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Zero-forcing, treewidth, and graph coloring

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Abstract

We show that certain types of zero-forcing sets for a graph give rise to chordal supergraphs and hence to proper colorings.

Zero-forcing was originally defined to provide a bound for matrix minimum rank problems [1], but is interesting as a graph-theoretic notion in its own right [5], and has applications to mathematical physics, such as quantum systems [3]. There are different flavors of zero-forcing, many corresponding to a minimum rank graph parameter, and each is typically defined via assignments of the colors black and white to vertices and a *color-change rule* that allows changing white vertices to black [2]; the associated *zero-forcing number* is then the smallest cardinality among sets of vertices that when colored black originally allow the entire graph to become colored black via (repeated) application of the color-change rule (*zero-forcing sets*).

Barioli et al. [2] showed that even treewidth can be defined as a zero-forcing parameter. Their proof uses a characterization of treewidth involving the game of cops and robbers. In this paper, we will show that a treewidth zero-forcing set Z for a graph G can be used to directly construct a $|Z|$-tree on the vertices of G that contains G as a subgraph. As an application, we will see that many different types of zero-forcing sets give easy constructions of proper colorings and proper list-colorings.

For a given coloring of the vertices of a graph using black and white, the treewidth color-change rule was defined as follows (standard definitions are taken from Diestel's *Graph Theory* [4]):

Definition. Let B be the set consisting of all the black vertices. Let $W_1, \ldots, W_k$ be the sets of vertices of the k components of $G - B$. For each component i, $1 \leq i \leq k$, let $C_i \subseteq B$ be the subset of black vertices that are considered to be *active* with regard to that component, where initially each $C_i = B$. If $w \in W_i$ and for each component X of $G[W_i] - w$ there is a vertex $u_X \in C_i$ with no white neighbor in $G[V_X \cup B]$, then change the color of w to black and associate to each connected component X of $G[W_i] - w$ a new active set equal to $(C_i - u_X) \cup \{w\}$.

When this color-change rule is applied, we will say that the u_X vertices force w and the u_X vertices and w together comprise a forcing.

In studying treewidth zero-forcing sets, we will find it advantageous to keep track of active sets for each vertex as well as the progress of the color-changes. If Z is a treewidth zero-forcing set of a graph G and $m = |G - Z|$, let $w_1, \ldots, w_m$ be the vertices of $G - Z$ in the order in which they are turned black (there may be more than one such order – we'll pick one). Our notational scheme will be subscripts that refer to the progress of the forcing: a subscript i, $1 \leq i \leq m$, will reference the state of things after i forces, that is, when w_i has become black and (if $i < m$) w_{i+1} is still white. For example, let $B_0 = Z$ and recursively define, for $1 \leq i \leq m$, $B_i = B_{i-1} \cup \{w_i\}$. Then B_i is the set of black vertices after i forces. Continuing in this spirit, if u is a vertex of $G - B_j$ for some j with $0 \leq j \leq m-1$, let C_j^u be the connected component of $G - B_j$ containing u and let A_j^u be the set of active vertices of C_j^u.

Proposition. Let Z be a treewidth zero-forcing set of a graph G and use the notation above. Let G_0 be the graph obtained from G by adding edges between any two vertices of B_0 that are not adjacent in G. Let G_i be the graph obtained from G_{i-1} by adding edges between w_i and any vertices of $A_{i-1}^{w_i}$ that are not neighbors of w_i in G_i. Then G_m is a $|Z|$-tree on the same vertices as G containing G as a subgraph. Moreover, Z is a treewidth zero-forcing set for G_m with the same forcings in the same order.

Proof. Since no vertices are added and no edges are removed, G is a subgraph of G_m and they share the same vertex set. To prove that G_m is a $|Z|$-tree, we will use the recursive definition of k-tree. Specifically, we claim that for each i with $1 \leq i \leq m$, each $G_i[B_i]$ is a $|Z|$-tree and that $G_i[B_i]$ is obtained from $G_{i-1}[B_{i-1}]$ by adding the vertex w_i, which is adjacent in $G_i[B_i]$ to the vertices of a $|Z|$-clique in $G_{i-1}[B_{i-1}]$.

We begin by collecting some useful facts. First, active sets (for both vertices and components) start with $|Z|$ vertices and only change via a one-for-one swap of vertices, so $|A_{i-1}^{w_i}| = |Z|$ for each i.

Let $N(w)$ denote the set of neighbors of vertex w in G. We next claim that $N(w_i) \cap B_{i-1} \subseteq A_{i-1}^{w_i}$ for each i with $1 \leq i \leq m$. To see this, suppose that $v \in N(w_i) \cap B_{i-1}$. There are two possibilities: either $v \in B_0$, in which case $v \in A_0^{w_i}$, or $v \notin B_0$, meaning $v = w_j$ for some $j < i$. In the latter case, since w_i and $v = w_j$ are adjacent in G, $C_{j-1}^{w_i} = C_{j-1}^{w_j}$, and so $v = w_j \in A_j^{w_i}$ due to the jth force. Either way, v is at some point an active vertex for w_i. Suppose $v \notin A_{i-1}^{w_i}$. Then for some $k < i$ (and $k > j$ if $v = w_j$), v was u_X for $X = C_k^{w_i}$, but this contradicts that $v \in N(w_i)$ since w_i would be a white neighbor of v in $G[V_X \cup B_{k-1}]$. Thus $v \in A_{i-1}^{w_i}$.

Finally, we claim that for i and j such that $0 \leq i < j \leq m$, each $A_i^{w_j}$ forms a clique in each G_k such that $0 \leq k < j$. First, note that A_0^v consists of the vertices of Z, which form a clique in G_0 by definition. Assume then that $0 < i < j \leq m$ and the vertices of the set $A_{i-1}^{w_j}$ form a clique in G_{i-1}. If $A_{i-1}^{w_j} = A_i^{w_j}$, the vertices of $A_i^{w_j}$ will still be a clique in G_i. If $A_{i-1}^{w_j} \neq A_i^{w_j}$, then $w_i \in C_{i-1}^{w_j}$. Thus $C_{i-1}^{w_i} = C_{i-1}^{w_j}$, which by the definition of the active sets for vertices implies $A^{w_i}_{i-1} = A^{w_j}_{i-1}$. By assumption, $A^{w_i}_{i-1}$ is a clique in G_{i-1}. By construction, $A^{w_i}_{i-1} \cup \{w_i\}$ is a clique in G_i, and, by definition, $A^{w_j}_i \subset A^{w_i}_{i-1} \cup \{w_i\}$, so that $A^{w_j}_i$ is also a clique in G_i. The claim follows by induction.

To start the induction for the main part of the proof, notice that G_0 is a $|Z|$-clique in $G_0[B_0] = G_0$ by construction, that $A^{w_1}_0 = B_0$ by definition, and thus $w_1 \in B_1$ is adjacent in $G_1[B_1]$ to the vertices of G_0 by construction. Thus $G_1[B_1]$ is a $|Z|$-tree.

Suppose now that $G_j[B_j]$ is a $|Z|$-tree for some j with $1 \leq j < m$. By construction, the vertices of $G_{j+1}[B_{j+1}]$ are those of $G_j[B_j]$ and the vertex w_{j+1}, which by construction and the second fact above is adjacent to exactly the vertices of $A^{w_{j+1}}_j$ in $G_{j+1}[B_{j+1}]$. By the first fact above, $|A^{w_{j+1}}_j| = |Z|$, and by the third and final fact above, $G_j[A^{w_{j+1}}_j]$ is a clique. Thus $G_{j+1}[B_{j+1}]$ is a $|Z|$-tree. By induction, $G_m = G_m[B_m]$ is a $|Z|$-tree.

We also claim that Z is a zero-forcing set of G_m using the same forces. The only way this can fail to be true is if an edge is added to G that will cause the color-change rule to no longer be applicable at some point. We will show this cannot happen. Consider a vertex z that is the u_X vertex for a connected component X of $G - B_i$ for some i such that $0 \leq i < m$. If an edge is added between z and a vertex that is in B_i, that edge does not affect the ability of z to be u_X. Thus we only need consider an edge added between z and some w_j where $j > i$. If $C^{w_j}_i \neq X$, then the edge to w_j does not affect the ability of z to be u_X. If $C^{w_j}_i = X$, then z is replaced by w_i in $A^{w_j}_i$. A vertex that has become inactive for another vertex can never become active again, so $z \notin A^{w_j}_i$ contradicts that $z = u_X$, since $z = u_X$ implies $z \in A^{w_i}_{i-1}$. □

In the hierarchy of color-change rules, the treewidth color-change rule is among the least restrictive. In particular, it is clear from the article by Barioli et al. [2] that, among others, standard zero-forcing sets and positive semidefinite zero-forcing sets are also treewidth zero-forcing sets. As an application of our proposition, many types of zero-forcing sets thus give proper colorings as in the following corollary:

Corollary. If G is a graph with a treewidth zero-forcing set Z, where the vertices of $G - Z$ are $w_1, \dots, w_m$ in the order they are forced, given an assignment of a list of $|Z| + 1$ colors to each vertex, a proper list-coloring of G may be selected by first choosing a proper list-coloring of $G[Z]$ then selecting an available color from the list of each w_i in order.

Proof. A proper list-coloring of $G[Z]$ exists since $|G[Z]| < |Z|+1$, and from the proof of the proposition, each w_i will be adjacent to at most $|Z|$ vertices whose colors have already been selected when its turn arrives. □

Remark. The treewidth color-change rule is significantly different from the original zero-forcing color-change rule in that its application requires knowledge of more than the graph and which vertices are black. We would be very interested to know if it possible to define a color-change rule that depends only on the graph and which vertices are black that will also give treewidth as its zero-forcing number.

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