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Embeddings in Eulerian graceful graphs

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Abstract

Let $G(V, E)$ be a graph of order n and size m. A *graceful labeling* of G is an injection $f : V(G) \to \{0, 1, 2, ..., m\}$ such that, when each edge uv is assigned the label $f(uv) = |f(u) - f(v)|$, the resultant edge labels are distinct. We focus on general results in graceful labelings, and provide an affirmative answer to the following open problem: *Can every connected graph be embedded as an induced subgraph in an Eulerian graceful graph?* As a result we infer that the problems on deciding whether the chromatic number is less than or equal to an integer k, for $k \geq 3$, and deciding whether the clique number is greater than or equal to an integer k, for $k \geq 3$, are NP-Complete even for Eulerian graceful graphs.

1 Introduction

Labeling, in general, is naming objects using precise symbolic format. Let $G(V, E)$ be a finite simple undirected graph of order $n = |V(G)|$ and size $m = |E(G)|$. The graph G can be labeled either by its vertices, or edges, or a combination of both. *Vertex labeling* of a graph G is an injective function $f : V(G) \to \mathbb{Z}^+ \cup \{0\}$, i.e. assignment of a set of non-negative integers to $V(G)$. Such a vertex labeling naturally induces an edge labeling, where an edge $e = uv$ gets the label $f(u) * f(v)$. The operator $*$ is usually defined to optimize the number of symbols used to label the entire graph.

In the past few decades many labelings have been studied, primarily originating from the following one introduced by Rosa [12]. Let $f : V(G) \to \{0, 1, 2, ..., m\}$ be an injective vertex labeling with edge labels defined as $f(uv) = |f(u) - f(v)|$, for every $uv \in E(G)$. Rosa defined such a labeling to be a $\beta - valuation$ if $f(E) = \{1, 2, ..., m\}$. This was later termed as *graceful labeling* by Golomb [9], which became widely accepted. The updated details of most graph labeling problems can be found in Gallian's dynamic survey [7].

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Before diving into the general results in graceful labeling, we shall define some related terminology. For an integer k, set $[k] = \{0, 1, 2, .., k\}$. An unpublished result of Erdős states that most graphs are not graceful ([7], page 5). If a graph of size m is not graceful, then its vertices cannot be labeled from $[m]$ to produce distinct edge labels. However for a suitable $k \geq m$, it is possible to label edges distinctly. The minimum value of such k for a graph G is its *index of gracefulness*, denoted by $\theta(G)$. Acharya [1, 4] proved that such vertex labelings always exist. If a graph G is graceful, then $\theta(G) = m$, else $\theta(G) > m$. So, the index of gracefulness measures how close the graph is to being graceful. Also, it is easy to see that there must be two vertices in $V(G)$ having labels 0 and $\theta(G) = k$ (else k won't be the index of gracefulness). It is well-known that $\theta(G) \sim O(n^2)$ [7].

For the reader's convenience, we recall some basic definitions about graphs which will be useful in this article. An *Eulerian cycle* is a closed walk in a graph that visits every edge exactly once. A graph possessing an Eulerian cycle is said to be *Eulerian.* In an Eulerian graph the degree of every vertex is even. The *chromatic number* $\chi(G)$ of a graph G is the smallest number of colors needed to color the vertices of G such that adjacent vertices have different colors. The *clique number* $\omega(G)$ of a graph G is the largest set of pairwise adjacent vertices. For other definitions we refer to the standard text by Harary [10].

Although a lot of papers have been published on various graph labelings, very few of them present general results on graceful labelings. This article is motivated by the following open problem, originally posed by Rao and, mentioned in such a paper by Acharya and Arumugum [3].

Can every connected graph be embedded as an induced subgraph in an Eulerian graceful graph?

We prove a stronger result which is an affirmation to this problem and can be stated as follows:

Theorem 1.1. *Every graph can be embedded as an induced subgraph in an Eulerian graceful graph.*

Our approach is to first embed the given graph G in an Eulerian graph G_1 as an induced subgraph. Then we shall add some triangles, squares and pentagons to one of the edges of G_1 to make it graceful. Finally we make it Eulerian (if necessary) resulting in an Eulerian graceful graph H. Since H induces G_1 and G_1 induces G, H induces G.

Results and Organization: In Section 2 we explore the general results known on embeddings of graphs in graceful graphs. In Section 3 we develop an exponential algorithm to embed G, as an induced subgraph, in an Eulerian graceful graph H. We also discuss cases where the complexity of this algorithm can be reduced, at the cost of a few families of graphs. It is then shown that deciding chromatic number is less than or equal to k, for $k \geq 3$; and deciding whether the clique number is greater than or equal to an integer k, for $k \geq 3$, are NP-complete even for Eulerian graceful graphs.

2 Known General Results

The following basic question on embeddings in graceful graphs has been stated by Acharya in [2] and proved by Acharya, Rao and Arumugam in [5]:

Any graph G can be embedded as an induced subgraph of a graceful graph.

The following results have been proved in [5]:

1. *The problem of deciding whether the following parameters are NP-complete for graceful graphs:*
   a. *The chromatic number $\chi(H)$ is less than or equal to k, for $k \geq 3$.*
   b. *The domination number $\gamma(H)$ is less than or equal to k, for $k \geq 3$.*
   c. *The clique number $\omega(H)$ is greater than or equal to k, for $k \geq 3$.*

The following resolves (see [5]) an unsolved problem by Chartrand and Lesniak in [6] (Page 266, see also the 5^{th} edition):

2. *Graceful graphs can have arbitrarily large chromatic number, domination number and clique number.*
3. *Any triangle free graph G can be embedded as an induced subgraph of a triangle free graceful graph. Also there exist triangle free graceful graphs with arbitrarily large chromatic number.*

3 Main Results

We now provide lemmas and constructions which are necessary for the proof of Theorem 1.1.

Lemma 3.1. *Every graph G can be embedded as an induced subgraph in an Eulerian graph G_1.*

Proof. We introduce a new vertex v_0 and join it to every vertex v_i of G, where $i = 1$ to $|V(G)|$. Now we join the vertices with odd degree to another new vertex v_{new}. Degree of v_{new} is even, since sum of degrees of vertices in a graph is always even. If v_0 and v_{new} are not adjacent, then we join them by an edge and also to a new vertex u. Hence the resultant graph G_1 is Eulerian and G is an induced subgraph of G_1. □

Remark 3.2. An alternate way to form G_1 is to add edges from vertices of the connected graph with odd degree to a new vertex. But the construction in Lemma 3.1 extends such an embedding to disconnected graphs.

Lemma 3.3. *Every Eulerian graph G_1 obtained by Lemma 3.1 can be embedded as an induced subgraph of a graceful graph such that the edge labels of G are different from the vertex labels of G.*

Proof. Let vertex v_0 be labeled 0, and the rest vertices are labeled using increasing powers of 3. None of the edges have the same labels since for distinct a and b, $3^a - 3^b = 3^p - 3^q$ if and only if $a = p,\ b = q$. Let $\{l_1, l_2, ..., l_k\}$ be the missing edge labels. Since v_0 is adjacent to every other vertex in $V(G_1)$, none of these l_i are vertex labels. Now in order to make the graph graceful we introduce new vertices with labels l_i and join them to v_0. Hence we have a graceful graph with G_1 as an induced subgraph. □

Remark 3.4. It should be noted that we shall just use the labeling of G_1 in Lemma 3.3 in our algorithm. An obvious question is why we choose 3^i instead of 2^i. A clear disadvantage of a 2^i labeling is that the presence of a path '*abc*' with vertices a, b and c labeled as 0, 2^{n-1} and 2^n respectively will result in edges '*ab*' and '*bc*' having the same labels. Also as we shall see later (in Subsection 3.2) we need our vertex labels to be odd, hence we use 3.

Let the Eulerian graph G_1 formed by the construction provided in Lemma 3.1 be of order n and size m. Let v_n be the vertex of $V(G_1) \setminus \{v_0\}$ with the highest degree. Our aim is to embed G_1 in an Eulerian graceful graph H, as an induced subgraph. So we modify the construction provided in Lemma 3.3 by adding triangles, squares and pentagons to one of the edges of G_1 such that all the vertices have even degree and the resultant graph is graceful. We begin with the 3^i labeling along with a few constructions.

3.1 3^i Labeling

We label $V(G_1)$ as described in Lemma 3.3 such that the maximum label 3^n is assigned to the vertex $v_n \in V(G_1) \setminus \{v_0\}$ with the highest degree. Care should be taken that vertex v_{n-1}, with label 3^{n-1}, is adjacent to both v_0 and v_n. Now the edges are labeled accordingly.

3.2 Nearly Graceful Graph

For $x = 1, 2, \ldots, \left\lfloor \frac{3^n}{2} \right\rfloor$ such that neither x nor $3^n - x$ are any of the edge labels of G_1, we join a new vertex u_x, with label x, to vertices v_0 and v_n. We call all such vertices u_x, *2-vertices*; and such a graph as a *nearly graceful graph* G_2. The label of edges $v_0 u_x$ and $v_n u_x$ will be x and $3^n - x$, respectively, which are distinct. This is possible because of the odd labeling of vertices. Now since v_0 and v_n have even degree in G_1, and both their degrees increase by the same amount *they have the same parity.* It should be noted that there are a few labels missing in the nearly graceful graph. We include these labelings by the following two constructions: *square construction*, for introducing missing even labels; and *pentagon construction*, for introducing missing odd labels.

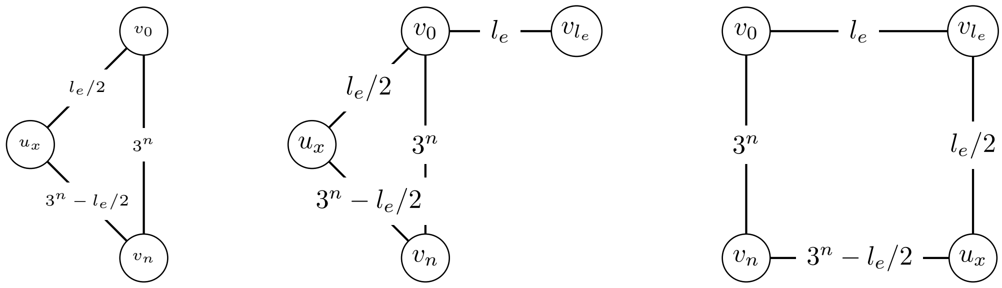

Figure 1: The stages in a Square construction

3.3 Square Construction

This is used to add a missing even edge label l_e in the graph. We join a new vertex v_{l_e} with label l_e to v_0. Hence edge $v_0 v_{l_e}$ has label l_e. We have labeled our vertices such that there already exists a *2-vertex* u_x in the graph with label $\frac{l_e}{2}$. Delete edge $u_x v_0$ and add edge $u_x v_{l_e}$, regaining the label $\frac{l_e}{2}$. Then the vertices v_0, v_{l_e}, u_x and v_n form a 4-cycle containing the missing label l_e, and no other edge label has been lost in the process. This is illustrated in Fig 1. We now prove the existence of the *2-vertex* u_x with label $\frac{l_e}{2}$.

Lemma 3.5. *Given a missing even label l_e in the nearly graceful graph G_2, there always exists a 2-vertex in this graph with label $\frac{l_e}{2}$.*

Proof. The missing even label l_e has form $3^n - 3^a$, corresponding to the edge label 3^a in G_1, and this happens only for edges $v_0 v_a$ where v_a is not adjacent to v_n. So u_x should have the label $\frac{3^n - 3^a}{2}$. Suppose such a vertex does not exist. The only labels missing in u_x have form $3^p - 3^q$ or 3^r. So either $\frac{3^n-3^a}{2} = 3^p - 3^q$ with $n \neq a$ and $p \neq q$, or $\frac{3^n-3^a}{2} = 3^r$ with $n \neq a$.

In the former case, let (n, a, p, q) be a solution; then $(n-1, a-1, p-1, q-1)$ is also a solution, and so on till we get $(x, y, z, 0)$ or $(x, 0, y, z)$ or $(x, 0, y, 0)$ as one of the solutions depending on whether $q < a$, $q > a$ or $q = a$ is smallest respectively. For the first two cases, one side of the equation is a multiple of 3 and the other is not, which is not possible. For the last case a simple rearrangement of powers of 3 in one side gives us the same result.

Now for the case when $\frac{3^n-3^a}{2} = 3^r$ with $n \neq a$, a similar argument proves that for $a \neq r$, this equation is not satisfied. For $a = r$, the only solution exists when $n = a + 1$. This is not possible, since in our construction we made sure v_{n-1} (with label 3^{n-1}) is adjacent to both v_0 and v_n. This ensures that $3^n - 3^{n-1}$ is not a missing even label.

Hence we have a contradiction in every case, and the lemma follows. □

We also observe the following result regarding parity of the degree of vertices v_0 and v_n. (By *parity* of a vertex, we mean *parity of the degree* of that vertex.)

Lemma 3.6. *Square construction does not change the parity of vertices v_0 and v_n.*

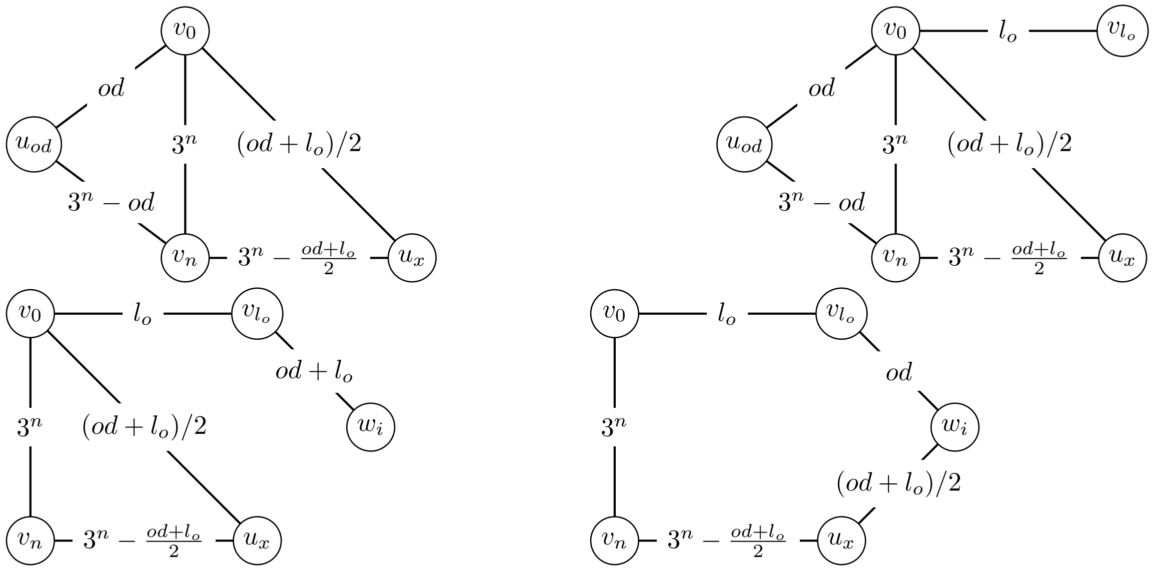

Figure 2: The stages in a Pentagon construction

Proof. For vertex v_0, the edge $v_0v_{l_e}$ is added and the edge u_xv_0 is deleted. For vertex v_n there is no change in the edges incident on it. Hence their parity remains intact. Refer Fig 1. □

3.4 Pentagon construction

This is used to add a missing odd edge label l_o to the graph. We join a new vertex v_{l_o} with label l_o to v_0. Hence edge $v_0v_{l_o}$ has label l_o. Now we intend to reuse the previous techniques described in the square construction, so we need an even label. Hence we find a *2-vertex* u_{od}, labeled $od \neq 1$ which is a small odd number, making $l_0 + od$ an even number. We have labeled our vertices such that there already exist *2-vertices* u_{od} and u_x in the graph with labels od and $\frac{l_o+od}{2}$. While choosing od, care must be taken to ensure there is a *2-vertex* with label $\frac{3^n-od}{2}$. Here one might wonder what if $l_0 + od > 3^n$. In that case we can do a subtraction to get the label $l_0 - od$. The idea still is to bring in a vertex with an even label. However with appropriate labeling of G_1 this can be avoided. We join a new vertex w_i, with label $l_o + od$, to v_{l_o}. Delete the vertex u_{od} and edges incident on it. Now the label $3^n - od$ is missing. Since it is an even label we can use the square construction. Now, we delete edge u_xv_0 and add u_xw_i regaining the lost label $\frac{l_o+od}{2}$. Then the vertices v_0, v_{l_o}, w_i, u_x and v_n form a 5-cycle containing the missing label l_o. No other edge label except $3^n - od$ has been lost in the process, which can be regained by a square construction. This is illustrated in Fig 2. We now prove the existence of a *2-vertices* u_{od} and u_x in the graph with labels od and $\frac{l_o+od}{2}$.

Lemma 3.7. *Given a missing odd label l_o in the nearly graceful graph G_2, there already exist* 2*-vertices u_{od} and u_x in this graph with labels od and $\frac{l_o+od}{2}$, where $od \neq 1$*

is a small odd number.

Proof. The missing label l_o has form $3^n - (3^a - 3^b)$ corresponding to the edge label $3^a - 3^b$ in G_1, where neither a nor b is 0 or n. Each of these missing odd labels are contributed by the even edge labels in original graph G or due to introduction of vertex v_{new} in Lemma 3.1 to make Eulerian graph G_1. So number of such missing labels is small ($< |E(G)| + |V(G)|$). Also due to the exponential nature of labeling, we have a lot of 2-vertices. Since the choice of u_{od} is up to us, we choose such an u_{od} such that u_x is also a 2-vertex where $x = \frac{l_o+od}{2}$. This completes the proof. □

We also observe the following result regarding parity of vertices v_0 and v_n.

Lemma 3.8. *Pentagon construction changes the parity of v_0 and v_n.*

Proof. For vertex v_0, edge $v_o v_{l_o}$ is added and edges $v_0 u_{od}$ and $v_0 u_x$ are deleted. For vertex v_n, edge $v_n u_{od}$ is deleted. Furthermore the square construction done to regain the label $3^n - od$ does not change the parity of v_0 and v_n. Therefore pentagon construction changes parity of v_0 and v_n. Refer Fig 2. □

3.5 Final Modifications

Now our graph is graceful and all vertices except possibly v_0 and v_n have even degree. So if only $d(v_0)$ and $d(v_n)$ are odd, we introduce a vertex p with label $3^n + 1$ and join it to v_0. Since $3^n + 1$ is even, we proceed with the square construction. But $\frac{3^n+1}{2}$ would not be available as a label of a 2-vertex, so we delete edge $v_0 u_x$ where u_x has the complement label i.e. $3^n - \frac{3^n+1}{2} = \frac{3^n-1}{2}$. Change the label of u_x to $\frac{3^n+1}{2}$. So edges pu_x and $u_x v_n$ have labels $\frac{3^n+1}{2}$ and $\frac{3^n-1}{2}$ respectively. Hence all missing labels are regained. But even after that $d(v_0)$ and $d(v_n)$ are odd, since square construction does not change the parity of v_0 and v_n (ref. Lemma 3.6). The graph is now gracefully labeled with labels $1, 2, \ldots, 3^n + 1$. So we add another vertex q with label $3^n + 2$ and join it to v_0. Since this label is odd we proceed with pentagon construction, but instead of adding od we subtract it, so that the maximum label is restricted to $3^n + 2$. The following lemma proves the availability of such 2-*vertices.*

Lemma 3.9. *There exist 2-vertices with labels $\frac{3^n-1}{2}$ and $\frac{3^n+2-od}{2}$, for a small odd number $od \neq 1$.*

Proof. Clearly $\frac{3^n-1}{2}$ and almost all the labels within $[3^{n-1}+1, \frac{3^n-1}{2}]$ are available. So for a small od we can find the 2-vertex with label $\frac{3^n+2-od}{2}$ within this range. Hence the given labels are available. □

Now finally the parity reverses due to the pentagon construction (cf. Lemma 3.8). Now all vertices have even degree and we have a graceful graph with labels $1, 2, \ldots, 3^n + 2$. Hence we obtained our Eulerian graceful graph H that embeds the given graph G. We now explicitly present the algorithm; illustrations are given in the Appendix.

Algorithm 1:
Outline:

$$G \xrightarrow{Eulerian} G_1 \xrightarrow{Nearly\ Graceful} G_2 \xrightarrow{Eulerian\ Graceful} H$$

Step 1. *Embedding G in an Eulerian graph G_1 using Lemma 3.1.*

Step 2. *Give 3^i Labeling to the intermediate Eulerian graph G_1.*

Step 3. *Adding vertices and making a nearly Graceful graph G_2.*

Step 4. *Adding missing labels and making the graph graceful.*

(a) Let $\{l_e\}$ and $\{l_o\}$ be the sets of missing even and odd labels respectively.

(b) We use the square construction for each l_e and pentagon construction for each l_o till all missing labels are added to the graph.

(c) Now we have a gracefully labeled graph. If degrees, $d(v_0)$ and $d(v_n)$ are even then it is Eulerian and we are done. Else proceed.

Step 5. *Making the graceful graph Eulerian as given in Subsection 3.5.*

Hence Algorithm 1 proves that every connected graph can be embedded as an induced subgraph in an Eulerian graceful graph, proving Theorem 1.1.

3.6 A remark on designing a polynomial algorithm

The backbone of Algorithm 1 lies in the proofs of Lemmas 3.5, 3.7 and 3.9. Previously in Remark 3.4, we saw why a 2^i labeling would fail. Now we shall develop a polynomial labeling which would work for graphs with a small clique number. We repeat the previous algorithm using a $(2n+1)^2$ labeling. Everything works out except the existence of odd numbers (a, b, c, d) satisfying $a^2 - b^2 = 2(c^2 - d^2)$ [ref. Lemma 3.5]. In fact, there are families of such 4-tuples. So we need to avoid having such cases in our labeling either by making sure that one of the odd numbers in each of the 4-tuple is missing, or that edge ab does not exist. However this turns out be impossible for a larger clique number. A counterexample where such a labeling would fail is when $G = K_n$, for a larger n. In fact, in this case, the 2^i labeling also fails. So in general, but not for all graphs, we can have a $(2n + 1)^r$ labeling for some integer $r > 1$. It is obvious to see that as r increases, the number of solutions to $a^r - b^r = 2(c^r - d^r)$, for odd tuples (a, b, c, d) decreases. Hence, the family of graphs G satisfying this polynomial labeling increases with r. The obvious question that would assure a $(2n + 1)^r$ labeling is whether $a^r - b^r \neq 2(c^r - d^r)$ for every 4-tuples (a, b, c, d) of odd numbers.

Now let us look at some of the graph characteristics of the Eulerian graceful graphs obtained from Algorithm 1.

Corollary 3.10. *The problem of deciding the following parameters is NP-complete for Eulerian graceful graphs:*

1. *The chromatic number* $\chi(H)$ *is less than or equal to* k*, for* $k \geq 3$.
2. *The clique number* $\omega(H)$ *is greater than or equal to* k*, for* $k \geq 3$.

Proof. Let $\mathcal{G}$ denote the class of Eulerian graphs and $\mathcal{G}'$ contain rest of the graphs. Let G be a graph with chromatic number $\chi(G) \geq 3$, and clique number $\omega(G)$. Let G_1 be the Eulerian graph, constructed in Lemma 3.1, that induces G. Let H be the Eulerian graceful graph that induces G. Clearly H also induces G_1.

Since in G_1, the vertex v_0 is joined to every vertex of G, it must be assigned a different color. If $G \in \mathcal{G}$, then the vertex v_{new} (in Lemma 3.1) also has to be assigned a different color, since it is also joined to every vertex of G and v_0. If $G \in \mathcal{G}'$, then the vertex v_{new} can be assigned one of the previous colors. So if $G \in \mathcal{G}$, $\chi(G_1) = \chi(G) + 2$; and if $G \in \mathcal{G}'$, $\chi(G_1) = \chi(G) + 1$. Let $\{c_1, c_2, c_3, ..\}$ be the colors of G_1. Without loss of generality let us assume that vertex v_0 and v_n have colors c_1 and c_2. Every vertex of $H - G_1$ either lie on a 3-cycle, 4-cycle or 5-cycle with v_0v_n as the common edge. So these vertices can be colored by c_1, c_2 and a third color c_3. So if $G \in \mathcal{G}$, $\chi(H) = \chi(G_1) = \chi(G) + 2$; and if $G \in \mathcal{G}'$, $\chi(H) = \chi(G_1) = \chi(G) + 1$. Since the problem of deciding whether the chromatic number $\chi(G) \leq k$ for $k \geq 3$ is NP-complete ([8], page 190), it is still NP-complete for graphs in $\mathcal{G}$ or/and for graphs in $\mathcal{G}'$. Hence it follows by restriction that deciding whether the chromatic number of an Eulerian graceful graph $\chi(H) \leq k$ for $k \geq 3$ is NP-complete.

For clique number, similarly if $G \in \mathcal{G}$, $\omega(G_1) = \omega(G) + 2$, since both v_0 and v_{new} are adjacent to every vertex of G; and if $G \in \mathcal{G}'$, $\omega(G_1) = \omega(G) + 1$. Also $\omega(H) = \omega(G_1)$, since the only vertices of G_1 adjacent to $H - G_1$ are v_0 and v_n. So if $G \in \mathcal{G}$, $\omega(H) = \omega(G_1) = \omega(G) + 2$; and if $G \in \mathcal{G}'$, $\omega(H) = \omega(G_1) = \omega(G) + 1$. Since the problem of deciding whether the clique number $\omega(G) \geq k$ is NP-complete ([8], page 194), using the previous argument, it follows that deciding whether the clique number of a Eulerian graceful graph $\omega(H) \geq k$ is NP-complete. □

Applying the construction described in Algorithm 1 to triangle free graphs, we have the following result.

Corollary 3.11. *Any triangle free non-Eulerian graph* G *can be embedded as an induced subgraph of a* K_4 *free Eulerian graceful graph.*

Due to the famous construction by Mycielski [11], triangle free graphs can have arbitrarily large chromatic number. Also the Mycielskian of any graph contains vertices with even degree as well as vertices with odd degree. Hence we have the following corollary.

Corollary 3.12. *There exists* K_4 *free Eulerian graceful graphs with arbitrarily large chromatic number.*

Also taking G as a clique of arbitrarily large size, and constructing H as given in Algorithm 1, we can have arbitrarily large chromatic number and given clique number for H. Hence we have the following stronger result than point 2 [in Sec. 2] for chromatic number, addressing a problem by Chartrand and Lesniak in [6] (Page 266, also see the 5^{th} edition), which was already settled in [5].

Corollary 3.13. *Eulerian graceful graphs can have arbitrarily large chromatic number and given clique number $\omega(H) \geq 3$.*

4 Conclusion

Although we believe that the size of H can be further reduced, the proofs of Lemma 3.5, 3.7 and 3.9 depend on the 3^i labeling of the vertices. We have also looked at cases where we can have a better algorithm. However, a simpler labeling does not work for all graphs, for example K_n. So at present this seems to be the only way of embedding a graph G, as induced subgraph, in an Eulerian graceful graph H. Although we have a polynomial algorithm for graphs with a smaller clique number, efforts can be made to reduce the number of vertices in H to some order of the *index of gracefulness*, so as to induce a polynomial labeling for all graphs.

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Appendix

Examples

Fig. 3 shows an embedding of P_3 in an Eulerian graceful graph. It uses the 3^i labeling. Fig. 4 shows the corresponding embedding using the $(2n+1)^2$ labeling. In both figures, vertices are labeled in a larger font, whereas edges are labeled in a smaller font. For the sake of clarity, edges are not labeled in the final Eulerian graceful graph.

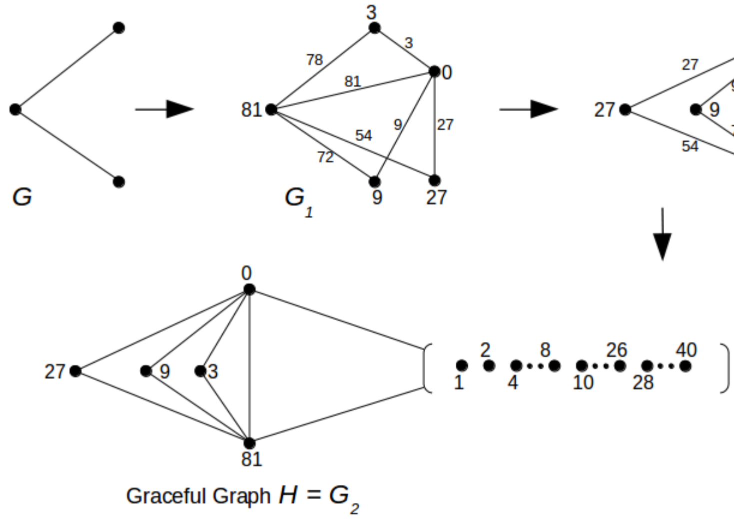

Figure 3: Embedding P_3 in an Eulerian graceful graph using 3^i labeling.

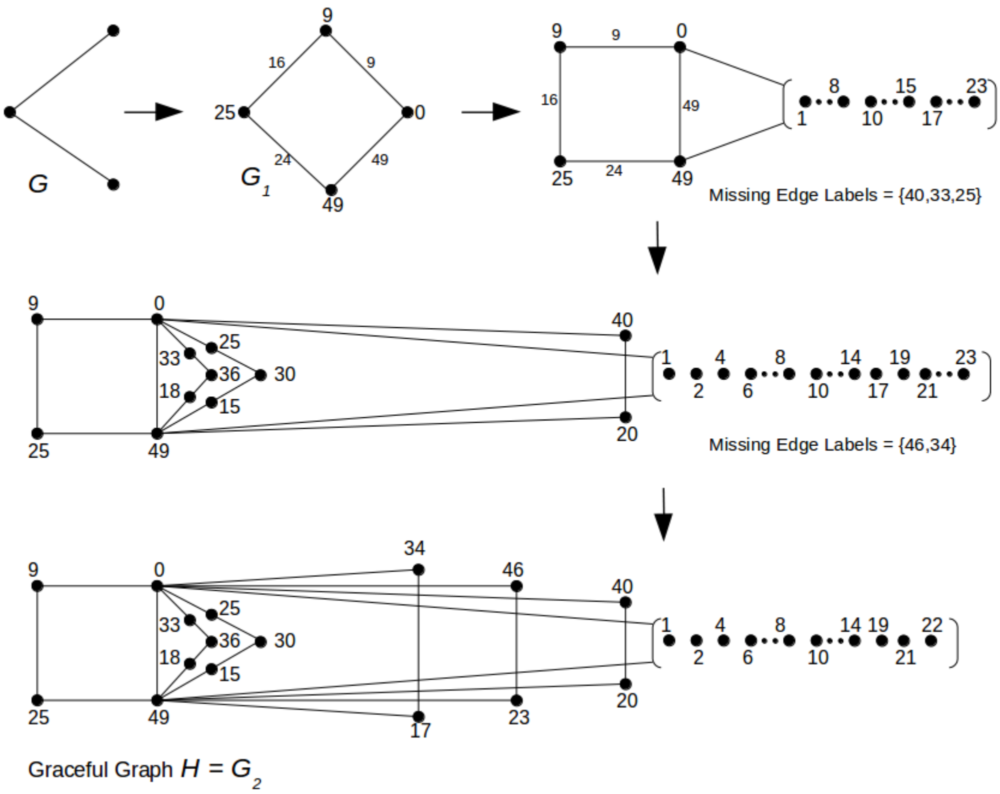

Figure 4: Embedding P_3 in an Eulerian graceful graph using $(2n + 1)^2$ labeling.

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