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On the diameter of domination bicritical graphs

Michitaka Furuya

Department of Mathematical Information Science
Tokyo University of Science
1-3 Kagurazaka, Shinjuku-ku, Tokyo 162-8601
Japan

Abstract

For a graph G, we let $\gamma(G)$ denote the domination number of G. A graph G is said to be k-bicritical if $\gamma(G) = k$ and $\gamma(G - \{x, y\}) < k$ for any two vertices $x, y \in V(G)$. Brigham et al. [*Discrete Math.* 305 (2005), 18–32] conjectured that the diameter of a connected k-bicritical graph is at most $k - 1$. However, in [*Australas. J. Combin.* 53 (2012), 53–65], counterexamples of the conjecture for $k \neq 4$ were constructed by this author. In this paper, we construct counterexamples of the conjecture for $k = 4$.

Our main aim is to give upper bounds of the diameter of a bicritical graph. In particular, we show that the diameter of a connected k-bicritical graph is at most $2k - 3$.

1 Introduction

All graphs considered in this paper are finite, simple, and undirected. Let G be a graph. We let $V(G)$ and $E(G)$ denote the vertex set and the edge set of G, respectively. For $u \in V(G)$, we let $N_G(u)$ and $N_G[u]$ denote the *open neighborhood* and the *closed neighborhood* of u, respectively; thus $N_G[u] = N_G(u) \cup \{u\}$. For $u, v \in V(G)$, we let $d_G(u, v)$ denote the *distance* between u and v in G. For $u \in V(G)$ and a non-negative integer i, let $N_G^{(i)}(u) = \{v \in V(G) \mid d_G(u, v) = i\}$; thus $N_G^{(0)}(u) = \{u\}$ and $N_G^{(1)}(u) = N_G(u)$. For $u \in V(G)$, we define the *eccentricity* $\mathrm{ecc}_G(u)$ of u in G by $\mathrm{ecc}_G(u) = \max\{d_G(u, v) \mid v \in V(G)\}$. The *diameter* of G is defined to be the maximum of $\mathrm{ecc}_G(u)$ as u ranges over $V(G)$, and is denoted by $\mathrm{diam}(G)$. For $X \subseteq V(G)$, we let $G[X]$ denote the subgraph of G induced by X. We let $\overline{G}$ denote the *complement* of G. For two graphs H_1 and H_2, we let $H_1 \cup H_2$ denote the *union* of H_1 and H_2. For a graph H and an integer $s \geq 2$, sH denote the disjoint union of s copies of H. We let K_n denote the *complete graph* of order n, and let P_n denote the *path* of order n. For terms and symbols not defined here, we refer the reader to [5].

Let G be a graph. For two subsets X, Y of $V(G)$, we say that X *dominates* Y if $Y \subseteq \bigcup_{u \in X} N_G[u]$. A subset of $V(G)$ which dominates $V(G)$ is called a *dominating* *set* of G. The minimum cardinality of a dominating set of G is called the *domination number* of G, and is denoted by $\gamma(G)$. A dominating set of G having cardinality $\gamma(G)$ is called a γ-*set* of G. Let $V^0(G) = \{x \in V(G) \mid \gamma(G-x) = \gamma(G)\}$, $V^+(G) = \{x \in V(G) \mid \gamma(G-x) > \gamma(G)\}$ and $V^-(G) = \{x \in V(G) \mid \gamma(G-x) < \gamma(G)\}$. A vertex x belonging to $V^-(G)$ is said to be *critical*. A graph G is *critical* if every vertex of G is critical (i.e., $V(G) = V^-(G)$), and G is k-*critical* if G is critical and $\gamma(G) = k$.

In this paper, we mainly study the relationship between the domination number and the diameter. For $k \geq 1$, it has been known that the diameter of a connected graph G with $\gamma(G) = k$ is at most $3k-1$ (see Theorem 2.24 of [8]), and the bound is best possible. On the other hand, domination criticality often decreases the upper bound on the diameter of a connected graph with domination number k. For example, Fulman, Hanson and MacGillivray [6] showed the following theorem which was conjectured in [2] (and for each $k \geq 2$, since there exist infinitely many k-critical graphs with diameter exactly $2k-2$, the bound in Theorem A is best possible).

Theorem A ([6]) *Let $k \geq 2$ be an integer, and let G be a connected k-critical graph. Then $diam(G) \leq 2k-2$.*

Now we introduce another domination critical concept, which was first introduced in [3]. A graph G is *bicritical* if $\gamma(G-\{x,y\}) < \gamma(G)$ for any two vertices $x, y \in V(G)$, and G is k-*bicritical* if G is bicritical and $\gamma(G) = k$. Note that every bicritical graph G satisfies $V^+(G) = \emptyset$. Brigham, Haynes, Henning and Rall [3] gave a conjecture concerning the upper bound on the diameter of bicritical graphs.

Conjecture 1 ([3]) *Let $k \geq 3$ be an integer, and let G be a connected k-bicritical graph. Then $diam(G) \leq k-1$.*

However, the author [7] constructed counterexamples for Conjecture 1 in the case where $k \neq 4$ as follows.

Theorem B ([7]) *Let $k \geq 3$ be an integer. Then there exist infinitely many connected k-critical and bicritical graphs G with*

$$diam(G) = \begin{cases} \frac{3k-3}{2} & (k \text{ is odd}) \\ \frac{3k-6}{2} & (k \text{ is even}). \end{cases}$$

In this paper, we refine Theorem B by considering only bicritical graphs (and by Theorem 1.1, Conjecture 1 is completely disproved).

Theorem 1.1 *Let $k \geq 3$ be an integer. Then there exist infinitely many connected k-bicritical graphs G with*

$$diam(G) = \begin{cases} 3 & (k = 3) \\ 6 & (k = 5) \\ \frac{3k-1}{2} & (k \text{ is odd and } k \geq 7) \\ \frac{3k-2}{2} & (k \text{ is even}). \end{cases}$$

Since Conjecture 1 is false, we are interested in a non-trivial upper bound of the diameter of bicritical graphs. One may consider that Theorem A gives a valuable information for the diameter of bicritical graphs. However, there exist bicritical graphs which are not critical; for example, the vertex-expansion of a critical and bicritical graph is bicritical and not critical (see [3]), and so we cannot apply Theorem A to all bicritical graphs. Furthermore, there exist infinitely many critical graphs which are not bicritical; for example, the coalescence of 2-critical graphs is critical and not bicritical. Thus it seems difficult to deal with criticality and bicriticality together. So we consider a non-trivial class of graphs which contains all critical graphs and all bicritical graphs.

A graph G is *weak bicritical* if $G - x$ is $\gamma(G)$-critical for every vertex $x \in V(G) - V^-(G)$, and G is *weak k-bicritical* if G is weak bicritical and $\gamma(G) = k$. By the definition, if a graph G is weak bicritical, then $V^+(G) = \emptyset$. Since all critical graphs and all bicritical graphs are weak bicritical, weak bicriticality seems a natural unification of criticality and bicriticality. Indeed, weak bicritical graphs have the same diameter-property as critical graphs. We show the following theorem which is an extension of Theorem A.

Theorem 1.2 *Let $k \geq 2$ be an integer, and let G be a connected weak k-bicritical graph. Then diam$(G) \leq 2k - 2$.*

Recall that for each integer $k \geq 2$, there exist infinitely many connected k-critical graphs with diameter $2k - 2$. Hence the bound in Theorem 1.2 is best possible.

By Theorem 1.2, the diameter of a connected k-bicritical graph is at most $2k - 2$. We further refine such upper bound as follows.

Theorem 1.3 *Let $k \geq 3$ be an integer, and let G be a connected k-bicritical graph. Then diam$(G) \leq 2k - 3$.*

In Section 3, we construct some bicritical graphs and prove Theorem 1.1. We prove Theorem 1.2 in Section 4, and prove Theorem 1.3 in Section 5.

By considering Theorem 1.1, Theorem 1.3 for the case where $k \in \{3, 4\}$ is sharp. We conclude this section with the following conjecture.

Conjecture 2 *Let $k \geq 5$ be an integer. Then there exist infinitely many connected k-bicritical graphs G with diam$(G) = 2k - 3$.*

2 Basic properties

In this section, we prepare some fundamental properties for our proof.

2.1 Weak bicritical graphs

In this subsection, we give some properties of weak bicritical graphs.

We first give a degree condition of weak bicritical graphs.

Proposition 2.1 *Let G be a connected weak bicritical graph of order at least three. Then the minimum degree of G is at least two.*

Proof. Suppose that there exists a vertex x of G of degree exactly one, and write $N_G(x) = \{y\}$. Note that y is not a critical vertex of G. Since $|V(G)| \geq 3$, y is adjacent to a vertex z $(\neq x)$. Let S be a γ-set of $G - \{y, z\}$. Since G is weak bicritical and y is not critical, $|S| \leq \gamma(G) - 1$. Furthermore, $x \in S$ because S dominates x. This implies that $S' = (S - \{x\}) \cup \{y\}$ is a dominating set of G with $|S'| \leq \gamma(G) - 1$, which is a contradiction. □

We next consider weak 2-bicritical graphs. A characterization of 2-critical graphs and 2-bicritical graphs was given in the following two theorems.

Theorem C ([1]) *A graph G is 2-critical if and only if $\overline{G} = nK_2$ for some $n \geq 1$ (i.e., G is the graph obtained from a complete graph of even order by deleting a perfect matching).*

Theorem D ([3]) *There is no 2-bicritical graph of order at least four.*

Now we characterize weak 2-bicritical graphs by using Theorem C.

Theorem 2.2 *A graph G is weak 2-bicritical if and only if $\overline{G} \in \{nK_2, nK_2 \cup K_3, (n-1)K_2 \cup P_3 \mid n \geq 1\}$.*

Proof. If $\overline{G} \in \{nK_2, nK_2 \cup K_3, (n-1)K_2 \cup P_3 \mid n \geq 1\}$, then we can easily check that G is a weak 2-bicritical graph. Thus it suffices to show that if G is weak 2-bicritical, then $\overline{G} \in \{nK_2, nK_2 \cup K_3, (n-1)K_2 \cup P_3 \mid n \geq 1\}$. If G is critical, then by Theorem C, $\overline{G} = nK_2$ for some $n \geq 1$, as desired. Thus we may assume that G has a non-critical vertex u. Since $\gamma(G) = 2$, $V(G) - N_G[u] \neq \emptyset$. Let $x \in V(G) - N_G[u]$. Since G is weak 2-bicritical, $G - u$ is 2-critical, and hence the complement of $G - u$ is isomorphic to nK_2 for some $n \geq 1$ by Theorem C. In particular, x is not adjacent to exactly one vertex y in $G - u$. Since there is no vertex of $G - y$ dominating $V(G) - \{y\}$, y is not a critical vertex of G. Since G is weak 2-bicritical, it follows that $\gamma(G - \{x, y\}) = 1$. This implies that u is adjacent to any vertices in $V(G) - \{u, x, y\}$ in G. If $uy \in E(G)$, then $\overline{G} = (n-1)K_2 \cup P_3$, as desired. Thus we may assume that $uy \notin E(G)$. Then $\overline{G} = (n-1)K_2 \cup K_3$. If $n = 1$, then G consists of three isolated vertices, and so $\gamma(G) = 3$, which is a contradiction. Thus $n \geq 2$, and hence $\overline{G} = mK_2 \cup K_3$ for some $m \geq 1$. □

2.2 Coalescence of two graphs

In this subsection, we introduce a way of constructing of a graph from two small graphs, which was defined in [2].

Let H_1 and H_2 be graphs, and for each $i \in \{1, 2\}$, let x_i be a vertex of H_i. Under this notation, we let $(H_1 \bullet H_2)(x_1, x_2 : x)$ denote the graph obtained from H_1 and H_2 by identifying x_1 and x_2 into a vertex labeled x. We call $(H_1 \bullet H_2)(x_1, x_2 : x)$ a *coalescence* of H_1 and H_2 (via x_1 and x_2).

Some properties for criticality or bicriticality in a coalescence of two graphs have been known. (Note that Lemma 2.3(iii) is a special case of Theorem 3.3 in [7].)

Lemma 2.3 ([2, 4, 7]) *Let H_1 and H_2 be two graphs, and for each $i \in \{1, 2\}$, let x_i be a non-isolated vertex of H_i. Let $G = (H_1 \bullet H_2)(x_1, x_2 : x)$.*

- (i) *If x_i is a critical vertex of H_i for each $i \in \{1, 2\}$, then $V^-(G) = ((V^-(H_1) \cup V^-(H_2)) - \{x_1, x_2\}) \cup \{x\}$ and $\gamma(G) = \gamma(H_1) + \gamma(H_2) - 1$.*
- (ii) *The graph G is critical if and only if both H_1 and H_2 are critical.*
- (iii) *The graph G is bicritical if and only if for some $i \in \{1, 2\}$,*
  - (a) *H_i is critical and bicritical,*
  - (b) *H_{3-i} is bicritical, and*
  - (c) *x_{3-i} is a critical vertex of H_{3-i}.*

In particular, G is critical or bicritical, then $\gamma(G) = \gamma(H_1) + \gamma(H_2) - 1$.

By combining Lemma 2.3(i) and (iii), we get the following result.

Lemma 2.4 *Let H_1 and H_2 be two graphs, and for each $i \in \{1, 2\}$, let x_i be a non-isolated vertex of H_i. If $G = (H_1 \bullet H_2)(x_1, x_2 : x)$ is bicritical, then $V^-(G) = ((V^-(H_1) \cup V^-(H_2)) - \{x_1, x_2\}) \cup \{x\}$.*

2.3 2-coalescence of two graphs

Lemma 2.5 *Let G be a graph, and let x_1 and x_2 be two distinct vertices of G with $N_G[x_1] \subseteq N_G[x_2]$. Then x_2 is not a critical vertex of G. Furthermore, if G is bicritical, then $V(G) - \{x_1, x_2\} \subseteq V^-(G)$.*

Proof. Let S be a γ-set of $G - x_2$. Since S dominates x_1, $S \cap N_G[x_1] \neq \emptyset$, and so $S \cap N_G[x_2] \neq \emptyset$. This implies that S is a dominating set of G, and hence $\gamma(G - x_2) \geq \gamma(G)$. Consequently x_2 is not a critical vertex of G.

Assume that G is bicritical. Let $u \in V(G) - \{x_1, x_2\}$, and let S' be a γ-set of $G - \{x_2, u\}$. Then $|S'| \leq \gamma(G) - 1$. Since S' dominates x_1, S' also dominates x_2. Hence S' is a dominating set of $G - u$, and so $\gamma(G - u) \leq |S'| \leq \gamma(G) - 1$. Since $u \in V(G) - \{x_1, x_2\}$ is arbitrary, we have $V(G) - \{x_1, x_2\} \subseteq V^-(G)$. □

Let H_1 and H_2 be graphs, and for each $i \in \{1, 2\}$, let $x_i^{(1)}$ and $x_i^{(2)}$ be two adjacent vertices of H_i. Under this notation, we let $(H_1 \bullet H_2)(x_1^{(1)}, x_2^{(1)} : x^{(1)})(x_1^{(2)}, x_2^{(2)} : x^{(2)})$

denote the graph obtained from H_1 and H_2 by identifying $x_1^{(i)}$ and $x_2^{(i)}$ into a vertex labeled $x^{(i)}$ for each $i \in \{1, 2\}$. We call $(H_1 \bullet H_2)(x_1^{(1)}, x_2^{(1)} : x^{(1)})(x_1^{(2)}, x_2^{(2)} : x^{(2)})$ a 2-*coalescence* of H_1 and H_2.

Proposition 2.6 *Let H_1 and H_2 be two bicritical graphs, and for each $i \in \{1, 2\}$, let $x_i^{(1)}$ and $x_i^{(2)}$ be two distinct vertices of H_i with $N_{H_i}[x_i^{(1)}] = N_{H_i}[x_i^{(2)}]$. Then the graph $G = (H_1 \bullet H_2)(x_1^{(1)}, x_2^{(1)} : x^{(1)})(x_1^{(2)}, x_2^{(2)} : x^{(2)})$ is bicritical and $\gamma(G) = \gamma(H_1) + \gamma(H_2) - 1$.*

Proof. Recall that deleting a vertex of bicritical graphs cannot increase the domination number. By Lemma 2.5, for each $i \in \{1, 2\}$, $V^0(H_i) = \{x_i^{(1)}, x_i^{(2)}\}$ and $V^-(H_i) = V(H_i) - \{x_i^{(1)}, x_i^{(2)}\}$. Since $H_i - x_i^{(1)}$ is critical for each $i \in \{1, 2\}$, $(H_1 - x_1^{(1)} \bullet H_2 - x_2^{(1)})(x_1^{(2)}, x_2^{(2)} : x^{(2)})(= G - x^{(1)})$ is critical by Lemma 2.3(ii). By Lemma 2.3,

$$\gamma(G - x^{(1)}) = \gamma(H_1 - x_1^{(1)}) + \gamma(H_2 - x_2^{(1)}) - 1 = \gamma(H_1) + \gamma(H_2) - 1.$$

Since $N_G[x^{(1)}] = N_G[x^{(2)}]$, we see that $x^{(1)} \in V^0(G)$. Hence $\gamma(G) = \gamma(H_1) + \gamma(H_2) - 1$.

We show that G is bicritical. Let u and v be two distinct vertices of G. It suffices to show that there exists a dominating set S of $G - \{u, v\}$ with $|S| \leq \gamma(H_1) + \gamma(H_2) - 2$.

Case 1: $\{u, v\} = \{x^{(1)}, x^{(2)}\}$.

For each $i \in \{1, 2\}$, let S_i be a γ-set of $H_i - \{x_i^{(1)}, x_i^{(2)}\}$. Then $S_1 \cup S_2$ is a dominating set of $G - \{u, v\}$. Since both H_1 and H_2 are bicritical, $|S_1 \cup S_2| = |S_1| + |S_2| \leq (\gamma(H_1) - 1) + (\gamma(H_2) - 1)$.

Case 2: $|\{u, v\} \cap \{x^{(1)}, x^{(2)}\}| = 1$.

We may assume that $u = x^{(1)}$ and $v \in V(H_1) - \{x_1^{(1)}, x_1^{(2)}\}$. Let S_1 be a γ-set of $H_1 - \{x_1^{(1)}, v\}$, and let S_2 be a γ-set of $H_2 - \{x_2^{(1)}, x_2^{(2)}\}$. If $x_1^{(2)} \notin S_1$, let $S = S_1 \cup S_2$; if $x_1^{(2)} \in S_1$, let $S = (S_1 - \{x_1^{(2)}\}) \cup \{x^{(2)}\} \cup S_2$. Then S is a dominating set of $G - \{u, v\}$. Since both H_1 and H_2 are bicritical, $|S| = |S_1| + |S_2| \leq (\gamma(H_1) - 1) + (\gamma(H_2) - 1)$.

Case 3: $\{u, v\} \cap \{x^{(1)}, x^{(2)}\} = \emptyset$ and $u, v \in V(H_i)$ for some $i \in \{1, 2\}$.

We may assume that $u, v \in V(H_1) - \{x_1^{(1)}, x_1^{(2)}\}$. Let S_1 be a γ-set of $H_1 - \{u, v\}$, and let S_2 be a γ-set of $H_2 - \{x_2^{(1)}, x_2^{(2)}\}$. Since $N_{H_1 - \{u,v\}}[x_1^{(1)}] = N_{H_1 - \{u,v\}}[x_1^{(2)}]$, we see that $|S_1 \cap \{x_1^{(1)}, x_1^{(2)}\}| \leq 1$. We may assume that $x_1^{(1)} \notin S_1$. If $x_1^{(2)} \notin S_1$, let $S = S_1 \cup S_2$; if $x_1^{(2)} \in S_1$, let $S = (S_1 - \{x_1^{(2)}\}) \cup \{x^{(2)}\} \cup S_2$. Then S is a dominating set of $G - \{u, v\}$. Since both H_1 and H_2 are bicritical, $|S| = |S_1| + |S_2| \leq (\gamma(H_1) - 1) + (\gamma(H_2) - 1)$.

Case 4: $\{u, v\} \cap \{x^{(1)}, x^{(2)}\} = \emptyset$ and $|V(H_i) \cap \{u, v\}| = 1$ for each $i \in \{1, 2\}$.

We may assume that $u \in V(H_1) - \{x_1^{(1)}, x_1^{(2)}\}$ and $v \in V(H_2) - \{x_2^{(1)}, x_2^{(2)}\}$. Let S_1 be a γ-set of $H_1 - \{u, x_1^{(1)}\}$, and let S_2 be a γ-set of $H_2 - \{u, x_2^{(1)}\}$. If $x_i^{(2)} \notin S_i$ for each $i \in \{1, 2\}$, let $S = S_1 \cup S_2$; if $x_i^{(2)} \in S_i$ for some $i \in \{1, 2\}$, let $S = ((S_1 \cup S_2) - \{x_1^{(2)}, x_2^{(2)}\}) \cup \{x^{(2)}\}$. Then S is a dominating set of $G - \{u, v, x^{(1)}\}$.

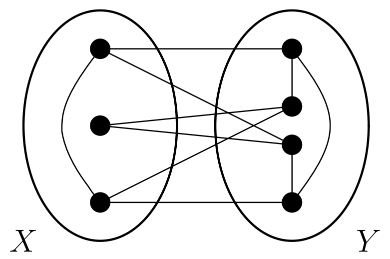

Figure 1: Graph A

Since S dominates $x^{(2)}$ and $N_G[x^{(1)}] = N_G[x^{(2)}]$, S also dominates $x^{(1)}$, and hence S is a dominating set of $G - \{u, v\}$. Since both H_1 and H_2 are bicritical, $|S| \leq |S_1| + |S_2| \leq (\gamma(H_1) - 1) + (\gamma(H_2) - 1)$.

This completes the proof of Proposition 2.6. □

3 Examples

In this section, we show Theorem 1.1. We first construct some bicritical graphs with small domination number.

Let A be the graph on $X \cup Y$ depicted in Figure 1. Let s be a positive integer, and let $A_i^{(j)}$ ($i \in \{1, 2\}, 1 \leq j \leq s$) be disjoint copies of A. For $i \in \{1, 2\}$ and $1 \leq j \leq s$, let $X_i^{(j)}$ (resp. $Y_i^{(j)}$) be the subset of $V(A_i^{(j)})$ which corresponds to the set X (resp. the set Y). Let $z^{(1)}, z^{(2)}, y$ be new vertices. We define some sets of edges as follows: Let

$$F_1 = \left\{ z^{(1)}u, z^{(2)}u \;\middle|\; u \in \bigcup_{1 \leq j \leq s} X_1^{(j)} \right\} \cup \left\{ yu \;\middle|\; u \in \bigcup_{1 \leq j \leq s} X_2^{(j)} \right\} \cup \{z^{(1)}z^{(2)}\},$$

$$F_2 = \left\{ uv \mid u \in Y_i^{(j)}, v \in V(A_i^{(j')}), i \in \{1, 2\}, j \neq j' \right\},$$

and

$$F_3 = \left\{ uv \;\middle|\; u \in \bigcup_{1 \leq j \leq s} Y_1^{(j)}, v \in \bigcup_{1 \leq j \leq s} Y_2^{(j)} \right\}.$$

Let L_s be the graph defined by

$$V(L_s) = \{z^{(1)}, z^{(2)}, y\} \cup \left(\bigcup_{i \in \{1,2\}} \left(\bigcup_{1 \leq j \leq s} V(A_i^{(j)}) \right) \right)$$

and

$$E(L_s) = \left(\bigcup_{1 \leq h \leq 3} F_h \right) \cup \left(\bigcup_{i \in \{1,2\}} \left(\bigcup_{1 \leq j \leq s} E(A_i^{(j)}) \right) \right).$$

Then we can verify that L_s is 4-bicritical graph with $\mathrm{diam}(L_s) = 5$ by tedious argument (and we omit the details). By Lemma 2.5, $V^0(L_s) = \{z^{(1)}, z^{(2)}\}$ and $V^-(L_s) = V(L_s) - \{z^{(1)}, z^{(2)}\}$. In particular, y is a critical vertex of L_s.

Let s be a positive integer, and let H_1 and H_2 be disjoint copies of L_s. For each $i \in \{1, 2\}$, let $x_i^{(1)}$ and $x_i^{(2)}$ be the distinct vertices of H_i with $N_{H_i}[x_i^{(1)}] = N_{H_i}[x_i^{(2)}]$. Then by Proposition 2.6, $L_s^* = (H_1 \bullet H_2)(x_1^{(1)}, x_2^{(1)} : x^{(1)})(x_1^{(2)}, x_2^{(2)} : x^{(2)})$ is a 7-bicritical graph with $\mathrm{diam}(L_s^*) = 10$. By Lemma 2.5, $V^0(L_s^*) = \{x^{(1)}, x^{(2)}\}$ and $V^-(L_s^*) = V(L_s^*) - \{x^{(1)}, x^{(2)}\}$.

Proof of Theorem 1.1. Let k be as in Theorem 1.1. If $k \in \{3, 5\}$, then Theorem B yields the desired result. Thus we may assume that $k \notin \{3, 5\}$. Fix a positive integer s. If k is even, let $G_1 = L_s$ and $m = (k-2)/2$; if k is odd, let $G_1 = L_s^*$ and $m = (k-5)/2$. In either case, there exists a vertex $w_1' \in V^-(G_1)$ with $\mathrm{ecc}_{G_1}(w_1') = \mathrm{diam}(G_1)$. By Theorem B, there exists a connected 3-critical and bicritical graph with diameter 3. For each $2 \le i \le m$, let G_i be a connected 3-critical and bicritical graph with diameter 3, and let w_i and w_i' be vertices of G_i which are at distance three apart. Let G be the graph obtained by concatenating $G_1, \ldots, G_m$ by letting G_{i-1} and G_i coalesce via w_{i-1}' and w_i for each $2 \le i \le m$. Then

$$\mathrm{diam}(G) = \sum_{1 \le i \le m} \mathrm{diam}(G_i) = \begin{cases} \frac{3k-1}{2} & (k \text{ is odd}) \\ \frac{3k-2}{2} & (k \text{ is even}). \end{cases}$$

Further by Lemma 2.3(iii), G is bicritical and $\gamma(G) = \gamma(G_1) + \sum_{2 \le i \le m} (\gamma(G_i) - 1) = k$. Since there exist infinitely many candidates for G_1, this yields the desired conclusion. □

4 Proof of Theorem 1.2

Let $l \ge 3$ be an integer, and let G be a connected graph. A pair (x, j) of a vertex $x \in V(G)$ and an integer $j \ge 2$ is *l-sufficient* if $\mathrm{ecc}_G(x) = \mathrm{diam}(G)$ and there exists a γ-set S of G with $|S \cap (\bigcup_{0 \le i \le j} N_G^{(i)}(x))| \ge (j + l)/2$.

Lemma 4.1 *Let $k \ge 3$ and $l \ge 3$ be integers, and let G be a connected weak k-bicritical graph having an l-sufficient pair. Then $\mathrm{diam}(G) \le 2k - l + 1$.*

Proof. Let (x, m) be an l-sufficient pair of G so that m is as large as possible. For each $i \ge 0$, let $X_i = N_G^{(i)}(x)$ and $U_i = X_0 \cup \cdots \cup X_i$. Then there exists a γ-set S_1 of G with $|S_1 \cap U_m| \ge (m + l)/2$. Suppose that $\mathrm{diam}(G) \ge 2k - l + 2$. Since $k \ge |S_1 \cap U_m| \ge (m+l)/2$, it follows that $\mathrm{diam}(G) \ge m+2$. Since $|S_1 \cap U_m| \ge (m+l)/2$ and $|S_1 \cap U_{m+2}| < ((m + 2) + l)/2$ by the maximality of m, $|S_1 \cap (X_{m+1} \cup X_{m+2})| = |S_1 \cap U_{m+2}| - |S_1 \cap U_m| < (m + l + 2)/2 - (m + l)/2 = 1$. This implies that $S_1 \cap (X_{m+1} \cup X_{m+2}) = \emptyset$, and hence $S_1 \cap X_{m+3} \ne \emptyset$. Since $\mathrm{diam}(G) \ge 2k - l + 2$ and $k \ge |S_1 \cap U_{m+3}| \ge (m + l)/2 + 1$, we have $\mathrm{diam}(G) \ge m + 4$.

Claim 4.1 *For every γ-set S_0 of G, $|S_0 \cap U_{m+2}| \geq (m+l)/2$ and $|S_0 \cap (X_{m+3} \cup X_{m+4})| \leq 1$.*

Proof. We first show that $S_0' = (S_0 \cap U_{m+2}) \cup (S_1 - U_{m+2})$ is a dominating set of G. Since S_0 dominates $V(G)$, $S_0 \cap U_{m+2}$ dominates U_{m+1}. Since S_1 dominates $V(G)$ and $S_1 \cap X_{m+1} = S_1 \cap X_{m+2} = \emptyset$, $S_1 - U_{m+2}$ dominates $V(G) - U_{m+1}$. Hence S_0' is a dominating set of G.

Since S_1 is a γ-set of G, $|S_1| \leq |S_0'|$. In particular, $|S_0 \cap U_{m+2}| = |S_0'| - |S_1 - U_{m+2}| \geq |S_1| - |S_1 - U_{m+2}| = |S_1 \cap U_{m+2}| \geq (m+l)/2$. Since S_0 is not $(m+4)$-sufficient by the maximality of m, $|S_0 \cap U_{m+4}| < ((m+4)+l)/2$. Therefore $|S_0 \cap (X_{m+3} \cup X_{m+4})| = |S_0 \cap U_{m+4}| - |S_0 \cap U_{m+2}| < (m+l+4)/2 - (m+l)/2 = 2$. □

Since $S_1 \cap X_{m+3} \neq \emptyset$, $|S_1 \cap X_{m+3}| = 1$ and $S_1 \cap X_{m+4} = \emptyset$ by Claim 4.1. Hence the unique vertex w in $S_1 \cap X_{m+3}$ dominates $X_{m+2} \cup X_{m+3}$. Let $w' \in X_{m+2}$. Note that $ww' \in E(G)$. If w is critical, let S_2 be a γ-set of $G - w$; if w is not critical, let S_2 be a γ-set of $G - \{w, w'\}$. Since G is weak bicritical, $|S_2| \leq \gamma(G) - 1$, and hence both $S_2 \cup \{w\}$ and $S_2 \cup \{w'\}$ are γ-sets of G. By Claim 4.1, $|S_2 \cap U_{m+2}| = |(S_2 \cup \{w\}) \cap U_{m+2}| \geq (m+l)/2$. Then $|(S_2 \cup \{w'\}) \cap U_{m+2}| = |S_2 \cap U_{m+2}| + 1 \geq ((m+2)+l)/2$, and hence $(x, m+2)$ is an l-sufficient pair of G, which contradicts the maximality of m.

This completes the proof of Lemma 4.1. □

Now we prove Theorem 1.2.

Proof of Theorem 1.2. Let k and G be as in Theorem 1.2. If $k = 2$, then $\mathrm{diam}(G) = 2 = 2k - 2$ by Theorem 2.2, as desired. Thus we may assume that $k \geq 3$. Suppose that $\mathrm{diam}(G) \geq 2k - 1$. Then by Lemma 4.1, there exists no 3-sufficient pair of G.

Let x be a vertex of G with $\mathrm{ecc}_G(x) = \mathrm{diam}(G)$. Since $\mathrm{diam}(G) \geq 2k - 1 \geq 5$, $N_G^{(4)}(x) \neq \emptyset$. Let $w \in N_G^{(3)}(x)$ and $w' \in N_G^{(4)}(x)$ be vertices so that $ww' \in E(G)$. If w is a critical vertex of G, let S be a γ-set of $G - w$; if w is not a critical vertex of G, let S be a γ-set of $G - \{w, w'\}$. In either case, since G is weak k-bicritical, $|S| \leq k - 1$ and $S \cup \{w\}$ is a γ-set of G. Since $(x, 3)$ is not a 3-sufficient pair of G, $|(S \cup \{w\}) \cap (\bigcup_{0 \leq i \leq 3} N_G^{(i)}(x))| < (3+3)/2 = 3$. Furthermore, $S \cap (\bigcup_{0 \leq i \leq 3} N_G^{(i)}(x))$ dominates $\bigcup_{0 \leq i \leq 2} N_G^{(i)}(x)$ in $G - w$ or $G - \{w, w'\}$ according as w is critical or not. This implies that $S \cap (\bigcup_{0 \leq i \leq 3} N_G^{(i)}(x))$ consists of exactly one vertex $a \in N_G(x)$ dominating $\bigcup_{0 \leq i \leq 2} N_G^{(i)}(x)$. Since $N_G[x] \subseteq N_G[a]$, a is not a critical vertex of G by Lemma 2.5. By Proposition 2.1, $N_G(x) - \{a\} \neq \emptyset$. Let $b \in N_G(x) - \{a\}$, and let S' be a γ-set of $G - \{a, b\}$. Since G is weak bicritical, $|S'| \leq \gamma(G) - 1$. Since S' dominates x in $G - \{a, b\}$, $S' \cap N_{G-\{a,b\}}[x] \neq \emptyset$. Let $c \in S' \cap N_{G-\{a,b\}}[x]$. Since a dominates $\bigcup_{0 \leq i \leq 2} N_G^{(i)}(x)$ in G, $N_G[c] \subseteq N_G[a]$, and hence $S'' = (S' - \{c\}) \cup \{a\}$ is a dominating set of G with $|S''| = |S'| \leq \gamma(G) - 1$, which is a contradiction. Therefore $\mathrm{diam}(G) \leq 2k - 2$. □

5 Proof of Theorem 1.3

In this section, we prove Theorem 1.3. We start with a lemma.

Lemma 5.1 *Let G be a bicritical graph, and let x be a non-isolated vertex of G. Then there exists a vertex $y \in N_G(x)$ such that no vertex in $N_G(x) - \{y\}$ dominates $N_G[x] - \{y\}$.*

Proof. Suppose that for every $y \in N_G(x)$, there exists a vertex in $N_G(x)-\{y\}$ which dominates $N_G[x] - \{y\}$. Let $y_1 \in N_G(x)$, and for each $i \geq 2$, let $y_i \in N_G(x) - \{y_{i-1}\}$ be a vertex which dominates $N_G[x] - \{y_{i-1}\}$. Suppose that $y_1 = y_3$. Let S_1 be a γ-set of $G - \{y_1, y_2\}$. Since S_1 dominates x, $S_1 \cap (N_G[x] - \{y_1, y_2\}) \neq \emptyset$. Since every vertex in $N_G[x] - \{y_1, y_2\}$ is adjacent to both y_1 and y_2 by the choice of y_1 $(= y_3)$ and y_2, this implies that S_1 is a dominating set of G with $|S_1| \leq \gamma(G) - 1$, which is a contradiction. Thus $y_1 \neq y_3$. Since y_2 dominates $N_G[x] - \{y_1\}$, $y_2y_3 \in E(G)$, and hence y_3 dominates $N_G[x]$. Let S_2 be a γ-set of $G - \{y_3, y_4\}$. Since S_2 dominates x, $S_2 \cap (N_G[x] - \{y_3, y_4\}) \neq \emptyset$. Since every vertex in $N_G[x] - \{y_3, y_4\}$ is adjacent to both y_3 and y_4, this implies that S_2 is a dominating set of G with $|S_2| \leq \gamma(G) - 1$, which is a contradiction. □

Proof of Theorem 1.3. Let k and G be as in Theorem 1.3. Suppose that $\mathrm{diam}(G) \geq 2k - 2$. Then by Theorem 1.2, $\mathrm{diam}(G) = 2k - 2$. By Lemma 4.1, there exists no 4-sufficient pair. Let $A = \{x \in V(G) \mid \mathrm{ecc}_G(x) = \mathrm{diam}(G)\}$. By Lemma 5.1, for each $x \in A$, there exists a vertex $y_x \in N_G(x)$ such that no vertex in $N_G(x) - \{y_x\}$ dominates $N_G[x] - \{y_x\}$.

Claim 5.1 *Let $x \in A$ be a vertex. Then the following hold:*

(i) *There exists a vertex in $N_G^{(2)}(x)$ which dominates $N_G(x) - \{y_x\}$.*

(ii) *The vertex y_x is not a critical vertex of G.*

(iii) $|N_G^{(2)}(x)| \geq 2$.

(iv) *There exists no vertex in $N_G(x)$ which dominates $N_G[x]$.*

Proof. We first show (i) and (ii). If y_x is a critical vertex of G, let S be a γ-set of $G - y_x$; if y is not a critical vertex of G, let S be a γ-set of $G - \{x, y_x\}$. In either case, $|S| \leq k - 1$ and $S \cup \{x\}$ is a γ-set of G. Since $(x, 2)$ is not a 4-sufficient pair of G, $|(S \cup \{x\}) \cap (\bigcup_{0 \leq i \leq 2} N_G^{(i)}(x))| < (2 + 4)/2 = 3$, and hence $|S \cap (\bigcup_{0 \leq i \leq 2} N_G^{(i)}(x))| \leq 1$. Since S dominates $N_G(x) - \{y_x\}$ and no vertex in $N_G(x) - \{y_x\}$ dominates $N_G(x) - \{y_x\}$, this implies that $|S \cap (\bigcup_{0 \leq i \leq 2} N_G^{(i)}(x))| = |S \cap N_G^{(2)}(x)| = 1$ and the unique vertex $w \in S \cap N_G^{(2)}(x)$ dominates $N_G(x) - \{y_x\}$. In particular, (i) holds. If y_x is a critical vertex of G, then S dominates x, which is a contradiction. Thus (ii) holds.

We next show (iii). Suppose that $|N_G^{(2)}(x)| = 1$, and let $H_1 = G[\bigcup_{0\le i\le 2} N_G^{(i)}(x)]$ and $H_2 = G - N_G[x]$. Then we can regard G as a coalescence of H_1 and H_2, and hence H_1 is bicritical by Lemma 2.3(iii). On the other hand, since $|V(H_1)| \geq 4$ and $\gamma(H_1) \leq 2$, H_1 is not bicritical by Theorem D, which is a contradiction.

We finally show (iv). Suppose that there exists a vertex in $N_G(x)$ which dominates $N_G[x]$ in G. By the definition of y_x, no vertex in $N_G(x) - \{y_x\}$ dominates $N_G[x]$ in G. Thus y_x dominates $N_G[x]$ in G. Let $x' \in V(G)$ with $d_G(x, x') = \mathrm{ecc}_G(x)$. Since $x' \in A$, the vertex $y_{x'}$ is not a critical vertex of G by (ii). Let S be a γ-set of $G - \{y_x, y_{x'}\}$. Since S dominates x, $S \cap (N_G[x] - \{y_x\}) \neq \emptyset$. Since y_x dominates $N_G[x]$ in G, S dominates y_x. In particular, S is a dominating set of $G - y_{x'}$, and hence $y_{x'}$ is a critical vertex of G, which is a contradiction. □

Let $x \in A$. For each $i \geq 0$, let $X_i = N_G^{(i)}(x)$ and $U_i = X_0 \cup \cdots \cup X_i$. By Claim 5.1(i), there exists a vertex $w_2 \in X_2$ which dominates $N_G(x) - \{y_x\}$. Let S_1 be a γ-set of $G - \{y_x, w_2\}$. If $S_1 \cap N_G(x) \neq \emptyset$, then S_1 is a dominating set of $G - y_x$, and hence y_x is a critical vertex of G, which contradicts Claim 5.1(ii). Thus $S_1 \cap N_G(x) = \emptyset$. Since S_1 dominates x, we have $x \in S_1$. Note that $S_1 \cup \{w_2\}$ is a γ-set of G. Since both $(x, 2)$ and $(x, 3)$ are not a 4-sufficient pair, $|(S_1 \cup \{w_2\}) \cap U_2| < (2 + 4)/2 = 3$ and $|(S_1 \cup \{w_2\}) \cap U_3| < (3 + 4)/2 = 7/2$. This forces $S_1 \cap U_2 = \{x\}$ and $|S_1 \cap X_3| \leq 1$. By Claim 5.1(iii), $X_2 - \{w_2\} \neq \emptyset$, and hence $|S_1 \cap X_3| = 1$ and the unique vertex w_3 in $S_1 \cap X_3$ dominates $X_2 - \{w_2\}$.

Let m be the maximum integer satisfying that $S_1 \cap X_{2j+1} \neq \emptyset$ for all j with $1 \leq j \leq m$. Choose a γ-set S_1 of $G - \{y_x, w_2\}$ so that m is as large as possible. Recall that $x \in S_1$. This together with the definition of m leads to $|S_1 \cap U_{2m+1}| \geq m + 1$. Since $(x, 2m+2)$ is not a 4-sufficient pair, $|(S_1 \cup \{w_2\}) \cap U_{2m+2}| < ((2m+2)+4)/2 = m+3$ (i.e., $|S_1 \cap U_{2m+2}| \leq m + 1$). This forces

- $|S_1 \cap U_{2m+1}| = m + 1$,
- $S_1 \cap X_{2j} = \emptyset$ for all j with $1 \leq j \leq m + 1$, and
- $|S_1 \cap X_{2j+1}| = 1$ for all j with $1 \leq j \leq m$.

By the maximality of m, we have $S_1 \cap X_{2m+3} = \emptyset$. Write $S_1 \cap X_{2m+1} = \{z\}$. Since $S_1 \cap X_{2m} = S_1 \cap X_{2m+2} = S_1 \cap X_{2m+3} = \emptyset$, z dominates $X_{2m+1} \cup X_{2m+2}$. Since $m + 2 = |(S_1 \cup \{w_2\}) \cap U_{2m+1}| \leq k$, $1 \leq m \leq k - 2$.

Suppose that $m = k - 2$ (i.e., $\mathrm{diam}(G) = 2k - 2 = 2m + 2$). Let $x' \in X_{2k-2}$. Then $x' \in A$. Since $N_G[x'] \subseteq X_{2k-3} \cup X_{2k-2}$, z dominates $N_G[x']$, which contradicts Claim 5.1(iv). Thus $m \leq k-3$, and so $\mathrm{diam}(G) = 2k-2 \geq 2m+4$. Let S_2 be a γ-set of $G - \{y_x, z\}$. If S_2 dominates z in G, then S_2 is a dominating set of $G - y_x$, and hence y_x is a critical vertex of G, which contradicts Claim 5.1(ii). Thus S_2 does not dominate z. This implies that $S_2 \cap (X_{2m+1} \cup X_{2m+2}) = \emptyset$. Since $\mathrm{diam}(G) \geq 2m + 4$, we have

$$S_2 \cap X_{2m+3} \neq \emptyset. \tag{5.1}$$

Claim 5.2 $|S_2 - U_{2m+1}| \leq k - m - 2$.

Proof. We first show that $S_0 = (S_2 \cap U_{2m}) \cup (S_1 - U_{2m})$ is a dominating set of $G - y_x$. Since S_2 dominates $V(G) - \{y_x, z\}$ and $S_2 \cap X_{2m+1} = \emptyset$, $S_2 \cap U_{2m}$ dominates $U_{2m} - \{y_x\}$. Recall that z dominates $X_{2m+1} \cup X_{2m+2}$. Since S_1 dominates $V(G) - \{y_x, w_2\}$ and $z \in S_1 - U_{2m}$, $S_1 - U_{2m}$ dominates $V(G) - U_{2m}$. Hence S_0 is a dominating set of $G - y_x$.

Since y_x is not a critical vertex of G by Claim 5.1(ii), $|S_2 \cap U_{2m}| + |S_1 - U_{2m}| = |S_0| \geq k$. On the other hand, $|S_1 - U_{2m}| = |S_1| - |S_1 \cap U_{2m}| = (k-1) - m$. Consequently, we have $|S_2 \cap U_{2m+1}| = |S_2 \cap U_{2m}| \geq k - (k - m - 1) = m + 1$. This together with $|S_2| \leq k - 1$ leads to $|S_2 - U_{2m+1}| = |S_2| - |S_2 \cap U_{2m+1}| \leq (k-1) - (m+1) = k - m - 2$. □

Set $S^* = (S_1 \cap U_{2m+1}) \cup (S_2 - U_{2m+1})$. Since S_1 is a dominating set of $G - \{y_x, w_2\}$ and $z \in S_1 \cap U_{2m+1}$, $S_1 \cap U_{2m+1}$ dominates $U_{2m+2} - \{y_x, w_2\}$ and $y_x, w_2 \notin S_1 \cap U_{2m+1}$. Since S_2 dominates $V(G) - \{y_x, z\}$, $S_2 - U_{2m+1}$ dominates $V(G) - U_{2m+2}$. Hence S^* is a dominating set of $G - \{y_x, w_2\}$. Since $|S^*| = |S_1 \cap U_{2m+1}| + |S_2 - U_{2m+1}| \leq (m+1) + (k-m-2) = k-1$ by Claim 5.2, S^* is a γ-set of $G - \{y_x, w_2\}$. Then by the definition of m and (5.1), $S^* \cap X_{2j+1} \neq \emptyset$ for all j with $1 \leq j \leq m+1$, which contradicts the maximality of m. Therefore $\mathrm{diam}(G) \leq 2k - 3$.

This completes the proof of Theorem 1.3. □

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