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Approaching the mixed Moore bound for diameter two by Cayley graphs

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Abstract

In a mixed (Δ, d)-regular graph, every vertex is incident with $\Delta \geq 1$ undirected edges and there are $d \geq 1$ directed edges entering and leaving each vertex. If such a mixed graph has diameter 2, then its order cannot exceed $(\Delta + d)^2 + d + 1$. This quantity generalizes the Moore bounds for diameter 2 in the case of undirected graphs (when $d = 0$) and digraphs (when $\Delta = 0$). For every d such that $d - 1$ is a prime power, Kautz digraphs of in- and out-degree d are Cayley digraphs of order missing the directed Moore bound by just 1. At the other extreme, the author and J. Širáň (2012) proved that the undirected Moore bound for diameter 2 and degree Δ can be asymptotically approached by Cayley graphs for an infinite set of values of Δ. We consider extensions of these results to mixed Cayley graphs, that is, mixed (Δ, d)-regular graphs admitting a group of automorphisms acting regularly on vertices.

1 Introduction

The degree-diameter problem has frequently been considered in its two "pure" versions, namely, undirected and directed. The version of the problem for graphs that may contain both undirected and directed edges, however, is of interest as well. In this paper we consider the degree-diameter problem for the latter variety, sometimes called *partially directed* or *mixed* graphs, and further restricting the attention to Cayley graphs of diameter two. We begin by setting up terminology and notation to be able to present further details about the subject and motivation of our research.

In this paper, a *graph* may contain both undirected and directed edges, called just *edges* and *darts*, respectively. An *undirected* graph contains no darts while a *directed* graph contains no edges. As usual, the *order* of a graph is the number of its vertices. The *degree* of a vertex is the sum of the number of edges incident to the vertex and the number of darts leaving the vertex. A path of length $\ell \geq 1$ from a vertex u to a vertex v, or an $u \to v$ path for short, is a sequence $u_0u_1 \ldots u_\ell$ of vertices such that $u_0 = u$, $u_\ell = v$, and for every $j \in \{0, \dots, \ell-1\}$ the vertices u_j and u_{j+1} are either joined by an edge or by a dart emanating from u_j and terminating at u_{j+1}. The *diameter* of a graph is the smallest positive integer k such that for every ordered pair (u, v) of distinct vertices there is a $u \to v$ path of length at most k; if such a k does not exist we set $k = \infty$.

We will confine ourselves to regular graphs in what follows, deferring the reader to [10] for general theory. Given non-negative integers Δ and d, we say that a graph is (Δ, d)-regular if every vertex is incident to Δ edges and is the tail of d darts. The order of a (Δ, d)-regular graph of diameter k is bounded above by the so-called *Moore bound* $M_k(\Delta, d)$, equal to the largest order of a unicentral tree of radius k each vertex of which is incident to at most Δ edges and at most d darts leaving the vertex. An explicit formula for $M_k(\Delta, d)$ may be found in [10] and will not be needed here in full generality. The (Δ, d)-regular graphs of order equal to $M_k(\Delta, d)$ are called *Moore graphs*.

It is well known that Moore graphs are rare. Indeed, by the classical results of [1] and [6] for undirected graphs, [12] and [3] for directed graphs, and a relatively recent result of [11], Moore graphs of diameter $k \geq 3$ and degree $\Delta + d$ do not exist at all except when $\Delta \leq 2$ and $d = 0$, or $\Delta = 0$ and $d = 1$. Setting these degree values and the case of the diameter $k = 1$ aside leaves us with the highly interesting instances of diameter $k = 2$ and degrees $\Delta + d$ such that either $\Delta \geq 3$ and $d = 0$, or $\Delta = 0$ and $d \geq 2$, or else both $\Delta, d \neq 0$. By results of [9], [3] and [2] for the undirected, directed and mixed cases, a Moore (Δ, d)-regular graph of diameter 2 and degree $\Delta + d$ as above can exist only if $\Delta \in \{3, 7, 57\}$ and $d = 0$, or when both $\Delta, d \neq 0$ and there is a positive integer divisor t of $(4d-3)(4d+5)$ such that $\Delta = (t^2+3)/4$. In particular, the Moore $(3,0)$- and $(7,0)$-regular undirected graphs are unique (the Petersen and the Hoffman-Singleton graphs) and the existence of a Moore $(57,0)$-regular undirected graph is still in doubt; there are no $(0,d)$-regular Moore directed graphs at all if $d \geq 2$, and the problem of a complete classification of mixed Moore graphs is still unresolved.

In the (almost) absence of Moore graphs, researchers have focused on constructions of infinite families of large (Δ, d)-regular graphs of a given diameter. This, however, was done almost exclusively for undirected and directed graphs; for more information we again refer to the recent update of the survey [10]. For manageable orders, a combined effort of a number of authors resulted in a number of computer-generated large undirected and directed graphs of given degree and diameter; the current results can be found in the on-line tables [17].

Not surprisingly, most of the extremely large undirected and directed graph orders listed in [17] have been found as Cayley graphs. We recall that given a group H and a unit-free subset S of H, the directed Cayley graph $\vec{C}(H, S)$ has vertex set H and for every $h \in H$ and $s \in S$ there is a dart emanating from the vertex h and terminating at the vertex hs. If S contains with every element also its inverse, then we 'collapse' the pair of darts from h to hs and from hs to $(hs)s^{-1} = h$ to a single undirected edge, obtaining thereby an undirected Cayley graph denoted by $C(H, S)$.

Generation (at both theoretical and computer-assisted level) of largest Cayley undirected and directed graphs of a given degree and diameter remains an open problem even for the smallest non-trivial diameter, $k = 2$. This is the main motivation of our paper. We will review the degree-diameter problem restricted to undirected and directed Cayley graphs and offer some extensions of the existing results to mixed Cayley graphs, containing both edges and darts.

2 Large undirected and directed Cayley graphs of diameter two and given degree

Let us begin our discussion with undirected graphs. The Moore bound for undirected regular graphs of degree Δ and diameter 2 is $M_2(\Delta, 0) = \Delta^2 + 1$. The known Moore graphs of diameter 2 and degree $\Delta \geq 3$, the Petersen graph and the Hoffman-Singleton graph, are well known to be vertex-transitive non-Cayley graphs. Although the existence or otherwise of an undirected Moore graph of degree 57 is unknown, it was proved by Higman (see [5]) that such a graph cannot be vertex-transitive. Combining these facts with the result of [7] implies that every Cayley graph of diameter 2 and degree $\Delta \geq 3$ has order at most $\Delta^2 - 1$.

The current best constructive result on large Cayley graphs of diameter 2 and given degree was obtained in [13]. We reproduce a substantial part of it for further reference.

For any integer $n \geq 1$ let F be a Galois field of order 2^{2n} and let F^+ and F^* be the additive and the multiplicative group of F. Take the one-dimensional affine group $H_n = AGL(1, F) \cong F^+ \rtimes F^*$ with multiplication given by $(a, b)(c, d) = (a + bc, bd)$ for all $a, c \in F^+$ and $b, d \in F^*$. Let $A = \{(x, x^2);\ x \in F^*\}$. The group F^+ can be identified with a vector space of dimension $2n$ over a field of order 2. Let B be the set of all elements of H_n of the form $(z, 1)$ where z ranges over all non-zero vectors in F^+ whose first n or last n coordinates are equal to zero. The group F^* is isomorphic to the cyclic group Z_m of order $m = 2^{2n} - 1$; let $\varphi :\ Z_m \to F^*$ be such an isomorphism. Let C be the subset of H_n consisting of all elements of the form $(0, \varphi(j))$ for $j \in \{\pm i;\ 1 \leq i \leq 2^{n-1} - 1\} \cup \{\pm(2^n - 1)i;\ 1 \leq i \leq 2^{n-1}\}$. It can be checked that $U_n = A \cup B \cup C$ is an inverse-closed subset of H_n, with $|U_n| = 2^{2n} + 2^{n+2} - 6$. By [13] we have:

Proposition 1 *For every $n \geq 1$ the (undirected) Cayley graph $C(H_n, U_n)$ has diameter 2, degree $\Delta_n = |U_n| = 2^{2n} + 2^{n+2} - 6$ and order $|H_n| = 2^{2n}(2^{2n} - 1) > \Delta_n^2 - 8\Delta_n^{3/2}$; in particular, $|H_n|/M_2(\Delta_n, 0) \to 1$ as $n \to \infty$.* □

In this sense we may say that the family of Cayley graphs $C(H_n, U_n)$ of diameter 2 and degree Δ_n asymptotically approaches the Moore bound $M_2(\Delta_n, 0) = \Delta_n^2 + 1$ as $\Delta_n \to \infty$. In somewhat more loose terms this shows that *the undirected Moore bound for diameter 2 can be asymptotically approached by Cayley graphs*.

Let us now switch to directed graphs, which we will also call *digraphs* for short. We begin with recalling from [10] that the directed Moore bound for diameter 2 and degree d is $M_2(0, d) = d^2 + d + 1$.

A beautiful short argument of [3] shows that a directed Moore graph of degree d and diameter k exists if and only if $d = 1$ or $k = 1$. In particular, there are no directed Moore graphs of diameter 2 and degree $d \geq 2$. Surprisingly, there are (even vertex-transitive) digraphs of diameter 2 for any degree $d \geq 2$ of order $d^2 + d$, namely, the line digraphs of complete digraphs $\vec{K}_{d+1}$, known also as *Kautz digraphs*. We recall that in a complete digraph $\vec{K}_{d+1}$ o order $d + 1$, for any ordered pair u, v of distinct vertices there is a dart from u to v, so that every such pair forms a directed cycle of length 2, or a *digon*. Its line digraph $L(\vec{K}_{d+1})$ has diameter 2, in- and out-degree d, and order $(d + 1)d$, which misses the directed Moore bound $M_2(0, d)$ just by 1. By a deep result of [8], the Kautz digraphs of degree d are the only digraphs of diameter 2, degree d and order $M_2(0, d) - 1$ for any $d \geq 3$.

Note that since the complete digraphs $\vec{K}_{d+1}$ are arc-transitive, the Kautz digraphs $L(\vec{K}_{d+1})$ are vertex-transitive. By [4] the graph $L(\vec{K}_{d+1})$ is a Cayley digraph if and only if the automorphism group of $\vec{K}_{d+1}$ contains a sharply 2-transitive subgroup on vertices. Since $Aut(\vec{K}_{d+1})$ is isomorphic to the symmetric group S_{d+1} of degree $d + 1$ in its natural action on vertices, we may use [14] to conclude that $L(\vec{K}_{d+1})$ is a Cayley digraph if and only if $d + 1$ is equal to a prime power q. In such a case the corresponding sharply 2-transitive subgroup can be identified with a copy of a 1-dimensional affine group over a near-field of order q, cf. [14] again. In particular, if $d + 1 = q$ and $F = GF(q)$ is a Galois field of order q, it is easy to check that the Kautz digraph $L(\vec{K}_{d+1})$ can be identified with a Cayley digraph $\vec{C}(H, D)$ for the group $H = AGL(1, F) \cong F^+ \rtimes F^*$ and the generating set $D = \{(ax + b, x); x \in F^*\}$ for any preassigned $a, b \in F$ such that $a + b \neq 0$. Summing up, we have the following result a special case of which (for $a = 0$ and $b = 1$) can also be found in [15, 16]:

Proposition 2 *A Kautz digraph $L(\vec{K}_n)$ is a Cayley digraph if and only if n is a prime power. Moreover, let (q_n) be an arbitrary sequence of prime powers, let $F_n = GF(q_n)$, let $H_n = AGL(1, F_n) \cong F_n^+ \rtimes F_n^*$ and let $D_n = D_n(a, b) = \{(ax + b, x); x \in F_n^*\}$ for fixed a, b such that $a + b \neq 0$. For every $n \geq 1$ the Cayley digraph $\vec{C}(H_n, D_n)$ has diameter 2, degree $d_n = q_n - 1$, order $d_n^2 + d_n$ and is isomorphic to a Kautz digraph.* □

The Kautz digraphs $\vec{C}(H_n, D_n)$ therefore provide a sequence showing that *the directed Moore bound for diameter 2 can be approached by Cayley digraphs*. This time we not only have $|H_n|/M_2(0, d_n) \to 1$ as $n \to \infty$ but also the *defect* $M_2(0, d_n) - |H_n|$ is equal to 1, the smallest non-zero defect. Comparing this with the undirected case, observe that Proposition 1 only guarantees that the ratio of the defect and the undirected Moore bound tends to zero.

3 Large mixed Cayley graphs of diameter two and given degree

A (Δ, d)-regular graph will be called *mixed* if $\Delta \geq 1$ and $d \geq 1$. Here we will investigate the natural question of extending the statements from the previous section about asymptotic approach of the undirected and directed Moore bounds for diameter 2 by infinite sequences of Cayley graphs and digraphs to the class of mixed graphs.

The Moore bound for (Δ, d)-regular graphs of diameter 2 has the form [10]

$$M_2(\Delta, d) = (\Delta + d)^2 + d + 1 \tag{1}$$

and, as indicated in the Introduction, can easily be obtained estimating the number of paths of length at most two from a fixed vertex in a (Δ, d)-regular graph. It generalizes the undirected Moore bound $\Delta^2 + 1$ for undirected graphs (in the case $d = 0$) as well as the directed Moore bound $d^2 + d + 1$ for directed graphs (i.e., when $\Delta = 0$). A (Δ, d)-regular graph of diameter 2 with both $\Delta, d \geq 1$ is called a *mixed Moore graph* if its order is equal to the value of the Moore bound $M_2(\Delta, d)$ from (1).

Examples of mixed Moore graphs exist for all degrees from two on and can be obtained from the Kautz digraphs $L(\vec{K}_n)$ of degree $n - 1$ by replacing every digon by an undirected edge. The resulting graphs have degree $\Delta + d$ for $\Delta = 1$ and $d = n - 2 \geq 1$ and order $n(n-1) = (\Delta + d)^2 + d + 1$, which is indeed the Moore bound $M_2(\Delta, d)$. In fact, this is the only infinite family of mixed Moore graphs of diameter 2 known to date, and by [8] these are the only mixed Moore graphs with $\Delta = 1$. Sporadic examples exist for some other values of Δ, cf. [11].

The concepts of a Cayley graph and a Cayley digraph have a natural common generalization in the universe of mixed graphs. Let H be a group and let X and Y be disjoint unit-free subsets of H such that X is closed under taking inverse elements, that is, $X = X^{-1}$. The *mixed Cayley graph* $C(H; X, Y)$ has vertex set H; for every vertex $h \in H$ there is an undirected edge joining h with hx for every $x \in X$ and a directed edge from h to hy for every $y \in Y$.

It is obvious that a mixed Cayley graph $C(H; X, Y)$ is (Δ, d)-regular for $\Delta = |X|$ and $d = |Y|$. The classical concepts of an undirected and a directed Cayley graph correspond to the extreme cases when $Y = \emptyset$ and $X = \emptyset$, respectively. Observe that our definition does not stipulate that Y cannot contain a pair of mutually inverse elements. If there is an $y \in H \backslash \{1\}$ such that both y and y^{-1} are in Y, then there is a directed edge from h to hy and also a directed edge from hy to h for every vertex $h \in H$, forming a digon between the two vertices.

Returning to the theme of this section we now show that the mixed Moore bound for diameter 2 can be approached by mixed Cayley graphs in a rather strong sense. The order of a (mixed) graph G is denoted by $|G|$ in what follows.

Theorem 1 *For every c such that $0 \leq c \leq +\infty$ there exists an infinite sequence of mixed (Δ_n, d_n)-regular Cayley graphs G_n of diameter 2 such that $|G_n|/M_2(\Delta_n, d_n) \to 1$ and $\Delta_n/d_n \to c$ as $n \to \infty$.*

Proof. For any $n \geq 1$ consider the undirected Cayley graph $C(H_n, U_n)$ of diameter 2 and degree $k_n = 2^{2n} + 2^{n+2} - 6$, described in Proposition 1. If $0 < c < +\infty$, take an arbitrary inverse-closed subset X_n of U_n such that $|X_n| = \lfloor \frac{c}{1+c} k_n \rfloor$. Note that this is possible independently in the parity of the integer on the right because U_n contains involutions. If $c = 0$ and $c = +\infty$ we may take any X_n such that $|X_n| = o(k_n)$ and $|X_n| = (1 - o(1))k_n < k_n$, respectively. Letting $Y_n = U_n \backslash X_n$, the crucial but easy observation to be made is that the diameters of the undirected Cayley graph $C(H_n, U_n)$ and the mixed Cayley graph $G_n = C(H_n, X_n, Y_n)$ are the same, namely, 2. Straightforward calculations with the help of Proposition 1 then show that $|G_n|/M_2(\Delta_n, d_n) \to 1$ and $\Delta_n/d_n \to c$. □

One might argue that the proof of Theorem 1 was based on 'cheating' in the sense that we took a rather strong existing result from [13] on undirected Cayley graphs $C(H_n, U_n)$ and replaced every edge joining a vertex $h \in H_n$ with hy for $y \in Y_n$ in the graph $C(H_n, U_n)$ by a digon formed by darts between the vertices h and hy in the mixed Cayley graph $C(H_n, X_n, Y_n)$. This raises the question if it is possible to strengthen Theorem 1 to *simple* mixed Cayley graphs, that is, graphs without any digon.

Before addressing this concern let us mention another type of 'cheating' one might use to produce statements about approaching the mixed Moore bound for diameter 2 by Cayley graphs. Instead of replacing edges by digons one may use in a sense a reverse procedure, namely, replacing darts by edges provided that the number of such replacements is negligible compared to the degree. More precisely, if $G_n = \vec{C}(H_n, D_n)$ is a family of Cayley digraphs of diameter 2 and degree k_n such that $|G_n|/k_n^2 \to 1$ as $n \to \infty$, let U_n be a subset of D_n such that $|U_n| = o(k_n)$. Letting now $X_n = U_n \cup U_n^{-1}$ and $Y_n = D_n \backslash U_n$, with $|X_n| = \Delta_n$ and $|Y_n| = d_n$, and considering the mixed Cayley graphs $G'_n = C(H_n, X_n, Y_n)$ of diameter 2 we still have $|G'_n|/(\Delta_n + d_n)^2 \to 1$ as $n \to \infty$. A way to avoid this is to require our mixed Cayley graphs $C(H, X, Y)$ have the property that the removal of any element from X increases the diameter. More generally, we will say that a mixed Cayley graph is *irredundant* if removal of any generator from the generating set increases the diameter of the graph.

Any proper strengthening of Theorem 1 should therefore address *simple* and *irredundant* mixed Cayley graphs. This appears to be much harder and we offer the following result in this direction.

Theorem 2 *There is an infinite sequence of simple and irredundant mixed* (Δ_n, d_n)*-regular Cayley graphs* G_n *of diameter* 2 *such that* $4d_n/\Delta_n^2 \to 1$ *and* $|G_n|/M_2(\Delta_n, d_n) \to 1$ *as* $n \to \infty$.

Proof. For Let (q_n) be an increasing infinite sequence of prime powers. Consider the Galois field $K = GF(q_n^2)$ and its unique subfield F isomorphic to $GF(q_n)$; as before, K^+, F^+, K^* and F^* will denote the corresponding additive and multiplicative groups. Let $H_n = AGL(1, K) = K^+ \rtimes K^*$, with the group operation $(a, b)(c, d) = (a + bc, bd)$ for $a, c \in K^+$ and $b, d \in K^*$. Let $A = \{(0, x);\ x \in F^*,\ x \neq 1\}$ and let $B = \{(z, 1);\ z \in F^*\}$. Observe that $X = A \cup B$ is an inverse-closed subset of H_n. Further, let $Y = \{(1, s);\ s \in K^* \backslash F^*\}$. We show that the mixed Cayley graph $G_n = C(H_n, X, Y)$ has all the properties of the statement of our theorem.

Let us begin by proving that the diameter of G_n is 2, which is equivalent to showing that every element $(a, b) \in H_n$ not equal to the unit element $(0, 1)$ and not contained in $X \cup Y$ is a sum of two elements from $X \cup Y$. We will consider two cases.

Firstly, let $a \in K \backslash F$; in particular, $a - 1 \notin F$ and so $a - 1 \neq 0$. Suppose $b \in K^*$ is such that $b(a-1)^{-1} \in K^* \backslash F^*$. Letting $s = a - 1$ and $t = b(a-1)^{-1}$ one can check that $(1, s)(1, t) = (a, b)$, so that (a, b) is a product of two elements from Y. The other possibility to consider is that $b(a-1)^{-1} \in F^*$; since $a - 1 \notin F$ we have $b \notin F$. Then, letting $z = (a-1)b^{-1} \in F^*$ and $s = b \notin F$ one sees that $(1, s)(z, 1) = (a, b)$, that is, (a, b) is now a product of an element of Y and an element of $B \subset X$.

Secondly, let $a \in F$. If $b \in K^* \backslash F^*$, we only need to consider $a \neq 1$ since we have $(1, b) \in Y$. Taking $z = a - 1 \neq 0$ and $s = b$ and realizing that $(z, 1)(1, s) = (a, b)$ we have (a, b) as a product of an element from $B \subset X$ and an element from Y. If $b \in F^*$ and $(a, b) \neq (0, 1)$, then either $(a, b) \in X$ or $(a, b) = (a, 1)(0, b)$ is a product of an element from A and an element from B.

It follows that the mixed Cayley graph $G_n = C(H_n, X, Y)$ has diameter 2. We now address the issue of redundancy. Suppose we remove an element $(1, u)$ from Y. Then there is no way to express the element $v = (1 + u, u^2)$ as a product of two elements from $X \cup Y$. Indeed, since $u \notin F$, the only way to do this would be as a product of two elements from Y, that is, $(1 + u, u^2) = (1, s)(1, t) = (1 + s, st)$, or as a product of an element from Y and an element from B, that is, $(1 + u, u^2) = (1, s)(z, 1) = (1 + sz, s)$, and both ways lead to immediate contradictions. If an element $(z, 1)$ for some $z \in F^*$ is removed, then choose $a, b \in K^* \backslash F^*$ such that $z = (a-1)b^{-1}$. It can be checked that then (a, b) would not be expressible as a product of two elements from $X \cup Y$. Finally, if $(0, x)$ is removed for some $x \in F^*$, $x \neq 1$, then $(1, x)$ would fail to be a product of two elements from $X \cup Y$. This shows that our mixed Cayley graph is *irredundant*.

To verify the remaining items of the statement of our theorem, observe that the sets X and Y are disjoint and $Y \cup Y^{-1} = \emptyset$, implying that G_n is *simple*. Further, we obviously have $d_n = |Y| = q_n^2 - q_n$ darts leaving every vertex, and $\Delta_n = |X| = 2q_n - 3$ undirected edges incident with every vertex; note that $4d_n/\Delta_n^2 \to 1$ as $n \to \infty$. Since $|G_n| = |H_n| = q_n^2(q_n^2 - 1) = (\Delta_n + d_n)^2 + O(q_n^3)$, we have $|G_n|/M_2(\Delta_n, d_n) \to 1$ as $n \to \infty$, completing the proof. □

In the proof of Theorem 2 we managed to replace just an asymptotically negligible proportion of darts with edges, maintaining simplicity and irredundancy. An analogous result for mixed Cayley graphs of affine one-dimensional groups over Galois fields of characteristic 2 can be stated and proved exactly as above by replacing the set A with the set $\{(x, x^2);\ x \in F^*\}$ used in Proposition 1, 'marrying' thus the generating sets from Propositions 1 and 2.

The problem of extending Theorem 1 to simple and irredundant mixed Cayley graphs remains i open. One of the issues is lack of suitable generating sets for Cayley graphs and digraphs approaching the Moore bound for diameter 2. In fact, to the best of our knowledge the only such generating sets known at the time of writing this article were those listed in Section 2.

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