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An involution on bicubic maps and $\beta(0,1)$-trees

Anders Claesson   Sergey Kitaev

Department of Computer and Information Sciences
University of Strathclyde
26 Richmond Street, Glasgow G1 1XH
United Kingdom
anders.claesson@strath.ac.uk   sergey.kitaev@strath.ac.uk

Anna de Mier*

Departament de Matemàtica Aplicada II
Universitat Politècnica de Catalunya
Jordi Girona 1–3, 08034 Barcelonax
Spain
anna.de.mier@upc.edu

Abstract

Bicubic maps are in bijection with $\beta(0,1)$-trees. We introduce two new ways of decomposing $\beta(0,1)$-trees. Using this we define an endofunction on $\beta(0,1)$-trees, and thus also on bicubic maps. We show that this endofunction is in fact an involution. As a consequence we are able to prove some surprising results regarding the joint equidistribution of certain pairs of statistics on trees and maps. Finally, we conjecture the number of fixed points of the involution.

1 Introduction

A *planar map* is an embedding of a connected multigraph in the sphere with no edge-crossings, considered up to continuous deformations. A map has *vertices*, *edges*, and *faces* (disjoint simply connected domains). The maps we consider shall be *rooted*, meaning that a directed edge has been distinguished as the root. The face that lies to the right of the root edge while following its orientation is the *root face*, whereas the vertex from which the root stems is the *root vertex*. When drawing a planar map on the plane, we usually follow the convention to choose the outer (unbounded) face as the root face. Tutte [10, Chapter 10] founded the enumerative theory of planar maps in a series of papers in the 1960s (see [9] and the references in [3]).

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A planar map in which each vertex is of degree 3 is *cubic*; it is *bicubic* if, in addition, it is bipartite, that is, if its vertices can be colored using two colors, say, black and white, so that adjacent vertices are assigned different colors.

The smallest bicubic map has two vertices and three edges joining them. It is well-known that the faces of a bicubic map can be colored using three colors so that adjacent faces have distinct colors, say, colors 1, 2 and 3, in a counterclockwise order around white vertices. We will assume that the root vertex is black and the root face has color 3. There are exactly three different bicubic maps with 6 edges and they are given in Figure 1. The number of bicubic maps with $2n$ vertices was given by

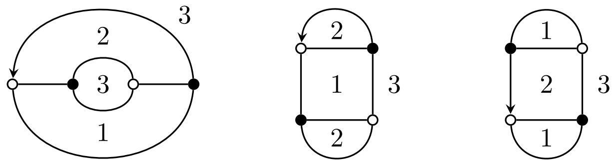


Figure 1: All bicubic maps with 4 vertices.

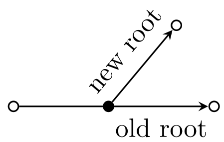
Tutte [9]:

$$\frac{3 \cdot 2^{n-1}(2n)!}{n!(n+2)!}.$$

Let M be a bicubic map. For $i = 1, 2, 3$, let $\mathcal{F}_i(M)$ be the set of i-colored faces of M. Let $R_1 \in \mathcal{F}_1(M)$, $R_2 \in \mathcal{F}_2(M)$, and $R_3 \in \mathcal{F}_3(M)$ be the three faces around the root vertex; in particular, R_3 is the root face. We shall now define two statistics on bicubic maps:

f1r3(M) is the number of faces in $\mathcal{F}_1(M)$ that touch R_3;
f3r2(M) is the number of faces in $\mathcal{F}_3(M)$ that touch R_2.

Consider the following transformation ϕ on bicubic maps. Recolor the faces by the mapping $\{1 \mapsto 2,\ 2 \mapsto 3,\ 3 \mapsto 1\}$. Keep the colors of the vertices. Keep, also, the root vertex, but let the new root edge be the first edge in counterclockwise direction from the old root edge:



It is easy to see that ϕ is a bijection; indeed, ϕ^3 is the identity transformation. Moreover, ϕ establishes the following result.

Proposition 1. *For any positive integer n, we have*

$$\sum_M x^{\mathrm{f1r3}(M)} = \sum_M x^{\mathrm{f3r2}(M)},$$

where both sums are over all bicubic maps on n vertices. In other words, the statistics f1r3 *and* f3r2 *are equidistributed.*

In this paper we show the following stronger result.

Theorem 2. *For any positive integer* n*, we have*

$$\sum_M x^{\mathrm{f1r3}(M)} y^{\mathrm{f3r2}(M)} = \sum_M x^{\mathrm{f3r2}(M)} y^{\mathrm{f1r3}(M)},$$

where both sums are over all bicubic maps on n *vertices. In other words, the two pairs of statistics* (f1r3, f3r2) *and* (f3r2, f1r3) *are jointly equidistributed.*

To prove Theorem 2 we first translate the statement to a corresponding statement on so called $\beta(0,1)$-trees; there is a one-to-one correspondence [4] between bicubic maps and such trees. We then provide two proofs of the theorem. Our first proof of Theorem 2 is based on generating functions (see the end of Section 4). Our combinatorial proof of the theorem (see Corollary 12 and the text following it) is based on defining an endofunction on the trees, and proving that it is an involution that respects the statistics in question (see Theorem 10). We also conjecture the number of fixed points of the involution.

The results in this paper can be seen as an extension to $\beta(0,1)$-trees and bicubic maps of studies conducted in [1, 2, 5, 6] on $\beta(1,0)$-trees and rooted non-separable planar maps.

2 $\beta(0,1)$-trees

Cori et al. [4] introduced description trees to give a framework for recursively decomposing several families of planar maps. A $\beta(0,1)$-tree is a particular kind of description tree; it is defined as a rooted plane tree whose nodes are labeled with nonnegative integers such that

1. leaves have label 0;
2. the label of the root is one more than the sum of its children's labels;
3. the label of any other node exceeds the sum of its children's labels by at most one.

The unique $\beta(0,1)$-tree with exactly one node (and no edges) will be called *trivial*; the root of the trivial tree has label 0. Any other $\beta(0,1)$-tree will be called *nontrivial*. In Figure 2 we have listed all $\beta(0,1)$-trees on 4 nodes. Let $\mathrm{root}(T)$ denote the root label of T, and let $\mathrm{sub}(T)$ denote the number of children of the root. We say that a $\beta(0,1)$-tree T is *reducible* if $\mathrm{sub}(T) > 1$, and *irreducible* otherwise. Any reducible tree can be written as a sum of irreducible ones, where the sum $U \oplus V$ of two trees U and V is defined as the tree obtained by identifying the roots of U and V into a new root with label $\mathrm{root}(U) + \mathrm{root}(V) - 1$. See Figure 3 for an example.

Note also that any irreducible tree with at least one edge is of the form $\lambda_i(T)$, where $0 \le i \le \mathrm{root}(T)$ and $\lambda_i(T)$ is obtained from T by joining a new root via an

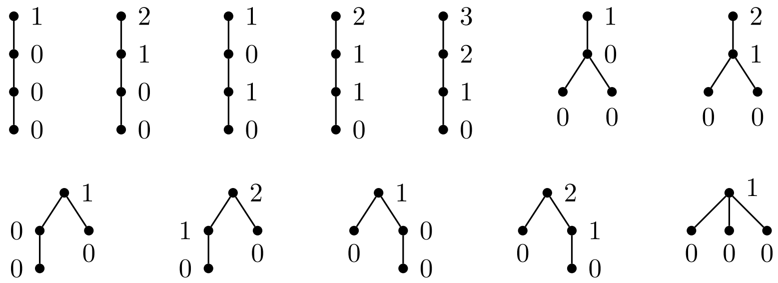


Figure 2: All $\beta(0,1)$-trees on 4 nodes.

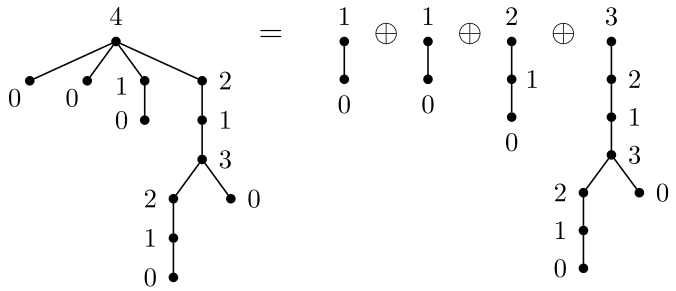
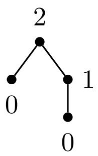
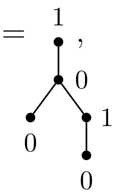
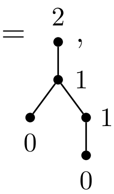
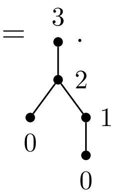


Figure 3: Decomposing a reducible $\beta(0,1)$-tree.

edge to the old root; the old root is given the label i, and the new root is given the label $i+1$. For instance,

if $T =$  then $\lambda_0(T) =$ , $\lambda_1(T) =$ , and $\lambda_2(T) =$ .

Let us now introduce a few more statistics on $\beta(0,1)$-trees. By the *rightmost path* we shall mean the path from the root to the rightmost leaf. We define $\mathrm{rzero}(T)$ as the number of zeros on the rightmost path. By definition, $\mathrm{rzero}(\bullet) = 0$.

A node is called *excessive* if its label exceeds the sum of its children's labels; it is called *moderate* otherwise. In particular, a leaf is a moderate node and the root is an excessive node. Assuming that T is nontrivial, we let $\mathrm{rmod}(T)$ be the number of moderate nodes on the rightmost path of T. For the case of the trivial tree we define $\mathrm{rmod}(\bullet) = 0$.

A node on the rightmost path, possibly the root, will be called *open* if its rightmost child (the child on the rightmost path), if any, is a non-leaf moderate node. In particular, the rightmost leaf is always an open node. Let $\mathrm{open}(T)$ denote the number of open nodes in T; we define $\mathrm{open}(\bullet) = 0$.

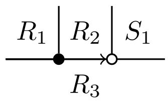
For the tree T in Figure 3 we see that $\mathrm{root}(T) = 4$, $\mathrm{sub}(T) = 4$, $\mathrm{rzero}(T) = 1$ and $\mathrm{rmod}(T) = \mathrm{open}(T) = 2$. That $\mathrm{rmod}(T)$ and $\mathrm{open}(T)$ agree is not a coincidence as demonstrated in the proof of the following lemma.

Lemma 3. *For any $\beta(0,1)$-tree T we have* $\mathrm{rmod}(T) = \mathrm{open}(T)$.

Proof. Since the right child of a non-leaf open node is non-leaf and moderate, and the root is not a moderate node, it follows that among non-leaves the numbers of open and moderate nodes agree. As the rightmost leaf is both open and moderate, the equality of both statistics follows. ☐

3 Bicubic maps as $\beta(0,1)$-trees

Following [3] we will now describe a bijection between bicubic maps and $\beta(0,1)$-trees. Let us first recall some definitions from the introduction. For any bicubic map M and $i = 1,2,3$, let $\mathcal{F}_i(M)$ be the set of i-colored faces of M. Let $R_1 \in \mathcal{F}_1(M)$, $R_2 \in \mathcal{F}_2(M)$, and $R_3 \in \mathcal{F}_3(M)$ be the three faces around the root vertex; in particular, R_3 is the root face. In addition, let $S_1 \in \mathcal{F}_1(M)$ be the 1-colored face that meets the vertex that the root edge points at:



Let us say that a face touches another face k times if there are k different edges each belonging to the boundaries of both faces. Define the following two statistics:

$\mathrm{b}(M)$ is the number of black vertices incident to both R_1 and R_2;
$\mathrm{s1r3}(M)$ is the number of times S_1 touches R_3.

For example, let M_1, M_2 and M_3 be the three maps in Figure 1 (in that order). Then $\mathrm{rzero}(M_1) = 2$ and $\mathrm{rzero}(M_2) = \mathrm{rzero}(M_3) = 1$. Also, $\mathrm{s1r3}(M_1) = 1$, $\mathrm{s1r3}(M_2) = 2$ and $\mathrm{s1r3}(M_3) = 1$.

We say that M is *irreducible* if $\mathrm{s1r3}(M) = 1$, or, in other words, if S_1 touches R_3 exactly once; we say that M is *reducible* otherwise. We shall introduce operations on bicubic maps that correspond to λ_i and $\oplus$ of $\beta(0,1)$-trees. This will induce the desired bijection ψ between bicubic maps and $\beta(0,1)$-trees.

To construct an irreducible bicubic map based on M, and having two more vertices than M, we proceed in one of two ways. The first way (1) corresponds to $\lambda_i(T)$ when $i = \mathrm{root}(T)$; the second way (2) corresponds to $\lambda_i(T)$ when $0 \leq i < \mathrm{root}(T)$.

(1) We create a new 1-colored face touching the root face exactly once, so $\mathrm{f1r3}(M') = \mathrm{f1r3}(M)+1$, by removing the root edge from M and adding two new vertices and four new edges that we connect to the map as in Figure 4.

(2) Assuming that $\mathrm{f1r3}(M) = k$, that is, M has k 1-colored faces touching the root face, we can create an irreducible map M' such that $\mathrm{f1r3}(M') = i$, where $1 \leq i \leq k$. To this end, we remove the root edge from M. Starting at the root node and counting in *clockwise direction*, we also remove the first edge of the i-th 1-colored face that touches the root face. In Figure 5 we schematically illustrate the case $i = 3$. Next we add two more vertices and respective edges, and assign a new root as shown in the figure.

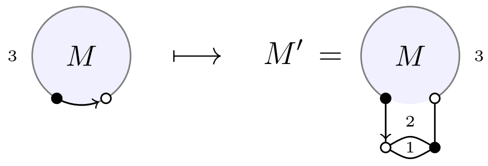


Figure 4: Constructing an irreducible map (Case 1).

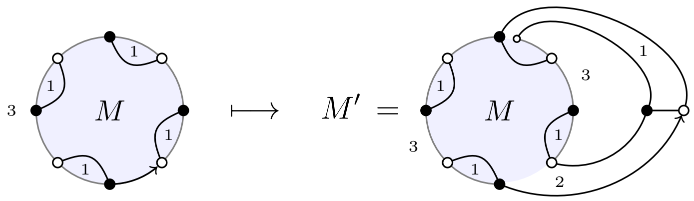


Figure 5: Constructing an irreducible map (Case 2).

Any irreducible bicubic map on $n + 2$ vertices can be constructed from some bicubic map on n vertices by applying operation (1) or (2) above.

We shall now describe how to create a reducible map based on irreducible maps $M_1, M_2, \ldots, M_k$. An illustration for $k = 3$ can be found in Figure 6. This corresponds to the $\oplus$-operation on $\beta(0, 1)$-trees.

(3) We begin by lining up the maps $M_1, M_2, \ldots, M_k$. Next, in each map M_i, we remove the first edge (in *counter-clockwise direction*) from the root edge on the root face. Then we connect the maps as shown in the figure, and define the root edge of the obtained map to be the root edge of M_k.

Any reducible bicubic map on n vertices can be constructed by applying the above operation (3) to some ordered list of irreducible bicubic maps whose total number of vertices is n.

By defining operations on bicubic maps corresponding to the operations λ_i and $\oplus$ we have now completed the definition of the bijection ψ between bicubic maps and $\beta(0, 1)$-trees. Two examples of applying ψ can be found in the appendix.

Proposition 4. *Let M be a bicubic map, and let* $\mathrm{one}(M) = |\mathcal{F}_1(M)|$ *be the number of* 1*-colored faces in M. Let T be a $\beta(0, 1)$-tree, and let* $\mathrm{exc}(T)$ *denote the number of excessive nodes in T. Let ψ be the map from bicubic maps to $\beta(0, 1)$-trees described*

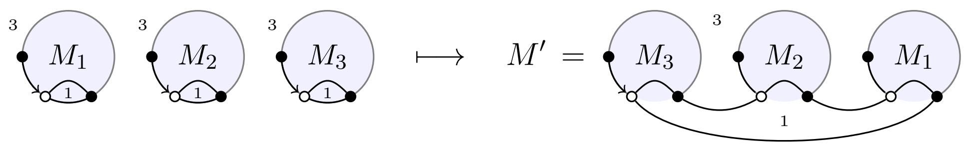
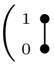
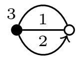


Figure 6: Constructing a reducible map.

above. Finally, assume that $T = \psi(M)$. Then

$$\begin{aligned} \mathrm{exc}(T) &= \mathrm{one}(M); \\ \mathrm{root}(T) &= \mathrm{f1r3}(M); \\ \mathrm{rmod}(T) &= \mathrm{f3r2}(M); \\ \mathrm{rzero}(T) &= \mathrm{b}(M); \\ \mathrm{sub}(T) &= \mathrm{s1r3}(M). \end{aligned}$$

Proof. The proofs of these five equalities are similar, and we will only detail the proof of $\mathrm{rzero}(T) = \mathrm{b}(M)$; the proofs of the other equalities are simpler. Clearly,

rzero () = b () = 1.

Let M' be a bicubic map with at least 4 vertices. Then M' can be constructed from one (M) or more ($M_1, \dots, M_k$) smaller bicubic maps as per the three rules above.

(1) Assume that T and T' are the trees corresponding to M and M', respectively. Then $T' = \lambda_i(T)$ with $i = \mathrm{root}(T)$. The labels on the rightmost path of T are preserved in T', and a new nonzero (root) node is added. Thus $\mathrm{rzero}(T') = \mathrm{rzero}(T)$. We need to show that $\mathrm{rzero}(M') = \mathrm{rzero}(M)$, but this easy to see from the picture above: the only black vertex added is not incident to R_1, and the status (incident or not incident to R_1 and R_2) of each of the other black vertices incident to both R_1 and R_2 is preserved.

(2) Here $T' = \lambda_i(T)$ with $0 \leq i < \mathrm{root}(T)$, and we distinguish two sub-cases.

   (a) Assume that $i = 0$. Comparing T to T' we see that one more zero appears on the rightmost path of T', namely the new root. Thus $\mathrm{rzero}(T') = \mathrm{rzero}(T) + 1$. On the map M' we have just one 1-colored face touching R_3 and this face must be R_1. Comparing M to M' we see that the black vertex added to M in order to form M' is incident to both R_1 and R_2. The status of each of the other black vertices is preserved. Thus $\mathrm{b}(M') = \mathrm{b}(M) + 1$.

   (b) Assume that $i > 0$. Clearly, $\mathrm{rzero}(T') = \mathrm{rzero}(T)$. The black vertex added to M in order to form M' is not incident with R_1, and the status of each of the other black vertices is preserved. Thus $\mathrm{b}(M') = \mathrm{b}(M)$.

(3) Assume that $T_1, \dots, T_k$ and T' are the trees corresponding to $M_1, \dots, M_k$ and M', respectively. Clearly, $\mathrm{rzero}(T') = \mathrm{rzero}(T_k)$. Consider M': no black vertex in $M_1, \dots, M_{k-1}$ can contribute to the b-statistic because such a vertex is neither incident to R_1 nor incident to R_2. Since the status of each of the black vertices in M_k is preserved it follows that $\mathrm{b}(M') = \mathrm{b}(M_k)$.

The result now follows by induction. □

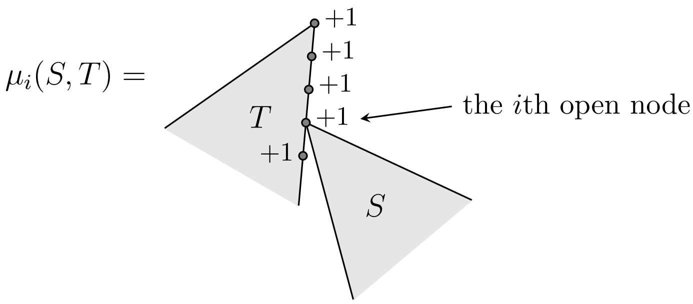


Figure 7: A schematic picture of μ_i.

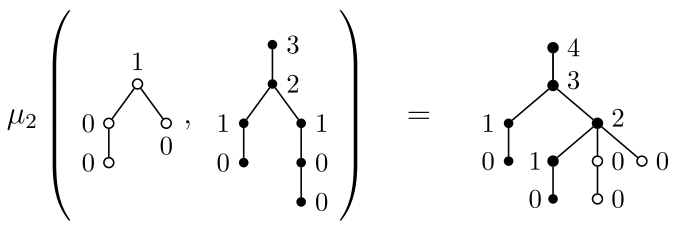


Figure 8: An example using μ_i.

4 New ways to decompose $\beta(0,1)$-trees

For any $\beta(0,1)$-trees T_1, T_2, $\ldots$, T_k define

$$\rho(T_1, T_2, \ldots, T_k) = \lambda_0(T_1) \oplus \lambda_0(T_2) \oplus \cdots \oplus \lambda_0(T_k).$$

Let S and T be $\beta(0,1)$-trees. Assume that $\mathrm{root}(S) = 1$ and that T is nontrivial. Let i be an integer such that $1 \leq i \leq \mathrm{open}(T)$, and let x denote the ith open node on the rightmost path of T. Also, let y be x if x is a leaf and let y be the rightmost child of x otherwise. We define $\mu_i(S,T)$ as the $\beta(0,1)$-tree obtained by identifying x with the root of S, keeping the label of x, and then adding one to each node on the rightmost path of T between the root and y. A schematic illustration can be found in Figure 7, and a specific example can be found in Figure 8. For convenience we also define that $\mu_1(S, \bullet) = S$.

Note that any $\beta(0,1)$-tree U with $\mathrm{root}(U) = 1$ is of the form $\rho(T_1, T_2, \ldots, T_k)$ for some $\beta(0,1)$-trees T_1, T_2, $\ldots$, T_k. On the other hand, any $\beta(0,1)$-tree U with $\mathrm{root}(U) > 1$ can be written $U = \mu_i(S,T)$, where $\mathrm{root}(S) = 1$ and T is nontrivial. Indeed, the node we call x above is the parent node of the first node labelled 0 on the rightmost path of U, and knowing x we trivially get S and T.

Thus we can completely decompose any $\beta(0,1)$-tree in terms of ρ and μ_i. As an example, the tree from Figure 3 can be written

$$\mu_2(\rho[\bullet], \mu_1(\rho[\mu_2(\rho[\bullet], \mu_1(\rho[\rho[\bullet]], \rho[\bullet]))], \mu_1(\rho[\bullet], \rho[\bullet, \bullet, \rho[\bullet]]))).$$

We shall now define two additional operations σ and ν_i on $\beta(0,1)$-trees that in a sense are dual to ρ and μ_i. We start with σ (see Figure 9 for an example):

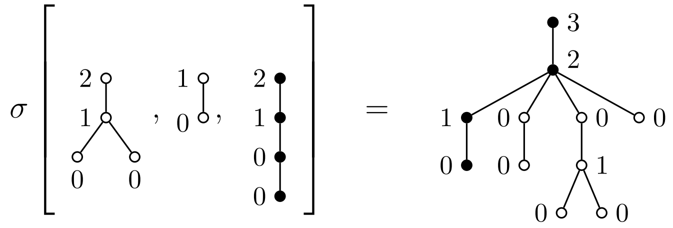


Figure 9: An example using σ.

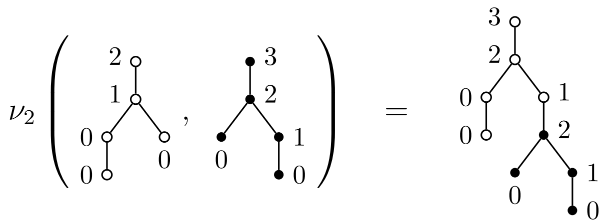


Figure 10: An example using ν_i.

Definition 1. For $\beta(0,1)$-trees $T_1, \ldots, T_k$ define

$$\sigma(T_1, \ldots, T_k) = \mu_1(\rho(T_{k-1}, \ldots, T_1, \bullet), T_k).$$

Let S and T be $\beta(0,1)$-trees. Assume that $\mathrm{open}(S) = 1$ and that T is nontrivial. Let i be an integer such that $1 \leq i \leq \mathrm{root}(T)$ and let x denote the rightmost leaf of S. Define $\nu_i(S,T)$ as the $\beta(0,1)$-tree obtained by identifying x with the root of T, keeping the (zero) label of x, and then adding $i-1$ to each node on the rightmost path of S between the root and x. See Figure 10 for an example. For convenience we shall also define that $\nu_1(S, \bullet) = S$.

Note that any $\beta(0,1)$-tree U with $\mathrm{open}(U) = 1$ is of the form $\sigma(T_1, T_2, \ldots, T_k)$ for some $\beta(0,1)$-trees $T_1, T_2, \ldots, T_k$, and any $\beta(0,1)$-tree U with $\mathrm{open}(U) > 1$ can be written $U = \nu_i(S,T)$, where $\mathrm{open}(S) = 1$ and T is nontrivial. Again, using the tree from Figure 3 as an example we have

$$\nu_2(\sigma[\sigma[\nu_1(\sigma[\bullet, \bullet, \bullet], \sigma[\bullet])]], \sigma[\nu_2(\sigma[\sigma[\bullet]], \sigma[\sigma[\bullet]])]).$$

The behaviour of the statistics root and open under ρ, μ_i, σ and ν_i follows easily from the definitions.

Lemma 5. *If $T_1, \ldots, T_k$, S and T are $\beta(0,1)$-trees, then*

$$\mathrm{open}(\rho(T_1, \ldots, T_k)) = 1 + \mathrm{open}(T_k), \tag{1}$$
$$\mathrm{root}(\mu_i(S,T)) = 1 + \mathrm{root}(T), \tag{2}$$
$$\mathrm{open}(\mu_i(S,T)) = i - 1 + \mathrm{open}(S), \tag{3}$$
$$\mathrm{root}(\sigma(T_1, \ldots, T_k)) = 1 + \mathrm{root}(T_k), \tag{4}$$
$$\mathrm{root}(\nu_i(S,T)) = i - 1 + \mathrm{root}(S), \tag{5}$$
$$\mathrm{open}(\nu_i(S,T)) = 1 + \mathrm{open}(T), \tag{6}$$

where in (2) *and* (3) *we assume that* $\mathrm{root}(S) = 1$*, and in* (5) *and* (6) *we assume that* $\mathrm{open}(S) = 1$.

We can now give a generating function proof of Theorem 2.

Proof of Theorem 2. Let $F(x, y) := F(t, x, y)$ be the generating function for $\beta(0, 1)$-trees where t marks the number of edges, x marks the root statistic, and y marks the rmod statistic. We claim that

$$F(x, y) = 1 + xS + \frac{x}{y-1} S\big(F(x, y) - F(x, 1)\big),$$

where $S := tyF(1, y)/(1 - tF(1, 1))$. Let us prove first that xS is the generating function for $\beta(0, 1)$-trees with root label equal to 1. Indeed, such a tree is of the form $\rho(T_1, \ldots, T_k)$ for some $k \geq 1$. By Lemma 5, the behaviour of the rmod statistic under ρ is known (recall rmod agrees with open), and it follows that

$$[x]F(x, y) = \sum_{k \geq 1} tyF(1, y)\,(tF(1, 1))^{k-1} = S,$$

as claimed. As for the last term, it corresponds to $\beta(0, 1)$-trees with root label greater than 1. They are of the form $\mu_i(S_0, T)$, for some $\beta(0, 1)$-trees S_0, T, and some integer i with $1 \leq i \leq \mathrm{open}(T)$ and $\mathrm{open}(S_0) = 1$. The behaviour of the root and rmod statistics gives

$$\begin{aligned} F(x, y) - (1 + [x]F(x, y)) &= \sum_T t^{\#\mathrm{edges}(T)} \sum_{i=1}^{\mathrm{open}(T)} x^{\mathrm{root}(T)+1} y^{i-1} S \\ &= \sum_T t^{\#\mathrm{edges}(T)} x^{\mathrm{root}(T)+1} S\, \frac{y^{\mathrm{open}(T)} - 1}{y - 1}, \end{aligned}$$

which easily sums to the claimed expression.

Let now $G(x, y) := G(t, x, y)$ be the generating function for $\beta(0, 1)$-trees where t marks the number of edges, x marks the rmod statistic, and y marks the root statistic. This time using the (σ, ν_i) decomposition we have

$$G(x, y) = 1 + xT + \frac{x}{y-1} T\big(G(x, y) - G(x, 1)\big),$$

where $T := tyG(1, y)/(1 - tG(1, 1))$. The proof is analogous to the one in the paragraph above; in this case the second and third summands correspond to $\beta(0, 1)$-trees with the rmod statistic equal to or greater than 1, respectively.

Since $F(x, y)$ and $G(x, y)$ satisfy the same equation with the same initial conditions $F(1, 1) = G(1, 1)$ being the generating function for $\beta(0, 1)$-trees, we must have $F(x, y) = G(x, y)$. On the other hand, by definition $F(x, y) = G(y, x)$. Thus, $F(x, y) = F(y, x)$ which proves Theorem 2 via the respective statistics on bicubic maps and $\beta(0, 1)$-trees. $\square$

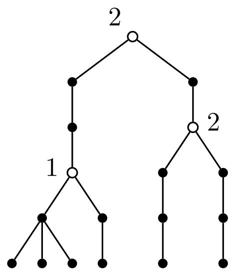


Figure 11: A bicolored tree.

5 Bicolored trees

If we look at the parse tree of an expression of a $\beta(0,1)$-tree in terms of σ and ν_i (or ρ and λ_i) we arrive at a new tree. For instance, writing the tree from Figure 3 in terms of σ and ν_i, as above, we arrive at the tree in Figure 11, where an internal black node corresponds to σ and a white node labeled i corresponds to ν_i.

Let $\mathcal{T}$ denote the set of trees that can be obtained from $\beta(0,1)$-trees in this manner. Then it is not hard to see that $\mathcal{T}$ has the following recursive characterization. A member of $\mathcal{T}$ is a rooted plane tree on white and black nodes such that either the root is black and is connected to a possibly empty list of trees in $\mathcal{T}$, or the root is white, has a label i, is connected to exactly two trees $T_1, T_2 \in T$, and $1 \leq i \leq \kappa(T_2)$, where κ is defined by recursion: κ of a tree consisting of a single leaf is 0; κ of a tree with black root connected to $T_1, \ldots, T_k$ is $1 + \kappa(T_k)$; and κ of a tree with white root labeled i, connected to T_1 and T_2, is $i - 1 + \kappa(T_1)$. If, in addition, we define the *weight* of a tree in $\mathcal{T}$ to be the number of black nodes minus the number of white nodes, then we have established that there is a one-to-one correspondence between $\beta(0,1)$-trees on n nodes and trees in $\mathcal{T}$ of weight n.

In the next section we shall define an endofunction on $\beta(0,1)$-trees. One way to understand this endofunction is that we map a $\beta(0,1)$-tree T to a $\beta(0,1)$-tree T' if the (σ, ν_i) parse tree of T is the same as the (ρ, μ_i) parse tree of T'. We will prove that this endofunction is an involution.

6 An involution on $\beta(0,1)$-trees

The following three lemmas are immediate from the definitions of ρ, μ_i, σ and ν_i; they will be used in the proof of Lemma 9.

Lemma 6. *For all $\beta(0,1)$-trees $T_1, \ldots, T_k$ we have*

$$\rho(T_1, \ldots, T_k) = \nu_1(\sigma(T_{k-1}, \ldots, T_1, \bullet), T_k).$$

Note the similarity between Lemma 6 and Definition 1.

Lemma 7. *Let R, S and T be $\beta(0,1)$-trees. If* $\mathrm{open}(R) = \mathrm{root}(S) = 1$*, and T is nontrivial, then, for integers $i \geq 1$ and $j \geq 1$, we have*

$$\nu_{i+1}(R, \mu_j(S, T)) = \mu_{j+1}(S, \nu_i(R, T)).$$

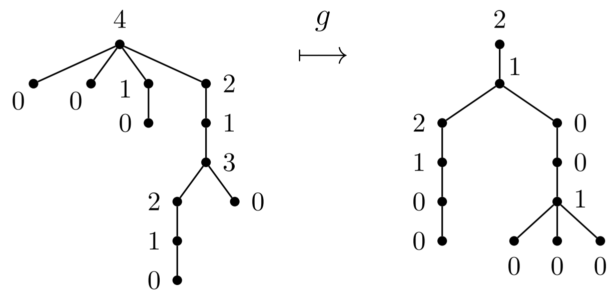


Figure 12: Applying the involution g.

Lemma 8. *Let R, S and T be $\beta(0,1)$-trees. If* $\mathrm{root}(R) = \mathrm{open}(R) = 1$*, then*

$$\mu_1(\nu_1(R,S),T) = \nu_1(\mu_1(R,T),S).$$

Definition 2. Let $T_1, \ldots, T_k$, S and T be $\beta(0,1)$-trees, and assume $\mathrm{root}(S) = 1$. Define the map g on $\beta(0,1)$-trees of size n by

1. $g(\bullet) = \bullet$;
2. $g(\rho(T_1,\ldots,T_k)) = \sigma(g(T_1),\ldots,g(T_k))$;
3. $g(\mu_i(S,T)) = \nu_i(g(S),g(T))$.

Note that there is a subtlety in this definition. In case (3), we apply ν_i to $g(S)$, so we need to make sure that $\mathrm{open}(g(S)) = 1$. But we are fine because, as $\mathrm{root}(S) = 1$ then S is $\rho(T_1,\ldots,T_k)$, so to compute $g(S)$ we would use case (2) and the image under σ of any sequence of trees has just one open node. Figure 12 gives an example of applying g. For a larger example see the appendix, where two $\beta(0,1)$-trees (and associated bicubic maps) corresponding to each other under g are given.

Lemma 9. *If $T_1, \ldots, T_k$, S and T are $\beta(0,1)$-trees, and* $\mathrm{open}(S) = 1$*, then*

1. $g(\sigma(T_1,\ldots,T_k)) = \rho(g(T_1),\ldots,g(T_k))$;

2. $g(\nu_i(S,T)) = \mu_i(g(S),g(T))$.

Proof. We have

$$\begin{aligned} g(\sigma(T_1,\ldots,T_k)) &= g(\mu_1(\rho(T_{k-1},\ldots,T_1,\bullet),T_k)) && \text{by Definition 1}\\ &= \nu_1(g(\rho(T_{k-1},\ldots,T_1,\bullet)),g(T_k)) && \text{by Definition 2}\\ &= \nu_1(\sigma(g(T_{k-1}),\ldots,g(T_1),\bullet)),g(T_k)) && \text{by Definition 2}\\ &= \rho(g(T_1),\ldots,g(T_k)) && \text{by Lemma 6} \end{aligned}$$

which proves (1). To prove (2) we first note that $\mathrm{root}(\nu_i(S,T)) = 1$ if, and only if, $\mathrm{root}(S) = 1$ and $i = 1$. Accordingly, the proof of (2) will be split into three cases:

(a) $i = 1$ and $\mathrm{root}(S) = 1$;

(b) $i = 1$ and $\text{root}(S) > 1$;

(c) $i > 1$.

Case (a): By assumption, $\text{open}(S) = 1$; if also $\text{root}(S) = 1$, then S must be of the form $S = \sigma(S_1, \ldots, S_{\ell-1}, \bullet)$ for some $\beta(0,1)$-trees $S_1, \ldots, S_{\ell-1}$, and thus

$$\begin{aligned} \nu_1(S,T) &= \nu_1(\sigma(S_1, \ldots, S_{\ell-1}, \bullet), T) \\ &= \rho(S_{\ell-1}, \ldots, S_1, T) && \text{by Lemma 6.} \end{aligned}$$

Therefore,

$$\begin{aligned} g(\nu_1(S,T)) &= \sigma(g(S_{\ell-1}), \ldots, g(S_1), g(T)) && \text{by Definition 2} \\ &= \mu_1(\rho(g(S_1), \ldots, g(S_{\ell-1}), \bullet), g(T)) && \text{by Definition 1} \\ &= \mu_1(g(\sigma(S_1, \ldots, S_{\ell-1}, \bullet)), g(T)) && \text{by (1)} \\ &= \mu_1(g(S), g(T)). \end{aligned}$$

Case (b): Since $\text{root}(S) > 1$ there are $\beta(0,1)$-trees U and V, and an integer j, such that $\text{root}(U) = 1$, V is nontrivial, and $S = \mu_j(U, V)$. By assumption $\text{open}(S) = 1$. Moreover, item (3) from Lemma 5 implies that $\text{open}(U) = 1$ and $j = 1$; thus we can use Lemma 8. The proof now proceeds by structural induction (the base case is trivial):

$$\begin{aligned} g(\nu_1(S,T)) &= g(\nu_1(\mu_1(U,V),T)) \\ &= g(\mu_1(\nu_1(U,T),V)) && \text{by Lemma 8} \\ &= \nu_1(g(\nu_1(U,T)), g(V)) && \text{by Definition 2} \\ &= \nu_1(\mu_1(g(U), g(T)), g(V)) && \text{by induction.} \end{aligned}$$

Observe now that $\text{root}(U) = \text{open}(U) = 1$ implies that U can be written as $\rho(T_1, \ldots, T_{k-1}, \bullet)$, and hence $g(U) = \sigma(g(T_1), \ldots, g(T_{k-1}), \bullet)$. Then $\text{open}(g(U)) = \text{root}(g(U)) = 1$ and we can apply Lemma 8 to the last expression.

$$\begin{aligned} g(\nu_1(S,T)) &= \mu_1(\nu_1(g(U), g(V)), g(T)) && \text{by Lemma 8} \\ &= \mu_1(g(\mu_1(U,V)), g(T)) && \text{by Definition 2} \\ &= \mu_1(g(S), g(T)). \end{aligned}$$

Case (c): If $i > 1$, then $\text{root}(T) > 1$ and we can write $T = \mu_j(U, V)$ for some $\beta(0,1)$-trees U and V with $\text{root}(U) = 1$ and V nontrivial. We can now proceed by either using structural induction or induction on i, the base case $i = 1$ being provided by cases (a) and (b) above:

$$\begin{aligned} g(\nu_i(S,T)) &= g(\nu_i(S, \mu_j(U,V))) \\ &= g(\mu_{j+1}(U, \nu_{i-1}(S,V))) && \text{by Lemma 7} \\ &= \nu_{j+1}(g(U), g(\nu_{i-1}(S,V))) && \text{by Definition 2} \\ &= \nu_{j+1}(g(U), \mu_{i-1}(g(S), g(V))) && \text{by induction} \\ &= \mu_i(g(S), \nu_j(g(U), g(V))) && \text{by Lemma 7} \\ &= \mu_i(g(S), g(\mu_j(U,V))) && \text{by Definition 2} \\ &= \mu_i(g(S), g(T)) \end{aligned}$$

which concludes the proof. Notice that in the second application of Lemma 7 we need again the fact that if $\mathrm{root}(U) = 1$ then $\mathrm{open}(g(U)) = 1$. Also, it is necessary that $\mathrm{root}(g(S)) = 1$; this follows from part (1) because $\mathrm{open}(S) = 1$ allows us to write $S = \sigma(T_1, \ldots, T_k)$. □

Theorem 10. *The map g is an involution.*

Proof. We use induction on size. The base case $g^2(\bullet) = \bullet$ is trivial. For the induction step we have

$$\begin{aligned} g^2(\rho(T_1, \ldots, T_k)) &= g(\sigma(g(T_1), \ldots, g(T_k))) && \text{by Definition 2} \\ &= \rho(g^2(T_1), \ldots, g^2(T_k)) && \text{by Lemma 9} \\ &= \rho(T_1, \ldots, T_k) && \text{by induction} \end{aligned}$$

and

$$\begin{aligned} g^2(\mu_i(S, T)) &= g(\nu_i(g(S), g(T))) && \text{by Definition 2} \\ &= \mu_i(g^2(S), g^2(T)) && \text{by Lemma 9} \\ &= \mu_i(S, T) && \text{by induction} \end{aligned}$$

which concludes the proof. □

Theorem 11. *On $\beta(0,1)$-trees with n nodes, the pair of statistics* $(\mathrm{root}, \mathrm{open})$ *has the same joint distribution as the pair* $(\mathrm{open}, \mathrm{root})$. *Equivalently,*

$$\sum_T x^{\mathrm{root}(T)} y^{\mathrm{open}(T)} = \sum_T x^{\mathrm{open}(T)} y^{\mathrm{root}(T)},$$

where the sum is over all $\beta(0,1)$-trees with n nodes.

Proof. Using induction we shall now prove that $\mathrm{root}(g(U)) = \mathrm{open}(U)$ for each $\beta(0,1)$-tree U. The base case is plain. For the induction step, assume that $T_1, \ldots, T_k$, S and T are $\beta(0,1)$-trees, $\mathrm{root}(S) = 1$, and that T is nontrivial. We have

$$\begin{aligned} \mathrm{root}(g(\rho(T_1, \ldots, T_k))) &= \mathrm{root}(\sigma(g(T_1), \ldots, g(T_k))) && \text{by Definition 2} \\ &= 1 + \mathrm{root}(g(T_k)) && \text{by (4) from Lemma 5} \\ &= 1 + \mathrm{open}(T_k) && \text{by induction} \\ &= \mathrm{open}(\rho(T_1, \ldots, T_k)) && \text{by (1) from Lemma 5.} \end{aligned}$$

Also,

$$\begin{aligned} \mathrm{root}(g(\mu_i(S, T))) &= \mathrm{root}(\nu_i(g(S), g(T))) && \text{by Definition 2} \\ &= i - 1 + \mathrm{root}(g(S)) && \text{by (5) from Lemma 5} \\ &= i - 1 + \mathrm{open}(S) && \text{by induction} \\ &= \mathrm{open}(\mu_i(S, T)) && \text{by (3) from Lemma 5.} \end{aligned}$$

Since g is an involution it follows that $\mathrm{open}(g(T)) = \mathrm{root}(T)$ as well, which concludes the proof. □

Corollary 12. *On $\beta(0,1)$-trees with n nodes, the pair of statistics* $(\mathrm{root}, \mathrm{rmod})$ *has the same joint distribution as the pair* $(\mathrm{rmod}, \mathrm{root})$. *Equivalently,*

$$\sum_T x^{\mathrm{root}(T)} y^{\mathrm{rmod}(T)} = \sum_T x^{\mathrm{rmod}(T)} y^{\mathrm{root}(T)},$$

where both sums are over all $\beta(0,1)$-trees with n nodes.

Proof. This is a direct consequence of Lemma 3 and Theorem 11. □

Our second proof of Theorem 2 now follows from Corollary 12 through the correspondence between bicubic maps and $\beta(0,1)$-trees.

Definition 3. Let $C_n = \binom{2n}{n}/(n+1)$ denote the nth Catalan number. Define

$$a(n) = 2^{n-1}C_n.$$

This is sequence A003645 in OEIS [8].

By computing the number of trees fixed by g, for $n \leq 12$, we arrive at the following conjecture.

Conjecture 13. *For $n > 1$, the number of $\beta(0,1)$-trees on n nodes fixed under g is $a(\lfloor n/2 \rfloor)$. This sequence starts* $1, 1, 4, 4, 20, 20, 112, 112, 672, 672, 4224, 4224, \ldots$

The number of fixed points under the involution h on $\beta(1,0)$-trees (introduced in [1, 2]) was found in [5]. These numbers also count *self-dual rooted non-separable planar maps* [6]. However, we were not able to exploit the ideas to count fixed points under h in order to prove Conjecture 13, because the involution g is more complex, and in general, $\beta(0,1)$-trees are more complex than $\beta(1,0)$-trees.

Proposition 14 (Tutte, Koganov, Liskovets and Walsh)**.** *The number of bicubic maps on $2n$ vertices with one distinguished 1-colored face is $a(n)$.*

Proof. Koganov, Liskovets and Walsh [7, Proposition 3.1] showed that the number of rooted eulerian planar maps with n edges and a distinguished vertex is given by the formula $a(n)$. Tutte's well-known "trinity mapping" sends eulerian planar maps with n edges to bicubic maps with $2n$ vertices. It is easy to see that under the same mapping vertices are sent to 1-colored faces. □

Proposition 15. *The number of $\beta(0,1)$-trees on $n+1$ nodes with one distinguished excessive node is $a(n)$.*

Proof. This is a direct consequence of Propositions 4 and 14. □

In light of this last proposition we can reformulate Conjecture 13 as follows.

Conjecture 16. *There is a bijection between $\beta(0,1)$-trees on n nodes fixed under g and $\beta(0,1)$-trees on $\lfloor n/2 \rfloor + 1$ nodes with one distinguished excessive node.*

We close this paper by making an additional conjecture.

Conjecture 17. *The two pairs of statistics* $(\mathrm{root}, \mathrm{rzero})$ *and* $(\mathrm{rmod}, \mathrm{sub})$ *are jointly equidistributed on $\beta(0,1)$-trees.*

We have verified Conjecture 17 for $\beta(0,1)$-trees on at most 11 nodes. This conjecture will imply, via the bijection described in Section 3, that the two pairs of statistics $(\mathrm{f1r3}, \mathrm{b})$ and $(\mathrm{f3r2}, \mathrm{s1r3})$ are jointly equidistributed on bicubic maps.

Acknowledgments

We wish to thank the anonymous referee (of an earlier version of this paper) who came up with the generating function based proof of Theorem 2.

Appendix

In Figures 13 and 14 we give examples of the mapping ψ from bicubic maps to $\beta(0,1)$-trees. The image of each large map at the top is the tree below it, and for each smaller map, its image is the subtree consisting of the edge next to it and all the edges below, with the root label adjusted if necessary.

Also, the two trees are the image of each other under the involution g. For the tree (T) and map (M) in Figure 13 we have $\mathrm{exc}(T) = \mathrm{one}(M) = 6$, $\mathrm{root}(T) = \mathrm{f1r3}(M) = 4$, $\mathrm{rmod}(T) = \mathrm{f3r2}(M) = 2$, $\mathrm{rzero}(T) = \mathrm{b}(M) = 1$, and $\mathrm{sub}(T) = \mathrm{s1r3}(M) = 4$. For the tree ($T$) and map ($M$) in Figure 14 we have $\mathrm{exc}(T) = \mathrm{one}(M) = 6$, $\mathrm{root}(T) = \mathrm{f1r3}(M) = 4$, $\mathrm{rmod}(T) = \mathrm{f3r2}(M) = 2$, $\mathrm{rzero}(T) = \mathrm{b}(M) = 3$, and $\mathrm{sub}(T) = \mathrm{s1r3}(M) = 1$.

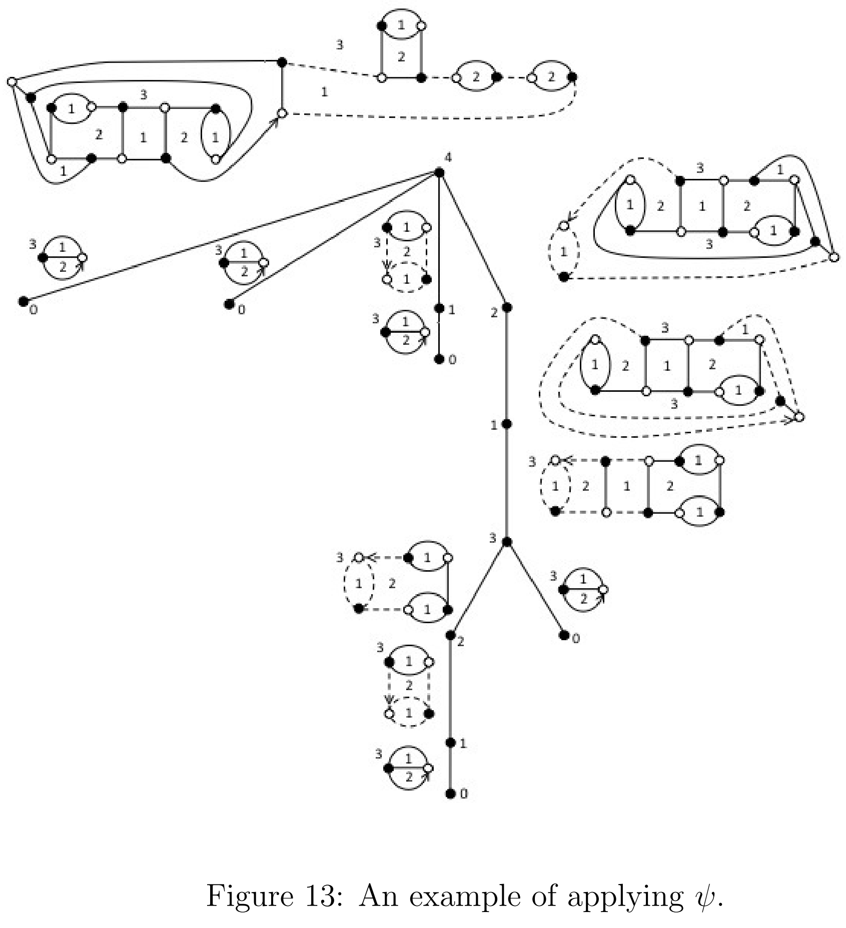


Figure 13: An example of applying ψ.

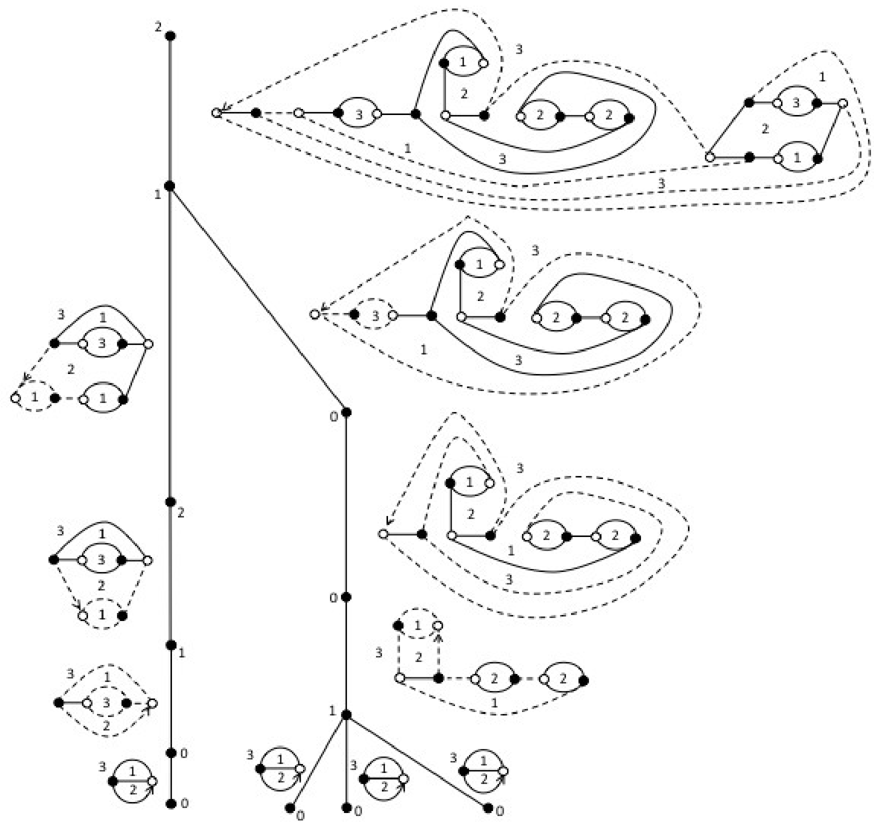


Figure 14: An example of applying ψ.

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