP-relation on G. We define relation R_j on $E(G_j)$ for every $j \in I$ by $(e, f) \in R_j$ for $e, f \in E(G_j)$ if $p_j(e') = e$, $p_j(f') = f$ for some $e', f' \in E(G)$ and $(e', f') \in R$. By above arguments, this is an RSP-relation on G_j. Notice that R corresponds to $\Box_{i \in I} R_i$ with possibly some joint equivalence classes, that emerge from different layers of $\Box_{i \in I} G_i$. Since R is a finest RSP-relation, $R = \Box_{i \in I} R_i$. If R_j is not a finest RSP-relation on G_j for some $j \in I$, then the product of a finer relation on G_j with $\Box_{i \in I \setminus \{j\}} R_i$ is a finer relation as R, a contradiction.

To see the converse, let $R = \Box_{i \in I} R_i$, where R_i is a finest RSP-relation on G_i. If Q is a finest relation on G, that is finer than R, by above arguments, $Q = \Box_{i \in I} Q_i$, where Q_i is finer or equal than R_i for every $i \in I$. Thus $Q = R$. $\Box$

Lemma 4.3 implies not only that $R = \Box_{i \in I} R_i$ is a finest RSP-relation on $E(G) = E(\Box_{i \in I} G_i)$ if R_i is a finest RSP-relation on $E(G_i)$, but also that any (finest) RSP-relation on a Cartesian product graph must reflect the layers w.r.t. its (prime) factorization. However, this is not true for $\circledast = \boxtimes$, as an example take $K_6 \cong K_3 \boxtimes K_2$ with the relation defined in Example 3.

Following [13], we introduce vertex partitions associated with an equivalence relation R on $E(G)$. In particular, we define for an equivalence class $\varphi \sqsubseteq R$ the partitions

$$\mathcal{P}^R_{\varphi} := \left\{ V(G^x_{\varphi}) \mid x \in V(G) \right\} \text{ and } \mathcal{P}^R_{\overline{\varphi}} := \left\{ V(G^x_{\overline{\varphi}}) \mid x \in V(G) \right\}.$$

Graham and Winkler showed in [9] that the Djoković-Winkler relation, or more precisely, the equivalence relation $R = \theta^*$ on $E(G)$ induces a canonical isometric embedding of a graph G into a Cartesian product $\Box_{\varphi \sqsubseteq R} G_{\varphi} / \mathcal{P}^R_{\overline{\varphi}}$. Moreover, Feder [5] showed that if we choose $R = (\theta \cup \tau)^*$ then $G \cong \Box_{\varphi \sqsubseteq R} G_{\varphi} / \mathcal{P}^R_{\overline{\varphi}}$ and thus, R coincides with the product relation σ.

In [20], we demonstrated that if R is an RSP-relation then

$$G / \mathcal{P}^R \cong \Box_{\varphi \sqsubseteq R} G_{\varphi} / \mathcal{P}^R_{\overline{\varphi}}, \tag{4.1}$$

where $\mathcal{P}^R$ denotes the common refinement of the partitions $\mathcal{P}^R_{\overline{\varphi}}$, $\varphi \sqsubseteq R$, i.e.,

$$\mathcal{P}^R := \left\{ \bigcap_{\varphi \sqsubseteq R} V(G^x_{\overline{\varphi}}) \mid x \in V(G) \right\},$$

which is again a partition of $V(G)$.

Lemma 4.4. *For $i \in I$ let G_i be connected graphs and let R_i be an RSP-relation on the edge set $E(G_i)$. The following hold.*

($\Box$) *If $G = \Box_{i\in I} G_i$ and $R = \Box_{i\in I} R_i$ then $G/\mathcal{P}^R = \Box_{i\in I} G_i/\mathcal{P}^{R_i}$.*

($\boxtimes$) *If $G = \boxtimes_{i\in I} G_i$ and $R = \boxtimes_{i\in I} R_i$ then $G/\mathcal{P}^R = \mathcal{L}K_1$.*

Proof: ($\Box$) By construction of R, $\psi \subseteq E(G)$ is an equivalence class of R if and only if there exists an $i \in I$ such that $p_i(e) \in E(G_i)$ and there exists $\varphi \in R_i$ with $p_i(e) \in \varphi$ for all $e \in \psi$. Hence, there exists a bijection $R = \circledast_{i\in I} R_i \to \dot{\bigcup}_{i\in I} R_i$. For $i \in I$ let $\varphi_1^i, \ldots, \varphi_{n_i}^i$ be the equivalence classes of R_i. Moreover, for $i \in I$ and $1 \leq j \leq n_i$ let ψ_j^i be the equivalence class of R such that $I_e = \{i\}$ and $p_i(e) \in \varphi_j^i$ for all $e \in \psi_j^i$. Thus, with Equation (4.1), we obtain $G/\mathcal{P}^R = \Box_{\psi \sqsubseteq R} G_\psi/\mathcal{P}^R_{\overline{\psi}} = \Box_{i\in I}(\Box_{j=1}^{n_i} G_{\psi_j^i}/\mathcal{P}^R_{\overline{\psi_j^i}})$. Furthermore, due to Equation (4.1), we have $\Box_{i\in I} G_i/\mathcal{P}^{R_i} = \Box_{i\in I}(\Box_{j=1}^{n_i} G_{i\varphi_j^i}/\mathcal{P}^{R_i}_{\overline{\varphi_j^i}})$.

Hence, we need to show $G_{\psi_j^i}/\mathcal{P}^R_{\overline{\psi_j^i}} \cong G_{i\varphi_j^i}/\mathcal{P}^{R_i}_{\overline{\varphi_j^i}}$ for all $i \in I$ and $1 \leq j \leq n_i$, to prove the assertion. Therefore, we show that $G_{\overline{\psi_j^i}}(x) \mapsto G_{i\overline{\varphi_j^i}}(p_i(x))$ for all $x \in V(G)$ defines an isomorphism $G_{\psi_j^i}/\mathcal{P}^R_{\overline{\psi_j^i}} \cong G_{i\varphi_j^i}/\mathcal{P}^{R_i}_{\overline{\varphi_j^i}}$. If $G_{\overline{\psi_j^i}}(x) = G_{\overline{\psi_j^i}}(y)$, there exists a path $P_{x,y} := (e_1, \ldots, e_k)$ from x to y in G, such that $e_l \notin \psi_j^i$ for $1 \leq l \leq k$. Then $p_i(P_x, y) = (p_1(e_1), \ldots, p_i(e_k))$ is a walk from $p_i(x)$ to $p_i(y)$ in G_i and by construction, the following holds: $p_i(e_l) \notin \varphi_j^i$ for $1 \leq l \leq k$, i.e., $G_{i\overline{\varphi_j^i}}(p_i(x)) = G_{i\overline{\varphi_j^i}}(p_i(y))$. Thus, this mapping is well defined. Moreover, by the projection properties of a Cartesian product into its factors, this mapping is surjective. Now, suppose $G_{i\overline{\varphi_j^i}}(p_i(x)) = G_{i\overline{\varphi_j^i}}(p_i(y))$, i.e., there exists a path $P_{p_i(x),p_i(y)} := (e_1, \ldots, e_k)$ from $p_i(x)$ to $p_i(y)$ in G_i such that $e_l \notin \varphi_j^i$ for $1 \leq l \leq k$. Let $w \in V(G)$ s.t. $p_i(w) = p_i(y)$ and $p_r(w) = p_r(x)$ for all $r \in I$, $r \neq i$. Hence, $w \in V(G_i^x)$. Thus, there exists a path $P'_{x,w} = (e'_1, \ldots, e'_k)$ in G with $p_i(e'_l) = e_l$ which implies $e'_l \notin \psi_j^i$ for $1 \leq l \leq k$ and thus $G_{\overline{\psi_j^i}}(x) = G_{\overline{\psi_j^i}}(w)$. Furthermore, by the properties of the Cartesian product, there exists a path $P''_{w,y} = (e''_1, \ldots, e''_s)$ from w to y in G such that $|p_i(e''_l)| = 1$ for $1 \leq l \leq s$, which implies $I_{e''_l} \neq \{i\}$ and consequently $e''_l \notin \psi_j^i$ for $1 \leq l \leq k$. Thus, $G_{\overline{\psi_j^i}}(y) = G_{\overline{\psi_j^i}}(w) = G_{\overline{\psi_j^i}}(x)$, that is, this mapping is injective and therefore bijective. It remains to show that $[G_{\overline{\psi_j^i}}(x), G_{\overline{\psi_j^i}}(y)]$ is an edge in $G_{\psi_j^i}/\mathcal{P}^R_{\overline{\psi_j^i}}$ if and only if $[G_{i\overline{\varphi_j^i}}(p_i(x)), G_{i\overline{\varphi_j^i}}(p_i(y))]$ is an edge in $G_{i\varphi_j^i}/\mathcal{P}^{R_i}_{\overline{\varphi_j^i}}$. By definition, $[G_{\overline{\psi_j^i}}(x), G_{\overline{\psi_j^i}}(y)]$ is an edge in $G_{\psi_j^i}/\mathcal{P}^R_{\overline{\psi_j^i}}$ if and only if there exists $x' \in V(G_{\overline{\psi_j^i}}(x)), y' \in V(G_{\overline{\psi_j^i}}(x))$ s.t. $[x', y'] \in \psi_j^i$, which, by the preceding and by construction, is equivalent to $p_i(x') \in V(G_{i\overline{\varphi_j^i}}(p_i(x))), p_i(y') \in V(G_{i\overline{\varphi_j^i}}(p_i(y)))$ and $[p_i(x'), p_i(y')] \in \varphi_j^i$, from what the assertion follows.

($\boxtimes$) To prove the assertion, we have to show that the spanning subgraph $G_{\overline{\varphi}}$ is connected for all $\varphi \sqsubseteq R$. For each $\varphi \sqsubseteq R$ we have $I_e = I_f$ for all $e, f \in \varphi$. We set $I_\varphi := I_e$ for some $e \in \varphi$. Moreover, define $\Phi := \{\psi \sqsubseteq R \mid I_\psi = I_\varphi\}$ Then for $\alpha := \bigcup_{\psi\in\Phi} \psi$, $G_{\overline{\alpha}}$ is a spanning subgraph of $G_{\overline{\varphi}}$. Therefore, it suffices to show that $G_{\overline{\alpha}}$ is connected. To be more precise, we have to show that for all $x, y \in V(G)$, there exists a walk $W_{x,y}$ from x to y in G such that for all $e \in E(W_{x,y})$ we have $I_e \neq I_\varphi$.

First, assume $|I_\varphi| > 1$. Since $\Box_{i\in I} G_i$ is a connected spanning subgraph of $\boxtimes_{i\in I} G_i$, there exists a walk $W_{x,y}$ from x to y in $\Box_{i\in I} G_i$. Then for all $e \in E(W_{x,y})$ we have $|I_e| = 1$ and thus, $I_e \neq I_\varphi$.

Now, let $|I_\varphi| = 1$, i.e., $I_\varphi = \{j\}$ for some $j \in I$. If $p_j(x) = p_j(y)$, then $y \in V((\Box_{i\in I\setminus\{j\}} G_i)^x)$. In this case, there exists a walk $W_{x,y}$ from x to y in $(\Box_{i\in I\setminus\{j\}} G_i)^x$ that has the desired properties. If $p_j(x) \neq p_j(y)$, let $y' \in V(G)$ such that $p_i(y') = p_i(x)$ for all $i \neq j$ and $p_j(y) = p_j(y')$. Then, as in the previous case, there exists a walk $W_{y,y'}$ from y to y' in $(\Box_{i\in I\setminus\{j\}} G_i)^y$ and hence $I_e \neq \{j\}$ for all $e \in E(W_{y,y'})$. By choice of y', it follows that $y' \in V(G_j^x)$. Let $P_{x,y'} := (x = x_0, x_1, \ldots, x_k = y')$ be a walk from x to y' that is entirely contained in G_i^x. Moreover, for arbitrary $i \in I$ with $i \neq j$ let $z \in V(G_i^x)$ such that $[p_i(x), p_i(z)] \in E(G_i)$ and let $w \in V(G_j^z)$ such that $p_j(w) = p_j(z)$. Then there exists a walk $P_{z,w} := (z = z_0, z_1, \ldots, z_k = w)$ from z to w in G_j^z such that $p_j(x_r) = p_j(z_r)$ for all $0 \leq r \leq k$. By definition of $\boxtimes$, $W_{x,y'} := (x_0, z_1, x_1, z_2, x_2, z_3, \ldots, x_{k-1}, z_k = w, x_k = y')$ is a walk from x to y' in G and for the edges $e \in E(W_{x,y'})$ we have $I_e = \{i, j\} \neq \{j\} = I_\varphi$ if e is of the form $[x_i, z_{i+1}]$, $0 \leq i \leq k-1$ and $I_e = \{i\} \neq \{j\} = I_\varphi$ if e is of the form $[x_i, z_i]$, $0 \leq i \leq k$. Hence, $W_{x,y} = W_{x,y'} \cup W_{y',y}$ is a walk from x to y that has the desired properties. $\Box$

In contrast to the Cartesian and strong products, no general statement can be obtained for the direct product $G = \times_{i\in I} G_i$ of graphs G_i since the structure of direct products strongly depends on additional properties such as bipartiteness.

5 RSP-Relations on Complete and Complete Bipartite Graphs

Since complete graphs and complete bipartite graphs contain large numbers of superimposed $K_{2,3}$ subgraphs they are responsible for much of the difficulties in finding finest RSP-relations. We therefore study their RSP-relations in some detail.

Lemma 5.1. *Let $V(K_m) = \{0, \ldots, m-1\}$. For $i = 1, \ldots, l := \lfloor \frac{m}{2} \rfloor$ define the set*

$$\varphi_i := \{[x, (x+i) \bmod m] \mid x \in \{0, \ldots m-1\}\} \subseteq E(K_m).$$

Then the sets $\varphi_1, \ldots, \varphi_l$ define an RSP-relation R on $E(K_m)$ with equivalence classes $\varphi_1, \ldots, \varphi_l$. If $m \neq 4$, then R is a finest RSP-relation.

Proof: At first we prove that R is an equivalence relation. That is, we have to show that $\varphi_i \cap \varphi_j = \emptyset$ for all $i \neq j$ and $E(K_m) = \bigcup_{i=1}^{l} \varphi_i$. For contraposition suppose, $\varphi_i \cap \varphi_j \neq \emptyset$ for some $i \neq j$. That is, there exists $x, y \in V(K_m) = \{0 \ldots, m-1\}$ such that $[x, (x+i) \bmod m] = [y, (y+j) \bmod m]$. Notice, $x + i < 2m$ as well as $y + j < 2m$. Thus, we have $x + i = p \cdot m + (x+i) \bmod m$ and $y + j = q \cdot m + (y+j) \bmod m$ with $p, q \in \{0, 1\}$. First assume $x = y$. Hence, $(x+i) \bmod m = (x+j) \bmod m$ and we obtain $|i - j| = |p - q| \cdot m$

with $|q-p| \in \{0,1\}$. If $|p-q|=0$ it follows $i=j$. Therefore suppose, $|p-q|=1$. This implies $|i-j|=m \geq 2l$ and moreover, $|i-j|<l$ since $i,j \in \{1,\ldots,l\}$, a contradiction.

Now, assume $x \neq y$. Then it must hold $x=(y+j) \bmod m$ and $y=(x+i) \bmod m$ if $[x,(x+i) \bmod m]=[y,(y+j) \bmod m]$. Hence, with our considerations above, we get $i+j=(p+q)\cdot m$ with $p+q \in \{0,1,2\}$. From $i,j \in \{1,\ldots,l\}$, we conclude $0<i+j \leq 2l$ which implies in particular $p+q>0$. It follows $2l \leq m \leq i+j \leq 2l$, hence $i=j=l$ which contradicts the choice of i,j. Thus, $\varphi_i \cap \varphi_j = \emptyset$ for all $i,j \in \{1,\ldots,l\}$ with $i \neq j$.

Next, we show $|\bigcup_{i=1}^{l} \varphi_i| = |E(K_m)|$. Since $\varphi_i \subseteq E(K_m)$ for all $i \in \{1,\ldots,l\}$, we then can conclude $\bigcup_{i=1}^{l} \varphi_i = E(K_m)$. First, let $i<\frac{m}{2}$. Assume, there exists $x \in \{0,\ldots,m-1\}$ such that $x=(x+i) \bmod m$. From previous considerations, it follows $i=p\cdot m$ with $p \in \{0,1\}$, which contradicts $0<1\leq i \leq l<m$. Now suppose, there are $x,y \in \{0,\ldots,m-1\}$ such that $[x,(x+i) \bmod m]=[y,(y+i) \bmod m]$. If $x \neq y$, it follows $x=(y+i) \bmod m$ and $y=(x+i) \bmod m$. As before, we conclude $2i=(p+q)\cdot m$ with $p+q \in \{0,1,2\}$ and since $i>0$, we have $p+q>0$. Thus, $m \leq 2i < m$, which is a contradiction. Hence, $|\varphi_i| = |\{0,\ldots,m-1\}| = m$ for all $i<\frac{m}{2}$. If $i=\frac{m}{2}$, and thus, m is even, we have $|\varphi_{\frac{m}{2}}| = \frac{m}{2}$, since for all $x<\frac{m}{2}$ it follows that $[x,x+\frac{m}{2}] = [x+\frac{m}{2},(x+\frac{m}{2}+\frac{m}{2}) \bmod m]$. It follows $|\bigcup_{i=1}^{l}\varphi_i| = \sum_{i=1}^{l}|\varphi_i| = l\cdot m = \frac{(m-1)\cdot m}{2} = |E(K_m)|$ if m is odd and $|\bigcup_{i=1}^{l}\varphi_i| = \sum_{i=1}^{l-1}|\varphi_i| + |\varphi_{\frac{m}{2}}| = (l-1)\cdot m + \frac{m}{2} = \frac{(m-1)\cdot m}{2} = |E(K_m)|$ if m is even. Therefore, R is an equivalence relation on $E(K_m)$.

It remains to show that R has the relaxed square property and there is no refinement of R with this property. Therefore, let $e=[x,y] \in \varphi_i$ and $f=[x,z] \in \varphi_j$, $i \neq j$. We have to show, that there exists a vertex $w \in V(K_m)$ such that $[y,w] \in \varphi_j$ and $[z,w] \in \varphi_i$. $[x,y] \in \varphi_i$ implies $y=(x+i) \bmod m$ or $x=(y+i) \bmod m$ and $[x,z] \in \varphi_j$ implies $z=(x+j) \bmod m$ or $x=(z+j) \bmod m$. If $y=(x+i) \bmod m$ and $z=(x+j) \bmod m$, we choose $w=(y+j) \bmod m$. It is clear, that $w \neq x,y,z$. By definition, we have $[y,w] \in \varphi_j$. Moreover, by simple calculation we get $w=(z+i) \bmod m$ and hence $[z,w] \in \varphi_i$. If $y=(x+i) \bmod m$ and $x=(z+j) \bmod m$, we choose $w=(z+i) \bmod m$, then $w \neq x,y,z$. Hence, $[w,z] \in \varphi_i$. In this case we get $y=(w+j) \bmod m$ that is $[y,w] \in \varphi_j$. If $x=(y+i) \bmod m$ and $z=(x+j) \bmod m$, we choose $w=(y+j) \bmod m$. Again $w \neq x,y,z$, and by definition $[y,w] \in \varphi_j$. Here, we obtain $z=(w+i) \bmod m$ and hence $[z,w] \in \varphi_i$. If $x=(y+i) \bmod m$ and $x=(z+j) \bmod m$, we choose w such that $z=(w+i) \bmod m$, that is $[z,w] \in \varphi_i$. In this case we have $w \neq x,y,z$ and moreover, $y=(w+j) \bmod m$ and hence $[y,w] \in \varphi_j$. That is, R has the relaxed square property.

We show now, that no equivalence class φ of R can be split into two classes $\varphi_i = \psi_{i_1} \cup \psi_{i_2}$, such that the equivalence relation S, that has classes $\varphi_1,\ldots,\varphi_{i-1},\psi_{i_1},\psi_{i_2},\varphi_{i+1},\ldots,\varphi_l$ is an RSP-relation. Therefore, notice that each vertex $x \in V(K_m)$ is incident to exactly two φ_i edges for all $i<\frac{m}{2}$, namely $[x,(x+i) \bmod m]$ and $[x,(x-i) \bmod m]$, thus the layers are all cycles for $i<\frac{m}{2}$. Moreover, each vertex $x \in V(K_m)$ is incident to exactly one $\varphi_{\frac{m}{2}}$-edge. Recalling Lemma 3.2, $\varphi_{\frac{m}{2}}$ cannot be split. For $k<\frac{m}{2}$ let C the φ_k-layer containing vertex 0. It has edges $[0,k],[k,2k],[2k,3k \bmod m],\ldots,[(q-1)\cdot k,0]$ with $q\cdot k \bmod m = 0$. By Lemma 3.3, any edge in C must be contained in a square, hence C itself must be a square and thus has edges $[0, k], [k, 2k], [2k, 3k], [3k, 0]$ with $4k = m$, since $k < \frac{m}{2}$ and $k > 1$ since $m \neq 4$. Because S is an RSP-relation, $([0, k], [2k, 3k]), ([k, 2k], [3k, 0]) \in S$ and $([0, k], [k, 2k]), ([2k, 3k], [3k, 0]) \notin S$ by Lemma 3.2. Consider the edges $[0, k] \in \varphi_k$ and $[0, 1] \in \varphi_1 \neq \varphi_k$, hence they are in different S-classes. Vertex $k \in V(K_m)$ is incident to exactly two φ_1-edges, namely $[k, k+1]$ and $[k, k-1]$. Since $[1, k-1] \in \varphi_{k-2} \neq \varphi_k$, the only possible square spanned by $[0, k]$ and $[0, 1]$ with opposite edges in the same S-class is $0-1-(k+1)-k$ with $[0, k], [k, k+1] \in S$. Now, consider edges $[k, 2k] \in \varphi_k$ and $[1, k] \in \varphi_{k-1}$. Vertex $2k \in V(K_m)$ is incident to exactly two φ_{k-1}-edges, namely $[2k, k+1]$ and $[2k, 3k-1]$. Since $[1, 3k-1] \in \varphi_{k+2} \neq \varphi_k$, the only possible square spanned by $[k, 2k]$ and $[1, k]$ with opposite edges in the same S-class is $1-k-2k-(k+1)$ with $([1, k+1], [k, 2k]) \in S$. Thus, $([0, k], [k, 2k]) \in S$, a contradiction. Hence, R is finest RSP-relation on K_m for all $m \neq 4$. □

Corollary 1. *For all $m > 3$ there exists a nontrivial RSP-relation on $E(K_m)$.*

Lemma 5.1 implies that the maximum number of classes of a finest RSP-relation is at least $\lfloor \frac{m}{2} \rfloor$. From Lemma 3.2, we infer that the maximum number of classes of a finest RSP-relation on K_m is at most $m-1$, the minimum degree of K_m. In the case of $m = 2^q$, this bound is sharp with the construction in Definition 4.1 and since $K_{2^q} = \boxtimes_{i=1}^{q} K_2$.

To show the large variety of possible finest RSP-relations on complete graphs we give a further example.

Example 3. For $n \geq 5$ and graph K_n, let G_1 be the induced subgraph on vertices $\{0, 1\}$ and G_2 the induced subgraph on $\{2, \ldots, n-1\}$. We claim that relation R with two equivalence classes $\varphi = E(G_1) \cup E(G_2)$ and $\overline{\varphi}$ is a finest RSP relation. It is easy to check that it is an RSP-relation. Equivalence class φ cannot be split into two equivalence classes since vertex 0 is incident with only one edge of φ. On the other hand, every vertex in $\{2, \ldots, n-1\}$ is incident with exactly two edges in $\overline{\varphi}$, therefore if $\overline{\varphi}$ can be split into two equivalence classes edges $[0, 2]$ and $[1, 2]$ must be in different equivalence classes. The definition of RSP-relations implies that $[0, 2]$ and $[2, 3]$ must lie on a common square with opposite edges in the same equivalence class. The only possible candidate is the square $0-2-3-1$, thus $[0, 2]$ and $[1, 3]$ must be in the same class. Similarly, $[1, 3]$ and $[3, 4]$ must lie on a common square with opposite edges in the same equivalence class. The only possible candidate is the square $1-3-4-0$, thus $[1, 3]$ and $[0, 4]$ must be in the same class. Now, we use the same arguments for edges $[0, 4]$ and $[4, 2]$ to find out that $[0, 4]$ and $[1, 2]$ are in the same class. Since the relation is transitive $[0, 2]$ and $[1, 2]$ must be in the same class, a contradiction with the assumption that $\overline{\varphi}$ can split.

Example 4. Consider the complete graph $K_9 = K_3 \boxtimes K_3$. Then the construction given in Lemma 5.1 and in Lemma 4.2 define two different RSP-relations $R \not\subseteq S$, for which $K_9/\mathcal{P}^R \simeq K_9/\mathcal{P}^S \simeq \mathcal{L}K_1$, by Lemma 4.4. Note, R and S have no RSP-relation as common refinement.

Let us now turn to complete bipartite graphs $K_{m,n}$. W.l.o.g. we may assume that $m \leq n$.

Lemma 5.2. *For $m = n$ let the vertex set of $K_{m,m}$ be given by $V(K_{m,m}) = V(K_2) \times V(K_m)$ and $E(K_{m,m}) = \{[x, y] \mid x, y \in V(K_{m,m})$ s.t. $p_1(x) \neq p_1(y)\}$. Furthermore, let S be an RSP-relation on $E(K_m)$. We define an equivalence relation R on $E(K_{m,m})$ as follows: $(e, f) \in R$ if and only if*

(1) $|p_2(e)| = |p_2(f)| = 1$, or

(2) $|p_2(e)| = |p_2(f)| = 2$ and $(p_2(e), p_2(f)) \in S$.

Then R has the relaxed square property. Moreover, R is a finest RSP-relation on $E(K_{m,m})$ if and only if S is finest RSP-relation on $E(K_m)$.

Proof: Notice, that with our notation we have $E(K_{m,m}) = E(K_2 \boxtimes K_m) \setminus (E(K_m^x) \cup E(K_m^y))$ with $x, y \in V(K_2) \times V(K_m)$ s.t. $p_1(x) \neq p_1(y)$. With Lemma 4.2 and Lemma 3.3, it follows that R is an RSP-relation on $E(K_{m,m})$. It is clear that any refinement of S leads to a refinement of R. Thus we just have to show the converse, i.e., that R is a finest RSP-relation if S is finest RSP-relation. Let φ denote the equivalence class defined by condition (1), i.e., $\varphi = \{e \in E(K_{m,m}) \mid |p_2(e)| = 1\}$. By construction, each vertex is adjacent to exactly one φ-edge, therefore, φ cannot be split by Lemma 3.2. Moreover, two adjacent edges e, f with $e \in \varphi$ and $f \in \psi \neq \varphi \sqsubseteq R$ span exactly one square with opposite edges in the same equivalence classes, namely the square with $p_2(f) = p_2(f')$, where f' is opposite edge of f. Therefore, $p_2(e) = p_2(e')$ implies $(e, e') \in Q$ for any refinement Q of R with relaxed square property. Furthermore, with our notations, any refinement Q of R leads also to a refinement $Q_{|E(K_2 \times K_m)}$ of $R_{|E(K_2 \times K_m)}$, the restrictions of Q and R to $E(K_2 \times K_m) \subseteq E(K_{m,m})$, respectively. If the refinement Q is proper and satisfies the relaxed square property on $E(K_{m,m})$, the same is true for $Q_{|E(K_2 \times K_m)}$ on $E(K_2 \times K_m)$ by Lemma 3.3 and our previous considerations. Moreover, we can conclude that Q determines an equivalence relation $p_2(Q)$ on K_m via $(p_2(e), p_2(f)) \in p_2(Q)$ if and only if $(e, f) \in Q$. It holds $p_2(C_4) \cong C_4$ for any square in $K_2 \times K_m$. Furthermore, $p_2(e) = p_2(e')$ implies $(e, e') \in Q$ if Q has the relaxed square property. Therefore, it follows, $p_2(Q)$ is a proper refinement of S with the relaxed square property if Q is a proper refinement of R with the relaxed square property. This completes the proof. $\square$

Lemma 5.3. *For $m < n$ let the vertex set of $K_{m,n}$ be given by $\{x_1, \ldots, x_m, y_1, \ldots, y_n\}$ such that $E(K_{m,n}) = \{[x_i, y_j] \mid 1 \leq i \leq m, 1 \leq j \leq m\}$. Furthermore, let S be an equivalence relation on the edge set of the induced subgraph $\langle\{x_1, \ldots, x_m, y_1, \ldots y_m\}\rangle \cong K_{m,m}$ of $K_{m,n}$. We extend S to an equivalence relation R on $E(K_{m,n})$ as follows: For each equivalence class $\varphi' \sqsubseteq S$ we extend φ' to an equivalence class $\varphi \sqsubseteq R$, i.e., we set $\varphi' \subseteq \varphi$ and moreover $[x_j, y_{m+i}]$ is an edge in equivalence class φ if and only if $[x_j, y_{k_i}]$ is an edge in φ' for fixed $k_i \in \{1, \ldots, m\}$ for all $i \in \{1, \ldots, n - m\}$. Then R has the relaxed square property.*

Proof: It is clear, that R is an equivalence relation. Thus, it remains to show that R has the relaxed square property. Therefore, let $e, f \in E(K_{m,n})$ such that $(e, f) \notin R$. Notice, by construction, $\psi' \neq \varphi'$ if and only if $\psi \neq \varphi$ for all $\psi', \varphi' \sqsubseteq S$ and $\psi, \varphi \sqsubseteq R$ with $\psi' \subseteq \psi$ and $\varphi' \subseteq \varphi$.

First, suppose that e and f are both incident to some vertex $y_r \in V(K_{m,n})$, $r \in \{1, \ldots, n\}$. That is, $e = [x_j, y_r]$ and $f = [x_l, y_r]$ for some $j, l \in \{1, \ldots, m\}, j \neq l$. If $r \leq m$ then by construction $e, f \in E(K_{m,m})$ and $(e, f) \notin S$, and hence they span a square with opposite edges in the same equivalence classes of S, which is also retained in $K_{m,n}$ with the same properties. If $r > m$, then $r = m + i$ for some $i \in \{1, \ldots, n - m\}$. By construction, there exists $k_i \in \{1, \ldots, m\}$ such that $([x_j, y_{k_i}], [x_j, y_{m+i}]) \in R$ and $([x_l, y_{k_i}], [x_l, y_{m+i}]) \in R$, which implies $([x_j, y_{k_i}], [x_l, y_{k_i}]) \notin R$ and hence, by construction, $([x_j, y_{k_i}], [x_l, y_{k_i}]) \notin S$. Since S has the relaxed square property, there exists $w \in V(K_{m,m}) \subset V(K_{m,n})$ such that $[x_j, y_{k_i}]$ and $[x_l, y_{k_i}]$ span a square $x_j - y_{k_i} - x_l - w$, such that $([x_l, w], [x_j, y_{k_i}]) \in S \subset R$ and $([x_j, w], [x_l, y_{k_i}]) \in S \subset R$. Then $x_j - y_{m+i} - x_l - w$ is a square spanned by e and f with opposite edges in the same equivalence class.

Now assume e and f are both incident to some vertex $x_j \in V(K_{m,n})$, $j \in \{1, \ldots, m\}$. That is, $e = [x_j, y_r]$ and $f = [x_j, y_s]$ for some $r, s \in \{1, \ldots, n\}, r \neq s$. If $r, s \leq m$, then by construction $e, f \in E(K_{m,m})$ and $(e, f) \notin S$, and hence they span a square with opposite edges in the same equivalence classes of S, which is also retained in $K_{m,n}$ with the same properties. If $r, s > m$, then $r = m + i$, $s = m + l$ for some $i, l \in \{1, \ldots, n - m\}$. By construction, there exists $k_i, k_l \in \{1, \ldots, m\}$ such that $([x_j, y_{m+i}], [x_j, y_{k_i}]) \in R$ as well as $([x_j, y_{m+l}], [x_j, y_{k_l}]) \in R$, from which we can conclude $([x_j, y_{k_i}], [x_j, y_{k_l}]) \notin R$. By construction we have $([x_j, y_{k_i}], [x_j, y_{k_l}]) \notin S$, and since S has the relaxed square property, there exists $w \in V(K_{m,m}) \subset V(K_{m,n})$ such that $[x_j, y_{k_i}]$ and $[x_j, y_{k_l}]$ span a square $(x_j, y_{k_i}, w, y_{k_l})$, such that $([w, y_{k_l}], [x_j, y_{k_i}]) \in S \subset R$ and $([w, y_{k_i}], [x_j, y_{k_l}]) \in S \subset R$. Moreover, by construction, we have $([w, y_{m+i}], [w, y_{k_i}]) \in R$ as well as $([w, y_{m+l}], [w, y_{k_l}]) \in R$. Thus $x_j - y_{m+i} - w - y_{m+l}$ is a square spanned by e and f with opposite edges in the same equivalence class. If $r > m, s \leq m$, then $r = m + i$ for some $i \in \{1, \ldots, n - m\}$. By construction, there exists $k_i \in \{1, \ldots, m\}$ such that $([x_j, y_{m+i}], [x_j, y_{k_i}]) \in R$ and thus, $([x_j, y_{k_i}], [x_j, y_l]) \notin R$, hence, $([x_j, y_{k_i}], [x_j, y_{k_l}]) \notin S$. Since S has the relaxed square property, there exists $w \in V(K_{m,m}) \subset V(K_{m,n})$ such that $[x_j, y_{k_i}]$ and $[x_j, y_l]$ span a square $x_j - y_{k_i} - w - y_l$, such that $([w, y_l], [x_j, y_{k_i}]) \in S \subset R$ and $([w, y_{k_i}], [x_j, y_l]) \in S \subset R$. Moreover, by construction, we have $([w, y_{m+i}], [w, y_{k_i}]) \in R$. Hence, $x_j - y_{m+i} - w - y_l$ is a square spanned by e and f with opposite edges in the same equivalence class. Analogously, one shows that e and f span a square with opposite edges in the same equivalence class if $r \leq m$ and $s > m$, which completes the proof. □

Obviously, any finer RSP-relation $S' \subset S$ on $E(K_{m,m})$ leads to a finer RSP-relation $R' \subset R$ on $E(K_{m,n})$, constructed from S' as in Lemma 5.3. It is not known yet, if the converse is also true.

Corollary 2. *For all $m, n \geq 2$ there exists a nontrivial RSP-relation on $E(K_{m,n})$.*

The constructions in Lemma 5.2 and Lemma 5.3 together with Lemma 5.1 imply that the maximum number of classes of a finest RSP-relation is at least $\lfloor \frac{m}{2} \rfloor + 1$. From Lemma 3.2, we infer that the maximum number of classes of a finest RSP-relation on $K_{m,n}$ is at most m, the minimum degree of $K_{m,n}$. In the case of $m = 2^q$, this bound is sharp with our considerations for complete graphs K_{2^q} and the constructions in Lemma 5.2 and Lemma 5.3.

6 RSP-relations and Covering Graphs

We are now in the position, to establish the close connection of covering graphs and (well-behaved) RSP-relations.

Definition 6.1. For a graph $G = (V, E)$, an RSP-relation R on E and $\varphi \sqsubseteq R$, let G^x_φ and G^y_φ be two distinct adjacent φ-layers. We define the graph $C_{G^x_\varphi,G^y_\varphi}$ in the following way:

1. Vertices $V(C_{G^x_\varphi,G^y_\varphi}) = \{[a, b] \in E \mid a \in V(G^x_\varphi), b \in V(G^y_\varphi)\}$ are precisely the edges of G connecting G^x_φ and G^y_φ.
2. Two vertices $[a_1, b_1], [a_2, b_2] \in V(C_{G^x_\varphi,G^y_\varphi})$ are adjacent if they are opposite edges of a square $a_1 - b_1 - b_2 - a_2$ in G with $[a_1, a_2] \in E(G^x_\varphi)$ and $[b_1, b_2] \in E(G^y_\varphi)$.

Lemma 6.2. *Let G be a graph, R an RSP-relation on $E(G)$, and G^x_φ and G^y_φ two distinct adjacent φ-layers for some $\varphi \sqsubseteq R$. Then $C_{G^x_\varphi,G^y_\varphi}$ is a quasi-cover of G^x_φ and G^y_φ. Moreover, if R is well-behaved, then $C_{G^x_\varphi,G^y_\varphi}$ is a cover of G^x_φ and G^y_φ.*

Proof: We define the map $f_1 : V(C_{G^x_\varphi,G^y_\varphi}) \to V(G^x_\varphi)$ by $f_1([a, b]) = a$ where $a \in V(G^x_\varphi)$ and $b \in V(G^y_\varphi)$ and show first that f_1 is a homomorphism, i.e., it maps neighbors in $C_{G^x_\varphi,G^y_\varphi}$ into neighbors in G^x_φ. Let $[a_1, b_1], [a_2, b_2] \in V(C_{G^x_\varphi,G^y_\varphi})$ be adjacent. By construction of edges in $C_{G^x_\varphi,G^y_\varphi}$, there is a square $a_1 - b_1 - b_2 - a_2$ in G with opposite edges $[a_1, b_1]$ and $[a_2, b_2]$. Hence, a_1 and a_2 are adjacent in G^x_φ. Now, let $a = f_1([a, b])$ and $c \in N_{G^x_\varphi}(a)$. Since $[a, c]$ and $[a, b]$ are adjacent edges of different equivalence classes, they span some square with opposite edges in relation R. Thus there exists a vertex $d \in V(G^y_\varphi)$, such that $[a, b]$ and $[c, d]$ are adjacent in $C_{G^x_\varphi,G^y_\varphi}$ and $f_1([c, d]) = c$. This proves that f_1 is locally surjective and therefore, that $C_{G^x_\varphi,G^y_\varphi}$ is a quasi-cover of G^x_φ.

Let f_1 be defined as above and assume that none of the subgraphs of G that are isomorphic to $K_{2,3}$ have a forbidden coloring. If $f_1([c_1, d_1]) = f_1([c_2, d_2])$, it holds that for $[c_1, d_1], [c_2, d_2] \in N_{C_{G^x_\varphi,G^y_\varphi}}([a, b])$ we have $c_1 = c_2$ by construction of f_1. If $d_1 \neq d_2$, then there is a subgraph of G isomorphic to $K_{2,3}$ with bipartition $\{b, c_1\} \cup \{a, d_1, d_2\}$. Moreover, since $[a, c_1], [b, d_1], [b, d_2] \in \varphi$ and the other edges are, by construction, in $\overline{\varphi}$ we conclude that this subgraph has a forbidden coloring, a contradiction. Thus, $d_1 = d_2$, i.e., the locally surjective map f_1 is also locally injective. Hence, $C_{G^x_\varphi,G^y_\varphi}$ is a cover of G^x_φ.

Arguing analogously for the map $f_2 : V(C_{G^x_\varphi,G^y_\varphi}) \to V(G^y_\varphi)$ with $f_2([a, b]) = b$, $a \in V(G^x_\varphi)$, $b \in V(G^y_\varphi)$, one obtains the desired results for $C_{G^x_\varphi,G^y_\varphi}$ and G^y_φ. □

To illustrate Lemma 6.2 consider the following example: Let $G_1 = C_6$ and $G_2 = C_9$ with vertex sets $\mathbb{Z}_6$ and $\mathbb{Z}_9$ and the canonical edge set definitions. To obtain G add the edges $[k, k \bmod 6]$ and $[k, k + 3 \bmod 6]$ for $0 \leq k \leq 8$ connecting G_1 with G_2. Construct an equivalence relation R with two classes $\varphi = E(G_1) \cup E(G_2)$, and $\overline{\varphi}$ comprising the connecting edges. R is a well-behaved RSP-relation on G. It is not hard to verify that C_{G_1,G_2} is a cover graph of C_6 and C_9 and is isomorphic to C_{18}.

For a similar result for the case when G^x_φ and G^y_φ are not distinct, that is $G^x_\varphi = G^y_\varphi$, but there are edges not in φ connecting its vertices, we have to be a bit more careful.

Definition 6.3. For a graph $G = (V, E)$, an RSP-relation R on E, and $\varphi \sqsubseteq R$, let G^x_φ be some φ-layer. We define the graph $C_{G^x_\varphi, G^x_\varphi}$ in the following way:

1. Vertices $V(C_{G^x_\varphi, G^x_\varphi}) = \{(a, b) \mid [a, b] \in E, a, b \in V(G^x_\varphi), [a, b] \in \overline{\varphi}\}$ are edges in $E(G)$ with superimposed orientation (a, b) from a to b, that are not contained in class φ, but that connect vertices of G^x_φ.
2. Two directed edges (a_1, b_1) and (a_2, b_2) in $V(C_{G^x_\varphi, G^x_\varphi})$ are adjacent if $[a_1, b_1]$, $[a_2, b_2]$ are opposite edges of a square $a_1 - b_1 - b_2 - a_2$ in G with $[a_1, a_2], [b_1, b_2] \in E(G^x_\varphi)$.

Remark 1. Since $[a, b] = [b, a]$, it holds that for all edges $[a, b] \in E$, we get two vertices in $V(C_{G^x_\varphi, G^x_\varphi})$ per edge $[a, b] \in E \setminus \varphi$, namely (a, b) and (b, a).

Lemma 6.4. *For a graph $G = (V, E)$, an RSP-relation R on E, and $\varphi \sqsubseteq R$, let G^x_φ be some φ-layer and assume that there are edges $[a, b] \in E \setminus \varphi$ with $a, b \in V(G^x_\varphi)$. Then $C_{G^x_\varphi, G^x_\varphi}$ is a quasi-cover of G^x_φ with two different locally surjective homomorphisms f_1 and f_2 such that $f_1(h) \neq f_2(h)$ for every $h \in C_{G^x_\varphi, G^x_\varphi}$. Moreover, if R is well-behaved, then $C_{G^x_\varphi, G^x_\varphi}$ is twice a cover of G^x_φ, i.e., there are at least two different covering maps.*

PROOF: Proof is the same as for Lemma 6.2 by defining $f_1((a, b)) = a$ and $f_2((a, b)) = b$. □

If every vertex of G^x_φ is incident with exactly one edge that is not in φ but connects two vertices of G^x_φ, then $G^x_\varphi \cong C_{G^x_\varphi, G^x_\varphi}$ and the edges in $\overline{\varphi}$ induce an automorphism of G^x_φ without fixed vertices by setting $f(a) = b$ whenever $[a, b] \in \overline{\varphi}$.

As an example consider the graph G with $V(G) = \mathbb{Z}_6$ and $E(G) = \varphi \cup \overline{\varphi}$ such that $\varphi = \{[k, k+1 \bmod 6] \mid 0 \leq k \leq 5\}$, i.e., $G_\varphi \cong C_6$ and $\overline{\varphi} = \{[1, 4], [2, 5], [3, 6]\}$. We then have $V(C_{G^x_\varphi, G^x_\varphi}) = \{(0, 3), (1, 4), (2, 5), (3, 0), (4, 1), (5, 2)\}$ and $C_{G^x_\varphi, G^x_\varphi}$ has edges $E(C_{G^x_\varphi, G^x_\varphi}) = \{[(0, 3), (1, 4)], [(1, 4), (2, 5)], [(2, 5), (3, 0)], [(3, 0), (4, 1)], [(4, 1), (5, 2)], [(5, 2), (0, 2)]\}$, that is $C_{G^x_\varphi, G^x_\varphi} \cong C_6 \cong G_\varphi$. The induced automorphism is given by $f(k) = k + 3 \bmod 6$, $k = 0, \dots, 5$.

Lemma 6.2 and Lemma 6.4 together highlight a connection between graph bundles and graphs with relaxed square property. For an RSP-relation R on G we see that the connected components G_φ correspond to fibers, while the graph $G_{\overline{\varphi}}/\mathcal{P}^R_\varphi$ has the role of the base graph. Such decomposition is a graph bundle if and only if edges connecting G^x_φ and G^y_φ for arbitrary x, y induce an isomorphism. In our language, this is equivalent to the condition $C_{G^x_\varphi, G^y_\varphi} \cong G^x_\varphi \cong G^y_\varphi$ for arbitrary x, y, provided that G^x_φ and G^y_φ are connected by an edge. Graphs with a nontrivial RSP-relation are therefore a natural generalization of graph bundles.

Corollary 3. *For a graph G and a well-behaved RSP-relation R on $E(G)$, let G^x_φ and G^y_φ be two (not necessarily distinct) φ-layers. Then*

$$|N_{G_{\overline{\varphi}}}(x) \cap V(G^y_\varphi)| = |N_{G_{\overline{\varphi}}}(u) \cap V(G^y_\varphi)| \tag{6.1}$$

is fulfilled for every $u \in V(G^x_\varphi)$.

PROOF: If there is no edge in $G_{\overline{\varphi}}$ connecting G^x_φ and G^y_φ the assertion is clearly true. Therefore assume now that they are connected by an edge. By Lemmas 6.2 and 6.4, $C_{G^x_\varphi, G^y_\varphi}$ is a cover of G^x_φ with covering map f_1 as defined in Lemmas 6.2 resp. 6.4. By definition of f_1, $|f_1^{-1}(u)| = |N_{G_{\overline{\varphi}}}(u) \cap V(G^y_\varphi)|$, which is the same for all $u \in V(G^x_\varphi)$. □

Corollary 3 indicates another property of well-behaved RSP-relations. It was shown in [20] that for a so-called USP-relation R on $E(G)$ the vertex partitions $P^R_{\overline{\varphi}}$ and P^R induced by equivalence classes $\varphi \sqsubseteq R$ are equitable partitions for the graphs G_φ and G, respectively. The key argument leading to this result was an analogue of Equation (6.1). Together with Lemma 3.3, the fact that if R is well-behaved on G then $R\backslash\varphi$ is well-behaved on $(V(G), E(G) \setminus \varphi)$, and since $|\dot\bigcup_\psi N_\psi(x)| = \sum_\psi |N_\psi(x)|$ for any set of pairwisely distinct equivalence classes ψ of R, we can use the same arguments as in [20] to obtain

Theorem 6.5. *Let R be (a coarsening of) a well-behaved RSP-relation on the edge set $E(G)$ of a connected graph G. Then:*

(1) $\mathcal{P}^R_{\overline{\varphi}} = \left\{V(G^x_{\overline{\varphi}}) \mid x \in V(G)\right\}$ is an equitable partition of the graph G_φ for every equivalence class φ of R.

(2) $\mathcal{P}^R = \left\{\bigcap_{\varphi\sqsubseteq R} V(G_{\overline{\varphi}}(x)) \mid x \in V(G)\right\}$ is an equitable partition of G.

As mentioned previously, while an RSP-relation R on $E(G)$ might be well-behaved and thus, has no forbidden $K_{2,3}$-coloring, this is no longer true for coarsenings of R in general. However, since the number of edges incident to a vertex is additive over equivalence classes of R, the latter theorem remains also true for coarsenings of relations *without* forbidden $K_{2,3}$-colorings.

Another interesting question is how two graphs G_1 and G_2 can be connected by additional edges so that $\varphi = E(G_1) \cup E(G_2)$ and $\overline{\varphi}$ comprises the connecting edges and $R = \{\varphi, \overline{\varphi}\}$ is an RSP-relation.

Lemma 6.6. *Let G_1, G_2, and G be graphs and $f_1 : G \to G_1$, $f_2 : G \to G_2$ be locally surjective homomorphisms. Then there exists a graph $H = (V, E)$ and an RSP-relation R on E with equivalence classes φ, $\overline{\varphi}$ such that*

$$V = V(G_1) \cup V(G_2) \quad \text{and} \quad \varphi = E(G_1) \cup E(G_2).$$

Note, it is allowed to have $G_1 = G_2$. In this case, H might have loops and double edges.

PROOF: For given graphs G_1, G_2, G and locally surjective homomorphisms $f_i : G \to G_i$, $i = 1, 2$ construct the graph H as follows: For $x \in V(G_1)$ and $y \in V(G_2)$ add an edge $[x, y]$ if and only if there exists $g \in V(G)$ such that $f_1(g) = x$ and $f_2(g) = y$. We set $[x, y] \in \overline{\varphi}$. It is clear, that R is an equivalence relation. We have to show, that R is an RSP-relation. Let $[x_1, x_2] \in E(G_1)$ and $[x_1, y_1]$ be an added edge. Then there exists $g_1 \in V(G)$, such that $f_1(g_1) = x_1$ and $f_2(g_1) = y_1$. Since f_1 is a locally surjective homomorphism, there exists a vertex g_2 as a neighbor of g_1, such that $f_1(g_2) = x_2$. Let $y_2 = f_2(g_2)$. Then y_2 and x_2

are connected by an added edge and y_1, y_2 are adjacent since f_2 is a homomorphism. Thus $[x_1, x_2]$ and $[x_1, y_1]$ lie on a square with opposite edges in relation R.

If $G_1 = G_2$, then just identify vertices of two copies of G_1. $\square$

Lemma 6.7. *Let G and G' be two graphs. Then there exists a graph $H = (V, E)$ and a well-behaved RSP-relation R with two equivalence classes φ, $\overline{\varphi}$ such that*

$$V = V(G) \cup V(G') \text{ and } \varphi = E(G) \cup E(G'),$$

and each vertex of $V(G)$ is incident to exactly one $\overline{\varphi}$-edge if and only if G is a cover of G'.

PROOF: Let $H = (V, E)$ be a graph with well-behaved RSP-relation R on E as claimed. Then, we can consider G, G' as φ-layers. By Lemma 6.2, $C_{G',G}$ is a cover of G' and G. Since each vertex in $V(G)$ is incident with exactly one $\overline{\varphi}$-edge, we see that for covering map $f_1 : C_{G',G} \to G$ holds $|f_1^{-1}(u)| = 1$ for all $u \in H$ which implies f_1 is also injective, thus an isomorphism.

For the converse, assume G is a cover of G'. Then G is a cover of G and G' and thus G and G' can be connected as in the prove of Lemma 6.6. Since clearly $G \cong G$ and thus the covering map $p : G \to G$ is in particular injective, each vertex is, by construction, incident to exactly one $\overline{\varphi}$-edge. This in turn implies, H contains no square $w - x - y - z$ such that $z \in V(G)$ and $[w, z], [y, z] \in \overline{\varphi}$. On the other hand, there is no square $w - x - y - z$ contained in H with $[w, x], [x, y] \in E(G) \subseteq \varphi$ and $[w, z], [y, z] \in \overline{\varphi}$, i.e., $z \in V(G')$, since otherwise the restriction of the covering map $p' : G \to G'$ to $N_G(x)$ (w.l.o.g. we can assume p to be the identity mapping) would not be injective, a contradiction. Hence, we can conclude that R is well-behaved. $\square$

Notice that checking if H is a cover graph of G is in general NP-hard [1]. Therefore, also connecting two graphs as described in Lemma 6.7 is NP-hard. On the other hand, one can connect two arbitrary graphs G_1, G_2 such that all vertices of G_1 are linked to all vertices of G_2. Then, the relation defined by the classes $\varphi = E(G_1) \cup E(G_2)$ and $\overline{\varphi}$ that consists of all added edges between G_1 and G_2 is an RSP-relation. This implies that any two graphs have a common finite quasi-cover. However, this is not true for covers, just take K_2 and K_3 as an example.

For a given graph G and an RSP-relation R, one can consider the subgraph G_φ, $\varphi \sqsubseteq R$ as one layer and all other edges of G not contained in G_φ as connecting edges. Notice, connectivity is not explicitly needed in Definition 6.3 and Lemma 6.4, and thus, they can be extended to C_{G_φ, G_φ}. Moreover, any spanning subgraph H of a graph G induces an equivalence relation R with two equivalence classes $E(H)$ and $E(G) \setminus E(H)$. Hence, $C_{H,H}$ is well defined and thus, Lemma 6.4 and 6.6 imply the following result.

Theorem 6.8. *A graph G has an RSP-relation with two equivalence classes if and only if there exists a (possibly disconnected) spanning subgraph $H \subsetneq G$ and $C_{H,H}$ is a quasi-cover of H.*

On the set of graphs $\mathfrak{G}$ we consider the relation $G_1 \sim G_2$ if G_1 and G_2 have a common finite cover.

Theorem 6.9. *The relation $\sim$ on $\mathfrak{G}$ is an equivalence relation.*

PROOF: Relation $\sim$ is clearly reflexive and symmetric. By assumption, the graphs G_1 and G_2 have a common cover H_{12} and G_2 and G_3 have a common cover H_{23}. By Lemma 6.7, H_{12}, G_2 and H_{23}, G_2 can be connected without forbidden colorings of $K_{2,3}$. Let E be the set of all edges connecting G_2 and H_{12} and E' edges connecting G_2 and H_{23}. Since every cover of H_{12} and H_{23} is a cover of G_1, G_2 and G_3, it is sufficient to find a cover of H_{12} and H_{23}. Therefore, it suffices to connect H_{12} and H_{23} without forbidden colorings of $K_{2,3}$. Define edges connecting H_{12} and H_{23} by connecting $h \in V(H_{12})$ and $h' \in V(H_{23})$ if there exists a vertex $v \in V(G_2)$ such that $[h, v] \in E$ and $[v, h'] \in E'$.

First we check that $E(H_{12}) \cup E(H_{23})$ and connecting edges form two equivalence classes of an RSP relation. Without loss of generality assume $[h_1, h_2] \in E(H_{12})$ and $[h_1, h_1']$, $h_1' \in V(H_{23})$ is a connecting edge. Then there exists $v_1 \in V(G_2)$ such that $[h_1, v_1] \in E$ and $[v_1, h_1'] \in E'$. Since edges E are defined by a local bijection between H_{12} and G_2, there exist $v_2 \in V(G_2)$, a neighbor of v_1, such that $[h_2, v_2] \in E$. Similarly, since E' is defined by a local bijection between H_{23} and G_2, there exists $h_2' \in V(H_{23})$, a neighbor of h_1', such that $[v_2, h_2'] \in E'$. Therefore there exists a square $h_1 - h_1' - h_2' - h_2$ with $[h_1, h_2], [h_1', h_2'] \in E(H_{12}) \cup E(H_{23})$ and $[h_1, h_2], [h_1', h_2']$ being connecting edges. This proves that relation R, with equivalence classes $E(H_{12}) \cup E(H_{23})$ and the set of connecting edges is an RSP relation.

It remains to prove that R is well-behaved. By symmetry, it is enough to prove that there exists no vertices $h_1, h_2, h_3 \in V(H_{12})$ and $h_1', h_2' \in V(H_{2,3})$ with $[h_1, h_2], [h_1, h_3] \in E(H_{12})$, $[h_1', h_2'] \in E(H_{23})$ and added edges $[h_1, h_1'], [h_2, h_2']$ and $[h_3, h_2']$. For the sake of contradiction, assume such vertices exist. By the construction of the added edges, there exist vertices $v_1, v_2, v_3 \in V(G_2)$ such that $[h_1, v_1], [h_2, v_2], [h_3, v_3] \in E$ and $[v_1, h_1'], [v_2, h_2'], [v_3, h_2'] \in E'$. Since edges in E are obtained from a covering map of H_{12} to G_2 we see that v_1, v_2 and v_3 are distinct vertices. But also the edges in E' are obtained from a covering map of H_{23} to G_2 therefore $[v_2, h_2'] = [v_3, h_2']$ and thus $v_2 = v_3$, a contradiction. □

We have proven Theorem 6.9 here by elementary means to keep this presentation self-contained. It also follows from a deep result of Leighton [19], who proved the following: A pair of finite connected graphs G_1 and G_2 has a common finite cover if and only if they have the same (possibly infinite) cover graph isomorphic to a tree. Such a cover is unique for every graph G and covers any other covering graph of G; it is therefore called the *universal cover* of G. On the other hand, a minimal common cover of two graphs need not be unique, as Imrich and Pisanski have shown [15].

Corollary 4. *Let G be a connected graph and let R be a well-behaved RSP-relation on $E(G)$. Then there exists a common covering graph for all φ-layers $G_\varphi^{x_i}$.*

PROOF: This result is an immediate consequence of the connectedness of G, Lemma 6.2 and Theorem 6.9. □

In terms of Leighton's theorem, the corollary could be read in the following way: For a graph G with a well-behaved RSP-relation on $E(G)$ and some fixed equivalence class φ all the graphs $\{G_\varphi^{x_i}\}$ have the same universal cover.

Under certain conditions it is possible to refine a given RSP-relation.

Lemma 6.10. *Let $G = (V, E)$ be a connected graph and R a well-behaved RSP-relation on E. Assume that for one equivalence class $\varphi \sqsubseteq R$ the graph G_φ has two connected components G^x_φ and G^y_φ. The next two statements are equivalent:*

1. *There is a well-behaved refined RSP-relation $R' \subsetneq R$ such that $\varphi = \chi_1 \cup \chi_2$ with $\chi_1, \chi_2 \sqsubseteq R'$*
2. *$C_{G^x_\varphi, G^y_\varphi}$ has a non-trivial RSP-relation Q such that $(e, f) \in Q$ if and only if $(e', f') \in R'$ for all $e, f \in p_1^{-1}(e') \cup p_1^{-1}(f') \cup p_2^{-1}(e') \cup p_2^{-1}(f')$ and for all $e, f \in E(G^x_\varphi) \cup E(G^y_\varphi)$, where $p_1 : C_{G^x_\varphi, G^y_\varphi} \to G^x_\varphi$, resp., $p_2 : C_{G^x_\varphi, G^y_\varphi} \to G^y_\varphi$.*

In other words, R can be refined to R' if and only if edges of G^x_φ, resp., G^y_φ that map on the same edges via the covering projection are in the same class w.r.t. Q.

Proof: If there is a finer RSP-relation R', every square $a_1 - b_1 - b_2 - a_2$ with $a_1, a_2 \in V(G^x_\varphi)$ and $b_1, b_2 \in V(G^y_\varphi)$ has edges $[a_1, a_2]$ and $[b_1, b_2]$ in the same class by the relaxed square property and since R is well-behaved. Thus, an equivalence relation on $E(G^x_\varphi)$ and $E(G^y_\varphi)$ can be lifted to an equivalence relation on $E(C_{G^x_\varphi, G^y_\varphi})$ in a natural way. One can check that it has the relaxed square property by using that the respective relations on $E(G^x_\varphi)$ and $E(G^y_\varphi)$ have the relaxed square property.

Conversely, we define a finer RSP-relation on $E(G^x_\varphi)$ and $E(G^x_\varphi)$ from the RSP-relation on $E(C_{G^x_\varphi, G^y_\varphi})$ by setting $(e', f') \in R'$ if and only if $(e, f) \in Q$ for some $e \in p_1^{-1}(e'), f \in p_1^{-1}(e')$. $\square$

Let R be a well-behaved RSP-relation on G, e.g., $R = \delta_0$, and suppose there is a finer RSP-relation R' in which an equivalence class φ is split into two equivalence classes φ_1 and φ_2. Let $\{G^{x_i}_\varphi\}$ be the connected components of G_φ. Then φ_1 and φ_2 induce an RSP-relation on each $G^{x_i}_\varphi$. Consider two components $G^{x_1}_\varphi$ and $G^{x_2}_\varphi$ that are connected by some edges (in other classes). From the proof of Lemma 6.10 we observe that an RSP-relation on $E(G^{x_1}_\varphi)$ already defines an RSP-relation on $C_{G^{x_1}_\varphi, G^{x_2}_\varphi}$, which in turn defines an RSP-relation on $G^{x_2}_\varphi$ and thus on all φ-layers $G^{x_i}_\varphi$. If multiple splits of φ exist, they are fixed by choosing one on any $G^{x_i}_\varphi$.

Now consider the graph G consisting of two copies of $K_{2,3}$ and all edges connecting them and the equivalence relation whose two classes are the edges of the two copies of $K_{2,3}$ and the connected edges, respectively. The discussion above implies that we can split the first class independently on the two copies of $K_{2,3}$. Thus, we cannot generalize the result above to RSP-relations with forbidden colorings.

7 Outlook and Open Questions

We discussed in this contribution in detail RSP-relations, the most relaxed type of relations fulfilling the square property. As it turned out, such relations are hard to handle in graphs that contain $K_{2,3}$-subgraphs. On the other hand, it is possible to determine finest RSP-relations in polynomial time in $K_{2,3}$-free graphs. Moreover, we showed how to determine (finest) RSP-relations in certain graph products, as well as in complete and complete-bipartite graphs. We finally established the close connection of (well-behaved) RSP-relations to graph covers and equitable partitions. Intriguingly, non-trivial RSP-relations can be characterized by means of the existence of spanning subgraphs that yield quasi-covers of the graph under investigation.

Still, many interesting problems remain open topics for further research. From the computational point of view, it would be worth to determine the complexity of the problem of determining a finest (well-behaved) RSP-relation. Since there is a close connection to graph covers, we suppose that the latter problem is NP-hard. If so, then fast heuristics need to be designed. It is also of interest to investigate, for which graph classes (that are more general than $K_{2,3}$-free graphs) the proposed algorithm determines well-behaved or finest RSP-relations.

From the mathematical point of view, one might ask, under which circumstances is it possible to guarantee that there is a non-trivial finest RSP-relation that is in addition well-behaved. Note, the graph $G = K_{2,3}$ has no such relation. However, there might be interesting graph classes that have one. In addition, it might be of particular importance (also for computational aspects) to distinguish RSP-relations. Let us say that two RSP-relations R and S on E are *equivalent*, $R \simeq S$, if there is an automorphism $f : V \to V$ such that $([x, y], [a, b]) \in R$ if and only if $([f(x), f(y)], [f(a), f(b)]) \in S$. Note, if $G = K_{2,3}$ then all finest RSP-relation consist of two equivalence classes and all such relations are equivalent.

Clearly, if $R \simeq S$, then $G/\mathcal{P}^R \simeq G/\mathcal{P}^S$. However, the converse is not true, i.e., $G/\mathcal{P}^R \simeq G/\mathcal{P}^S$ does not imply $R \simeq S$, see Example 4. This suggests to consider under which conditions finest RSP-relations are unique or for which graphs the equivalence of RSP-relations can be expressed in terms of isomorphism of quotient graphs.

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