AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **62(1)** (2015), Pages 18–34

The metric dimension of small distance-regular and strongly regular graphs

Robert F. Bailey

Division of Science (Mathematics), Grenfell Campus
Memorial University of Newfoundland
University Drive, Corner Brook, NL A2H 6P9
Canada
rbailey@grenfell.mun.ca

Abstract

A *resolving set* for a graph Γ is a collection of vertices S, chosen so that for each vertex v, the list of distances from v to the members of S uniquely specifies v. The *metric dimension* of Γ is the smallest size of a resolving set for Γ.

A graph is *distance-regular* if, for any two vertices u, v at each distance i, the number of neighbours of v at each possible distance from u (i.e. $i-1$, i or $i+1$) depends only on the distance i, and not on the choice of vertices u, v. The class of distance-regular graphs includes all distance-transitive graphs and all strongly regular graphs.

In this paper, we present the results of computer calculations which have found the metric dimension of all distance-regular graphs on up to 34 vertices, low-valency distance transitive graphs on up to 100 vertices, strongly regular graphs on up to 45 vertices, rank-3 strongly regular graphs on under 100 vertices, as well as certain other distance-regular graphs.

1 Introduction

A *resolving set* for a graph $\Gamma = (V, E)$ is a set of vertices $S = \{v_1, \ldots, v_k\}$ such that for each vertex $w \in V$, the list of distances $(\mathrm{d}(w, v_1), \ldots, \mathrm{d}(w, v_k))$ uniquely determines w. Equivalently, S is a resolving set for Γ if, for any pair of vertices $u, w \in V$, there exists $v_i \in S$ such that $\mathrm{d}(u, v_i) \neq \mathrm{d}(w, v_i)$; we say that v_i *resolves* u and w. The *metric dimension* of Γ is the smallest size of a resolving set for Γ. This concept was introduced to the graph theory literature in the 1970s by Harary and Melter [30] and, independently, Slater [39]; however, in the context of arbitrary metric spaces, the concept dates back at least as far as the 1950s (see Blumenthal [9], for instance). In more recent years, there has been a considerable number of papers written about the metric dimension of graphs. For further details, the reader is referred to the survey paper [3].

When studying metric dimension, distance-regular graphs are a natural class of graphs to consider. A graph Γ is *distance-regular* if, for all i with $0 \leq i \leq \mathrm{diam}(\Gamma)$ and any vertices u, v with $\mathrm{d}(u, v) = i$, the number of neighbours of v at distances $i-1$, i and $i+1$ from u depend only on the distance i, and not on the choices of u and v. For more information about distance-regular graphs, see the book of Brouwer, Cohen and Neumaier [10] and the forthcoming survey paper by van Dam, Koolen and Tanaka [16]. Note that the class of distance-regular graphs contains the distance-transitive graphs (i.e. those graphs Γ with the property that for any vertices u, v, u', v' such that $\mathrm{d}(u, v) = \mathrm{d}(u', v')$, there exists an automorphism g such that $u^g = u'$ and $v^g = v'$) and the connected strongly regular graphs (which are the distance-regular graphs of diameter 2).

For any graph Γ with diameter d, consider the partition of $V \times V$ into $d+1$ parts, given by the pairs of vertices at each possible distance in Γ. If Γ is distance-regular, this partition is an example of an *association scheme*. These are much more general objects, and ones which are inconsistently named in the literature; see [3, Section 3.3] for more details. An association scheme is said to be *P-polynomial* if it arises from a distance-regular graph; however, more than one graph may give rise to the same P-polynomial association scheme (see [10, 16]). It is not difficult to see that two graphs arising in this way must have the same metric dimension (see [3, Section 3.5]).

Since the publication of the survey paper by Cameron and the present author [3], a number of papers have been written on determining, or bounding, the metric dimension of various families of distance-regular graphs: see [1, 2, 4, 6, 18, 24, 25, 26, 31], for instance. The purpose of this paper is to give the results of a number of computer calculations, using the GAP computer algebra system [19], which have obtained the metric dimension for all "small" distance-regular graphs (i.e. on up to 34 vertices), for distance-regular graphs of valency 3 and 4 (on up to 189 vertices), low-valency distance-transitive graphs (up to valency 13, and up to 100 vertices), and certain other distance-regular graphs. For strongly regular graphs, this has included an independent verification of earlier computations by Kratica *et al.* [33] (which used an entirely different approach via linear programming).

2 Known results

In this section, we summarize the relevant known values of the metric dimension of various families of distance-regular graphs. It is a straightforward exercise to verify that the complete graph K_n has metric dimension $n-1$, that the complete bipartite graph $K_{m,n}$ has metric dimension $m+n-2$, and that a cycle C_n with $n \geq 3$ vertices has metric dimension 2. The following result is also straightforward, yet the author is not aware of it appearing anywhere in the literature.

Proposition 1. *Consider a complete multipartite graph $\Gamma = K_{m_1,\ldots,m_r}$ with r parts of sizes $m_1, \ldots, m_r$, for $r > 1$. Then the metric dimension of Γ is $\sum_{i=1}^{r}(m_i - 1)$.*

In particular, in the special case where a complete multipartite graph has r parts of size m (and thus is strongly regular), this shows that the metric dimension is $r(m-1)$.

Proof. Suppose the vertex set of Γ is $V = V_1 \cup \cdots \cup V_r$, where the V_i are disjoint sets of sizes $m_1, \ldots, m_r$; every possible edge exists from V_i to V_j (for $i \neq j$), and no edges exist inside any V_i. Let T be a transversal of $V_1, \ldots, V_r$, and let $S = V \setminus T$. It is straightforward to verify that S is a resolving set for Γ of size $\sum_{i=1}^{r}(m_i - 1)$. Furthermore, no smaller resolving set may exist: suppose for a contradiction that R is a subset of V with size smaller than the above. By the pigeonhole principle there exists an index i for which V_i contains two vertices u, v not in R, and no vertex in R will resolve this pair of vertices. $\square$

The following results are somewhat less trivial. Recall that the *Johnson graph* $J(n,k)$ is the graph whose vertex set consists of all k-subsets of an n-set, and two k-subsets are adjacent if and only if they intersect in a $(k-1)$-subset. The *Kneser graph* $K(n,k)$ has the same vertex set as $J(n,k)$, but adjacency is defined by two k-sets being disjoint. The Johnson graph is always distance-regular, whereas the Kneser graph only is in two special cases, namely $K(n,2)$ (which is the complement of $J(n,2)$) and $K(2k+1,k)$ (known as the *Odd graph*, and usually denoted O_{k+1}). The following result was obtained by Cameron and the present author in 2011.

Theorem 2 (Bailey and Cameron [3, Corollary 3.33])**.** *For $n \geq 6$, metric dimension of the Johnson graph $J(n,2)$ and Kneser graph $K(n,2)$ is $\frac{2}{3}(n-i)+i$, where $n \equiv i \pmod 3$, $i \in \{0,1,2\}$.*

It is easy to determine the metric dimension of $J(3,2)$, $J(4,2)$ and $J(5,2)$ by hand: these values are 2, 3 and 3, respectively. Further results about resolving sets for Johnson and Kneser graphs may be found in [2].

We also recall that the *Hamming graph* $H(d,q)$ has as its vertex set the collection of all d-tuples over an alphabet of size q, and two d-tuples are adjacent if and only if they differ in exactly one position; these graphs are distance-transitive. Two important examples are the hypercube $H(d,2)$ and the square lattice graph $H(2,q)$. The following result was obtained by Cáceres *et al.* in 2007.

Theorem 3 (Cáceres *et al.* [13, Theorem 6.1])**.** *For all $q \geq 1$, the metric dimension of the square lattice graph $H(2,q)$ is $\lfloor \frac{2}{3}(2q-1) \rfloor$.*

This is the only infinite family of Hamming graphs for which the metric dimension is known precisely. Further details about the metric dimension of Hamming graphs can be found in [3, Section 3.6]; for the hypercubes, see also Beardon [6]. Some precise values were computed by Kratica *et al.* [34], using genetic algorithms.

Finally, we mention a recent result of the present author, which helps to eliminate the need for some additional computations. The *bipartite double* of a graph $\Gamma = (V,E)$ is a bipartite graph $D(\Gamma)$, whose vertex set consists of two disjoint copies of V, labelled V^+ and V^-, with v^+ adjacent to w^- in $D(\Gamma)$ if v and w are adjacent in Γ.

Theorem 4 (Bailey [1]). *Suppose Γ is a distance-regular graph of diameter d, and whose shortest odd cycle has length $2d+1$. Then Γ and its bipartite double $D(\Gamma)$ have the same metric dimension.*

In particular, the graph $K_{n,n} - I$ (obtained by deleting a perfect matching from $K_{n,n}$) is the bipartite double of the complete graph K_n, which satisfies the conditions of Theorem 4, and thus $K_{n,n} - I$ has metric dimension $n-1$.

3 The method

Suppose $\Gamma = (V, E)$ is a graph with n vertices, labelled $v_1, \ldots, v_n$. The *distance matrix* of Γ is the $n \times n$ matrix A, whose (i, j) entry is the distance in Γ from v_i to v_j. Note that if there are two distance-regular graphs arising from the same P-polynomial association scheme, their distance matrices are equivalent, up to a relabelling of the distance classes. For instance, the metric dimension of a primitive strongly regular graph and that of its complement will be equal.

Suppose A is the distance matrix of Γ, and let S be a subset of V. Denote by $[A]_S$ the submatrix formed by taking the columns of A indexed by elements of S. It is clear that S is a resolving set for Γ if and only if the rows of $[A]_S$ are distinct. Furthermore, if one has obtained the distance matrix of a relatively small graph, it is near-instantaneous for a computer to verify if a given submatrix has this property.

The results in this paper were obtained using the computer algebra system GAP [19] and various packages developed for it, in particular the GRAPE package of Soicher [44], the functions for association schemes of Hanaki [27], and also some functions of Cameron [15]. However, the most useful tool for these computations is the SetOrbit package of Pech and Reichard [36]. Given a set V and a group G acting on it, this provides an efficient method for enumerating a canonical representative of each orbit of G acting on the subsets of V of a given size. It is clear that S is a resolving set for a graph Γ if and only if its image $S^g = \{x^g : x \in S\}$ is a resolving set, for any $g \in \mathrm{Aut}(\Gamma)$. Consequently, when searching for a resolving set of a particular size, it suffices to test just one representative of each orbit on subsets of that size. Therefore, the methods of Pech and Reichard (which are explained in detail in [35]) are precisely what is needed to dramatically reduce the search space.

3.1 Data sources

Hanaki and Miyamoto's library of small association schemes [28] contains all distance-regular graphs on up to 34 vertices: one merely has to filter out the P-polynomial examples from their lists. Association schemes with primitive automorphism groups may be constructed by using GAP's internal libraries of primitive groups. Graphs (such as incidence graphs and point graphs) obtained from generalized polygons may be constructed using the FinInG package of Bamberg *et al.* [5], while for graphs associated with block designs (including projective and affine geometries) the DESIGN package of Soicher [45] is used. Other useful data sources include Spence's catalogue of strongly regular graphs [46], Royle's catalogue of symmetric (i.e. arc-transitive) cubic graphs [38], the online Atlas of Finite Group Representations [47], and Sloane's libraries of Hadamard matrices [40].

4 Results

4.1 Distance-regular graphs on up to 34 vertices

As mentioned above, all distance-regular graphs on up to 34 may all be obtained from the catalogue of association schemes of Hanaki and Miyamoto [28]. For 31 and 32 vertices, a complete classification of association schemes is not available; however, as there are no feasible parameter sets for strongly regular graphs with these numbers of vertices, and (other than K_{31} and C_{31}) no distance-transitive graphs with 31 vertices, one can see that the data available contains all distance-regular graphs with 31 and 32 vertices. These graphs are all small enough that no sophisticated computations were required to determine the metric dimension; the results are given in Tables 1 and 2. Cycles, complete graphs, complete bipartite graphs and complete multipartite graphs are omitted from these tables. Those graphs which are not distance-transitive are indicated †; cases where there is a unique distance-transitive example are indicated ‡. The abbreviation IG is used to denote an incidence graph. In cases where there are multiple examples, N denotes the number of such graphs.

4.2 Distance-regular graphs of valency 3 and 4

The distance-transitive graphs of valency 3 were determined by Biggs and Smith in 1971 [8] (see also Gardiner [20]); this classification was extended to all distance-regular graphs of valency 3 by Biggs, Boshier and Shawe-Taylor in 1986 [7]. In addition to K_4 and $K_{3,3}$, there are eleven such distance-regular graphs, of which all but one are distance-transitive, with the exception being Tutte's 12-cage. The metric dimension of each of these graphs is given in Table 3.

The distance-transitive graphs of valency 4 were determined in 1974 by Smith [41, 42, 43] (see also Gardiner [21]); this classification was extended to distance-regular graphs by Brouwer and Koolen in 1999 [12]. The classification is complete, except for relying on a classification of generalized hexagons $GH(3,3)$, which give rise to 4-regular distance-regular graphs on 728 vertices. (In any case, these would be beyond the scope of the computations in this paper.) The metric dimension of each of these graphs is given in Table 4.

Of the graphs with more than 34 vertices, the Foster graph was obtained from Royle's catalogue [38], the Biggs–Smith graph from the group $\mathrm{PSL}(2,17)$, and Tutte's 12-cage as the incidence graph of the generalized hexagon $GH(2,2)$ (constructed using FinInG). The Odd graph O_4 lies inside the Johnson scheme $J(7,3)$, which may be constructed using GRAPE, as can line graphs and bipartite doubles. The incidence graph of the $GQ(3,3)$ may be constructed using FinInG.

Vertices	Graph	Valency	Diameter	Met. dim.
6	Octahedron $J(4,2)$	4	2	3
8	Cube $Q_3 \cong K_{4,4} - I$	3	3	3
9	Paley graph $P_9 \cong H(2,3)$	4	2	3
10	Petersen graph $O_3 = K(5,2)$	3	2	3
	$J(5,2)$	6	2	3
	$K_{5,5} - I$	4	3	4
12	Icosahedron	5	3	3
	$K_{6,6} - I$	5	3	5
13	Paley graph P_{13}	6	2	4
14	Heawood (IG of PG$(2,2)$)	3	3	5
	Non-IG of PG$(2,2)$	4	3	5
	$K_{7,7} - I$	6	3	6
15	Line graph of Petersen graph	4	3	4
	$K(6,2)$	6	2	4
	$J(6,2)$	8	2	4
16	4-cube Q_4	4	4	4
	$H(2,4)$	6	2	4
	Complement of $H(2,4)$	9	2	4
	Shrikhande graph†	6	2	4
	Complement of Shrikhande graph†	9	2	4
	Clebsch graph	5	2	4
	(Complement of) Clebsch graph	10	2	4
	$K_{8,8} - I$	7	3	7
17	Paley graph P_{17}	8	2	4
18	Pappus graph	3	4	4
	$K_{9,9} - I$	8	3	8
20	Dodecahedron	3	5	3
	Desargues graph $D(O_3)$	3	5	3
	$J(6,3)$	9	3	4
	$K_{10,10} - I$	9	3	9
21	Line graph of Heawood graph	4	3	4
	$J(7,2)$	10	2	4
	$K(7,2)$	10	2	4
22	IG of biplane	5	3	6
	Non-IG of biplane	6	3	6
	$K_{11,11} - I$	10	3	10
24	Symplectic cover [10, p. 386]†	7	3	5
	$K_{12,12} - I$	11	3	11

Table 1: Metric dimension of distance-regular graphs on up to 24 vertices

Vertices	Graph	Valency	Diameter	Met. dim.
25	$H(2,5)$	8	2	6
	Paley graph P_{25}	12	2	5
	Other srg$(25,12,5,6)$ *($N=14$, 7 pairs)*†	12	2	5
	Complement of $H(2,5)$	16	2	6
26	srg$(26,10,3,4)$ *($N=10$)*†	10	2	5
	IG of PG$(2,3)$	4	3	8
	Non-IG of PG$(2,3)$	9	3	8
	$K_{13,13}-I$	12	3	12
	Complements of srg$(26,10,3,4)$ *($N=10$)*†	15	2	5
27	$H(3,3)$	6	3	4
	$GQ(2,4)$ minus spread *($N=2$)*	8	3	5
	Complement of Schläfi graph	10	2	5
	Schläfi graph	10	2	5
28	Coxeter graph	3	4	4
	$J(8,2)$	12	2	6
	Chang graphs *($N=3$)*†	12	2	6
	Taylor graph from P_{13}	13	3	5
	$K_{14,14}-I$	13	3	13
	$K(8,2)$	15	2	6
	Complements of Chang graphs *($N=3$)*†	15	2	6
29	Paley graph P_{29}	14	2	6
	Other srg$(29,14,6,7)$ *($N=40$, 20 pairs)*†	14	2	5
30	Tutte's 8-cage	3	4	6
	IG of PG$(3,2)$	7	3	8
	Non-IG of PG$(3,2)$	8	3	8
	IGs of Hadamard designs *($N=3$)*‡	7	3	8
	Non-IGs of Hadamard designs *($N=3$)*‡	8	3	8
	$K_{15,15}-I$	14	3	14
32	IG of truncated AG$(2,4)$	4	4	6
	5-cube Q_5	5	5	4
	Armanios–Wells graph	5	4	5
	IGs of biplanes *($N=3$)*†	6	3	8
	Hadamard graph	8	4	7
	Non-IGs of biplanes *($N=3$)*†	10	3	8
	Taylor graph from $J(6,2)$	15	3	5
	Taylor graph from $K(6,2)$	15	3	5
	$K_{16,16}-I$	15	3	15
34	$K_{17,17}-I$	16	3	16

Table 2: Metric dimension of distance-regular graphs on 25 to 34 vertices

Graph	Vertices	Diameter	Metric dimension
Cube $Q_3 \cong K_{4,4} - I$	8	3	3
Petersen graph O_3	10	2	3
Heawood graph	14	3	5
Pappus graph	18	4	4
Dodecahedron	20	5	3
Desargues graph $D(O_3)$	20	5	3
Coxeter graph	28	4	4
Tutte's 8-cage	30	4	6
Foster graph	90	8	5
Biggs–Smith graph	102	7	4
Tutte's 12-cage†	126	6	8

Table 3: Metric dimension of distance-regular graphs of valency 3

Graph	Vertices	Diameter	Metric dimension
Octahedron $J(4,2)$	6	2	3
Paley graph $P_9 \cong H(2,3)$	9	2	3
$K_{5,5} - I$	10	3	5
Distance-3 graph of Heawood graph	14	3	5
Line graph of Petersen graph	15	3	4
4-cube Q_4	16	4	4
Line graph of Heawood graph	21	3	4
Incidence graph of PG$(2,3)$	26	3	8
Incidence graph of truncated AG$(2,4)$	32	4	6
Odd graph O_4	35	3	5
Line graph of Tutte's 8-cage	45	4	4
Doubled Odd graph $D(O_4)$	70	7	5
Incidence graph of $GQ(3,3)$†	80	4	10
Line graph of Tutte's 12-cage†	189	6	6
Incidence graph of $GH(3,3)$	728	6	unknown

Table 4: Metric dimension of distance-regular graphs of valency 4

4.3 Distance-transitive graphs of valencies 5 to 13

The distance-transitive graphs of valencies 5, 6 and 7 were determined in 1986 by Faradjev, Ivanov and Ivanov [17], and independently (for valencies 5 and 6) by Gardiner and Praeger [22, 23]. For valencies 8 to 13, the classification was obtained in 1988 by Ivanov and Ivanov [32].

Tables 5 to 12 contain the metric dimension of all such graphs on up to 100 vertices (apart one exception on 98 vertices), as well as larger graphs when the computations succeeded. No table is provided for valency 11, as the only example under 100 vertices (other than K_{12} or $K_{11,11}$) is $K_{12,12} - I$, which has metric dimension 11. An asterisk indicates that the metric dimension was not computed directly, but rather that Theorem 4 was applied to the result of an earlier computation.

Of the examples with more than 34 vertices, many of the graphs are incidence graphs of designs or geometries, so can be constructed using the DESIGN or FinInG packages; the resolvable transversal designs RT$[8, 2; 4]$ and RT$[9, 3; 3]$ are given in the paper of Hanani [29]. Otherwise, graphs were constructed in GAP from their automorphism groups, either from the internal libraries of primitive groups, or using permutation representations in the ATLAS.

Graph	Vertices	Diameter	Metric dimension
$K_{6,6} - I$	12	3	5
Icosahedron	12	3	3
Clebsch graph	16	2	4
Incidence graph of biplane	22	3	6
5-cube Q_5	32	5	4
Armanios–Wells graph	32	4	5
Sylvester graph from Aut(S_6)	36	3	5
Incidence graph of PG$(2, 4)$	42	3	10
Incidence graph of truncated AG$(2, 5)$	50	4	9
Odd graph O_5	126	4	6
Incidence graph of $GQ(4, 4)$	170	4	unknown
Doubled Odd graph $D(O_5)$	252	9	6*

Table 5: Metric dimension of distance-transitive graphs of valency 5

4.4 Strongly regular graphs on up to 45 vertices

Recall that a strongly regular graph Γ has *parameters* (n, k, a, c), where n is the number of vertices, k is the valency, a is the number of common neighbours of a pair of adjacent vertices, and c is the number of common neighbours of a pair of non-adjacent vertices. In this subsection, we consider the strongly regular graphs on between 35 and 45 vertices, for parameters where a complete classification of graphs are known, and obtain their metric dimension; the results are given in Table 13. (Smaller strongly

Graph	Vertices	Diameter	Metric dimension
$K_{7,7} - I$	14	3	6
$J(5, 2)$	10	2	3
Paley graph P_{13}	13	2	4
$K(6, 2)$	15	2	4
$H(2, 4)$	16	2	4
Non-incidence graph of biplane	22	3	6
$H(3, 3)$	27	3	4
Incidence graph of biplane	32	3	8
Hexacode graph	36	4	7
2^{nd} subconstituent of Hoffman–Singleton	42	3	7
Halved Foster graph	45	4	6
Flag graph of $\mathrm{PG}(2, 3)$	52	3	6
Perkel graph	57	3	6
Incidence graph of $\mathrm{PG}(2, 5)$	62	3	15
Point graph of $GH(2, 2)$	63	3	6
Point graph of dual $GH(2, 2)$	63	3	6
6-cube Q_6	64	6	5

Table 6: Metric dimension of distance-transitive graphs of valency 6

Graph	Vertices	Diameter	Metric dimension
$K_{8,8} - I$	16	3	7
Incidence graph of $\mathrm{PG}(3, 2)$	30	3	8
Hoffman–Singleton graph	50	2	11
Folded 7-cube	64	3	6
Incidence graph of truncated $\mathrm{AG}(2, 7)$	98	4	unknown
Doubled Hoffman–Singleton graph	100	5	11^*
7-cube Q_7	128	7	6

Table 7: Metric dimension of distance-transitive graphs of valency 7

Graph	Vertices	Diameter	Metric dimension
$J(6,2)$	15	2	4
Paley graph P_{17}	17	2	4
$K_{9,9} - I$	18	3	8
$H(2,5)$	25	2	6
Point graph of $GQ(2,4)$ minus spread	27	3	5
Non-incidence graph of PG$(3,2)$	30	3	8
Hadamard graph	32	4	7
Incidence graph of RT$[8,2;4]$	64	4	10
$H(4,3)$	81	4	5
Flag graph of PG$(2,4)$	105	3	7
Folded 8-cube	128	4	11
8-cube Q_8	256	8	6

Table 8: Metric dimension of distance-transitive graphs of valency 8

Graph	Vertices	Diameter	Metric dimension
Complement of $H(2,4)$	16	2	4
$K_{10,10} - I$	20	3	9
$J(6,3)$	20	3	4
Non-incidence graph of PG$(2,3)$	26	3	8
Incidence graph of RT$[9,3;3]$	54	4	10
$H(3,4)$	64	3	6
Unitals in PG$(2,4)$ (from PSL$(3,4)$)	280	4	5
9-cube Q_9	512	9	7

Table 9: Metric dimension of distance-transitive graphs of valency 9

Graph	Vertices	Diameter	Metric dimension
Clebsch graph	16	2	4
$J(7,2)$, $K(7,2)$	21	2	4
$K_{11,11} - I$	22	3	10
Complement of Schläfi graph	27	2	5
Non-incidence graph of biplane	32	3	8
$H(2,6)$	36	2	7
Gewirtz graph	56	2	9
Conway-Smith graph from $3.S_7$	63	4	6
Hall graph from PΣL$(2,25)$	65	3	6
Doubled Gewirtz graph	112	5	9^*
$H(5,3)$	243	5	5
Hall–Janko near octagon from $\mathrm{J}_2.2$	315	4	8

Table 10: Metric dimension of distance-transitive graphs of valency 10

Graph	Vertices	Diameter	Metric dimension
Paley graph P_{25}	25	2	5
$K_{13,13} - I$	26	3	12
$J(8,2)$	28	2	6
$J(7,3)$	35	3	5
Point graph of $GQ(3,3)$	40	2	7
Point graph of dual $GQ(3,3)$	40	2	8
Point graph of $GQ(4,2)$	45	2	8
Hadamard graph	48	4	8
$H(2,7)$	49	2	8
Doro graph from $\mathrm{P\Sigma L}(2,16)$	68	3	6
$H(3,5)$	125	3	7
$H(4,4)$	256	4	7

Table 11: Metric dimension of distance-transitive graphs of valency 12

Graph	Vertices	Diameter	Metric dimension
$K_{14,14} - I$	28	3	13
Taylor graph from P_{13}	28	3	5
Incidence graph of $\mathrm{PG}(3,3)$	80	3	14

Table 12: Metric dimension of distance-transitive graphs of valency 13

regular graphs were considered in Tables 1 and 2 above.) These graphs were obtained from the online catalogue of Spence [46]. This forms an independent verification of the earlier calculations of Kratica *et al.* [33]; however, we do not consider strongly regular graphs with parameters $(37, 18, 8, 9)$, as it is unknown if the 6760 known graphs form the complete set. (The Paley graphs on 37 and 41 vertices are considered in the next subsection.)

4.5 Rank-3 strongly regular graphs with up to 100 vertices

A graph which is both strongly regular and distance-transitive is called a *rank*-3 *graph*, as its automorphism group has permutation rank 3 (see [14]). Other than a complete multipartite graph, such a graph necessarily has a primitive automorphism group, and as GAP contains a library of all primitive groups on up to 2499 points (as obtained by Roney-Dougal [37]), it is straightforward to construct rank-3 graphs with a (relatively) small number of vertices. For this paper, we considered the rank-3 graphs with up to 100 vertices; the metric dimension of each of these graphs is given in Table 14. (For the Higman–Sims graph on 100 vertices, the exact value was not determined, but an upper bound of 14 was obtained; the author suspects this is the exact value.)

Parameters	No. of graphs	Metric dimension	Notes
$(35, 18, 9, 9)$	3854	6	
$(36, 10, 4, 2)$	1	7	$H(2, 6)$
$(36, 14, 4, 6)$	180	6	
$(36, 14, 7, 4)$	1	6	$J(9, 2)$
$(36, 15, 6, 6)$	32,548	6	
$(40, 12, 2, 4)$	27	7	
	1	8	point graph of dual $GQ(3, 3)$
$(45, 12, 3, 3)$	57	7	
	21	8	
$(45, 16, 8, 4)$	1	7	$J(10, 2)$

Table 13: Metric dimension of strongly regular graphs on up to 45 vertices

4.6 Hadamard graphs

A *Hadamard matrix* of order k is a $k \times k$ real matrix H with entries ± 1 and the property that $HH^t = kI$. Such a matrix must have order 1, 2 or a multiple of 4 (it is conjectured that all multiples of 4 are admissible). From a Hadamard matrix H of order $k \geq 4$, the associated *Hadamard graph* $\Gamma(H)$ has $4k$ vertices $\{r_i^+, r_i^-, c_i^+, c_i^- : 1 \leq i \leq k\}$, with $r_i^+ \sim c_i^+$ and $r_i^- \sim c_i^-$ if $H_{ij} = 1$, and $r_i^+ \sim c_i^-$ and $r_i^- \sim c_i^+$ if $H_{ij} = -1$. This graph is bipartite and distance-regular, with diameter 4 and valency k; for further details, see [10, Section 1.8].

Using Sloane's library of Hadamard matrices, one can easily construct the corresponding Hadamard graphs in GRAPE. The metric dimension for the Hadamard graphs arising from Hadamard matrices of orders from 4 to 20 is given in Table 15. In each case considered, Hadamard graphs of the same order had the same metric dimension.

Acknowledgements

Most of this work was carried out while the author was a postdoctoral fellow at Ryerson University. The author would like to thank Leonard Soicher and Alexander Hulpke for their assistance with GAP, and the Faculty of Science, Ryerson University for financial support.

Graph	Parameters	Metric dimension
Paley graph P_{37}	$(37, 18, 8, 9)$	5
Paley graph P_{41}	$(41, 20, 9, 10)$	7
$H(2, 7)$	$(49, 12, 5, 2)$	8
Paley graph P_{49}	$(49, 24, 11, 12)$	7
Self-complementary graph from $7^2 : (3 \times D_{16})$	$(49, 24, 11, 12)$	7
Hoffman–Singleton graph	$(50, 7, 0, 1)$	11
Paley graph P_{53}	$(53, 26, 12, 13)$	7
$J(11, 2)$	$(55, 18, 9, 4)$	8
Gewirtz graph	$(56, 10, 0, 2)$	9
Paley graph P_{61}	$(61, 30, 14, 15)$	7
(from $\mathrm{PSp}(6, 2)$)	$(63, 30, 13, 15)$	6
$H(2, 8)$	$(64, 14, 6, 2)$	10
(from $2^6 : (3.S_6)$)	$(64, 18, 2, 6)$	10
(from $2^6 : (S_3 \times \mathrm{GL}(3, 2))$	$(64, 21, 8, 6)$	9
Affine polar graph $\mathrm{VO}^-(6, 2)$	$(64, 27, 10, 12)$	7
(from $2^6 : S_8$)	$(64, 28, 12, 12)$	7
$J(12, 2)$	$(66, 20, 10, 4)$	8
Paley graph P_{73}	$(73, 36, 17, 18)$	7
M_{22} graph	$(77, 16, 0, 4)$	11
$J(13, 2)$	$(78, 22, 11, 4)$	9
$H(2, 9)$	$(81, 16, 7, 2)$	11
Brouwer–Haemers (from $3^4 : 2.\mathrm{P\Gamma L}(2, 9)$) [11]	$(81, 20, 1, 6)$	11
(from $3^4 : \mathrm{GL}(2, 3) : S_4$)	$(81, 32, 13, 12)$	8
Paley graph P_{81}	$(81, 40, 19, 20)$	7
Self-complementary graph from $3^5 : (4.S_5)$	$(81, 40, 19, 20)$	8
(from $PSp(4, 4).2$)	$(85, 20, 3, 5)$	12
Paley graph P_{89}	$(89, 44, 21, 22)$	8
$J(14, 2)$	$(91, 24, 12, 4)$	10
Paley graph P_{97}	$(97, 48, 23, 24)$	8
$H(2, 10)$	$(100, 18, 8, 2)$	12
Higman–Sims graph	$(100, 22, 0, 6)$	≤ 14
Hall–Janko graph	$(100, 36, 14, 12)$	9

Table 14: Metric dimension of rank-3 strongly regular graphs on up to 100 vertices

Order (=valency)	Vertices	No. of examples	Metric dimension
4	16	1 ($\cong Q_4$)	4
8	32	1	7
12	48	1	8
16	64	5	10
20	80	3	10

Table 15: Metric dimension of Hadamard graphs on up to 80 vertices

References

[1] R. F. Bailey, On the metric dimension of imprimitive distance-regular graphs, preprint (2014), `arXiv:1312:4971`.

[2] R. F. Bailey, J. Cáceres, D. Garijo, A. González, A. Márquez, K. Meagher and M. L. Puertas, Resolving sets in Johnson and Kneser graphs, *European J. Combin.* **34** (2013), 736–751.

[3] R. F. Bailey and P. J. Cameron, Base size, metric dimension and other invariants of groups and graphs, *Bull. London Math. Soc.* **43** (2011), 209–242.

[4] R. F. Bailey and K. Meagher, On the metric dimension of Grassmann graphs, *Discrete Math. Theor. Comput. Sci.* **13:4** (2011), 97–104.

[5] J. Bamberg et al., *FinInG — a finite geometry package for GAP*, Version 1.0, 2011; `cage.ugent.be/fining/`.

[6] A. F. Beardon, Resolving the hypercube, *Discrete Appl. Math.* **161** (2013), 1882–1887.

[7] N. L. Biggs, A. G. Boshier and J. Shawe-Taylor, Cubic distance-regular graphs, *J. London Math. Soc. (2)* **33** (1986), 385–394.

[8] N. L. Biggs and D. H. Smith, On trivalent graphs, *Bull. London Math. Soc.* **3** (1971), 155–158.

[9] L. M. Blumenthal, *Theory and Applications of Distance Geometry*, Clarendon Press, Oxford, 1953.

[10] A. E. Brouwer, A. M. Cohen and A. Neumaier, *Distance-Regular Graphs*, Springer-Verlag, Berlin, 1989.

[11] A. E. Brouwer and W. H. Haemers, Structure and uniqueness of the $(81, 20, 1, 6)$ strongly regular graph, *Discrete Math.* **106/107** (1992), 77–82.

[12] A. E. Brouwer and J. H. Koolen, The distance-regular graphs of valency four, *J. Algebraic Combin.* **10** (1999), 5–24.

[13] J. Cáceres, C. Hernando, M. Mora, I. M. Pelayo, M. L. Puertas, C. Seara and D. R. Wood, On the metric dimension of cartesian products of graphs, *SIAM J. Discrete Math.* **21** (2007), 423–441.

[14] P. J. Cameron, *Permutation Groups*, London Math. Soc. Student Texts **45**, Cambridge University Press, Cambridge, 1999.

[15] P. J. Cameron, *Functions for association schemes and permutation groups*, 2003, `www.maths.qmul.ac.uk/~pjc/gapprogs.html`.

[16] E. R. van Dam, J. H. Koolen and H. Tanaka, Distance-regular graphs, preprint (2014), `arXiv:1410.6294`.

[17] I. A. Faradjev, A. A. Ivanov and A. V. Ivanov, Distance-transitive graphs of valency 5, 6 and 7, *European J. Combin.* **7** (1986), 303–319.

[18] M. Feng and K. Wang, On the metric dimension of bilinear forms graphs, *Discrete Math.* **312** (2012), 1266–1268.

[19] The GAP Group, *GAP — Groups, Algorithms, and Programming*, Version 4.6.2, 2013, `www.gap-system.org`.

[20] A. Gardiner, On trivalent graphs, *J. London Math. Soc. (2)* **10** (1975), 507–512.

[21] A. Gardiner, An elementary classification of distance-transitive graphs of valency four, *Ars Combin.* **19A** (1985), 129–141.

[22] A. Gardiner and C. E. Praeger, Distance-transitive graphs of valency six, *Ars Combin.* **21A** (1986), 195–210.

[23] A. Gardiner and C. E. Praeger, Distance-transitive graphs of valency five, *Proc. Edinburgh Math. Soc. (2)* **30** (1987), 73–81.

[24] J. Guo, F. Li and K. Wang, Resolving sets for four families of distance-regular graphs, *Adv. Geom.* **14** (2014), 129–134.

[25] J. Guo, K. Wang and F. Li, Metric dimension of symplectic dual polar graphs and symmetric bilinear forms graphs, *Discrete Math.* **313** (2013), 186–188.

[26] J. Guo, K. Wang and F. Li, Metric dimension of some distance-regular graphs, *J. Comb. Optim.* **26** (2013), 190–197.

[27] A. Hanaki, *Elementary functions for association schemes on GAP*, `math.shinshu-u.ac.jp/~hanaki/as/gap/`.

[28] A. Hanaki and I. Miyamoto, *Classification of association schemes with small vertices*, `math.shinshu-u.ac.jp/~hanaki/as/`.

[29] H. Hanani, On transversal designs, in Combinatorics (Proc. NATO Advanced Study Inst., Breukelen, 1974), Part 1: Theory of designs, finite geometry and coding theory (eds. M. Hall Jr. and J. H. van Lint), *Math. Centre Tracts* **55**, Math. Centrum, Amsterdam, 1974, pp. 42–52.

[30] F. Harary and R. A. Melter, On the metric dimension of a graph, *Ars Combin.* **2** (1976), 191–195.

[31] T. Héger and M. Takáts, Resolving sets and semi-resolving sets in finite projective planes, *Electronic J. Combin.* **19:4** (2012), #P30, 21 pp.

[32] A. A. Ivanov and A. V. Ivanov, Distance-transitive graphs of valency k, $8 \leq k \leq 13$, in *Algebraic, extremal and metric combinatorics, 1986* (eds. M.-M. Deza, P. Frankl and I. G. Rosenberg), London Math. Soc. Lecture Note Series **131**, Cambridge University Press, Cambridge, 1988, pp. 112–145.

[33] J. Kratica, D. Cvetković, M. Čangalović, V. Kovačević-Vujčić and J. Kojić, The metric dimension of strongly regular graphs, *Proc. SYM-OP-IS 2008*, Soko Banja, 2008, pp. 341–344.

[34] J. Kratica, V. Kovačević-Vujčić and M. Čangalović, Computing the metric dimension of graphs by genetic algorithms, *Comput. Optim. Appl.* **44** (2009), 343–361.

[35] C. Pech and S. Reichard, Enumerating set orbits, in *Algorithmic algebraic combinatorics and Gröbner bases* (eds. M. Klin, G. A. Jones, A. Jurišić, M. Muzychuk and I. Ponomarenko), Springer-Verlag, Berlin, 2009; pp. 137–150.

[36] C. Pech and S. Reichard, *The SetOrbit package for GAP*, `www.math.tu-dresden.de/~pech/SetOrbit/`.

[37] C. M. Roney-Dougal, The primitive permutation groups of degree less than 2500, *J. Algebra* **292** (2005), 154–183.

[38] G. F. Royle, *Symmetric cubic graphs (Foster census)*, `school.maths.uwa.edu.au/~gordon/`.

[39] P. J. Slater, Leaves of trees, *Congr. Numer.* **14** (1975), 549–568.

[40] N. J. A. Sloane, *A library of Hadamard matrices*, `neilsloane.com/hadamard/`.

[41] D. H. Smith, On tetravalent graphs, *J. London Math. Soc. (2)* **6** (1973), 659–662.

[42] D. H. Smith, Distance-transitive graphs of valency four, *J. London Math. Soc. (2)* **8** (1974), 377–384.

[43] D. H. Smith, On bipartite tetravalent graphs, *Discrete Math.* **10** (1974), 167–172.

[44] L. H. Soicher, *The GRAPE package for GAP*, Version 4.6.1, 2012, `www.maths.qmul.ac.uk/~leonard/grape/`.

[45] L. H. Soicher, *The DESIGN package for GAP*, Version 1.6, 2011, `designtheory.org/software/gap_design/`.

[46] E. Spence, *Strongly Regular Graphs on at most 64 vertices*, `www.maths.gla.ac.uk/~es/srgraphs.php`.

[47] R. A. Wilson et al., Atlas of Finite Group Representations (version 3), `brauer.maths.qmul.ac.uk/Atlas/v3/`.

(Received 18 Dec 2013; revised 29 Jan 2015)