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Elementary proofs of congruences for the cubic and overcubic partition functions

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Abstract

In 2010, Hei-Chi Chan introduced the cubic partition function $a(n)$ in connection with Ramanujan's cubic continued fraction. Chan proved that

$$\sum_{n\geq 0} a(3n+2)q^n \quad = \quad 3\prod_{i\geq 1}\frac{(1-q^{3n})^3(1-q^{6n})^3}{(1-q^n)^4(1-q^{2n})^4}$$

which clearly implies that, for all $n \geq 0$, $a(3n+2) \equiv 0 \pmod 3$.

In the same year, Byungchan Kim introduced the overcubic partition function $\overline{a}(n)$. Using modular forms, Kim proved that

$$\sum_{n\geq 0} \overline{a}(3n+2)q^n \quad = \quad 6\prod_{i\geq 1}\frac{(1-q^{3n})^6(1-q^{4n})^3}{(1-q^n)^8(1-q^{2n})^3}.$$

More recently, Hirschhorn has proven Kim's generating function result above using elementary generating function methods. Clearly, this generating function result implies that $\overline{a}(3n+2) \equiv 0 \pmod 6$ for all $n \geq 0$.

In this note, we use elementary means to prove functional equations satisfied by the generating functions for $a(n)$ and $\overline{a}(n)$, respectively. These lead to new representations of these generating functions as products of terms involving Ramanujan's ψ and φ functions. In the process, we are able to prove the congruences mentioned above as well as numerous arithmetic properties satisfied by $\overline{a}(n)$ modulo small powers of 2.

1 Introduction

In 2010, Hei-Chi Chan [2, 3] introduced the cubic partition function $a(n)$ in connection with Ramanujan's cubic continued fraction. The generating function for $a(n)$ is given by

$$A(q) := \sum_{n\geq 0} a(n)q^n = \frac{1}{(q;q)_\infty(q^2;q^2)_\infty} = \frac{1}{(q;q^2)_\infty(q^2;q^2)_\infty^2} \tag{1}$$

where we have used the notation that

$$(a;q)_\infty = \prod_{i\geq 1}(1-aq^{i-1}).$$

Note that $a(n)$ counts the number of partitions of weight n such that the even parts can appear in two colors. So, for example, $a(3)=4$ where the colored partitions in question are

$$3, \quad 2_1+1, \quad 2_2+1, \quad \text{and} \quad 1+1+1.$$

(The subscripts 1 and 2 denote the "colors" in question.)

Among other results, Chan [2] proved that

$$\sum_{n\geq 0} a(3n+2)q^n = 3\frac{(q^3;q^3)_\infty^3(q^6;q^6)_\infty^3}{(q;q)_\infty^4(q^2;q^2)_\infty^4}.$$

This generating function result implies the following Ramanujan–like congruence:

Theorem 1.1 *For all $n \geq 0$, $a(3n+2) \equiv 0 \pmod 3$.*

Chan [3] went on to prove an infinite family of congruences satisfied by $a(n)$ modulo powers of 3.

In the same year, Byungchan Kim [5] introduced the overcubic partition function $\overline{a}(n)$ whose generating function is given by

$$\overline{A}(q) := \sum_{n\geq 0} \overline{a}(n)q^n = \frac{(-q;q)_\infty(-q^2;q^2)_\infty}{(q;q)_\infty(q^2;q^2)_\infty} = \frac{(q^4;q^4)_\infty}{(q;q)_\infty^2(q^2;q^2)_\infty}. \tag{2}$$

From this generating function we see that $\overline{a}(n)$ counts all of the overlined versions of the cubic partitions counted by $a(n)$. In this case, the first instance of each part is allowed to be overlined (although such overlining is not required). So, for example, $\overline{a}(3) = 12$. Based on the four cubic partitions of 3 mentioned above, we see that the corresponding 12 overcubic partitions are given by the following:

$$3, \quad \overline{3}, \quad 2_1+1, \quad \overline{2_1}+1, \quad 2_1+\overline{1}, \quad \overline{2_1}+\overline{1},$$
$$2_2+1, \quad \overline{2_2}+1, \quad 2_2+\overline{1}, \quad \overline{2_2}+\overline{1}, \quad 1+1+1, \quad \overline{1}+1+1$$

Using modular forms, Kim [5] proved that

$$\sum_{n\geq 0} \overline{a}(3n+2)q^n = 6\frac{(q^3;q^3)_\infty^6(q^4;q^4)_\infty^3}{(q;q)_\infty^8(q^2;q^2)_\infty^3}. \tag{3}$$

More recently, Hirschhorn [4] proved (3) using elementary generating function manipulations. Clearly, the generating function result (3) implies the following:

Theorem 1.2 *For all* $n \geq 0$, $\overline{a}(3n+2) \equiv 0 \pmod{6}$.

In this note, we use elementary means to prove functional equations satisfied by $A(q)$ and $\overline{A}(q)$ which lead to new representations of these generating functions as product of terms involving Ramanujan's ψ and φ functions. In the process, we provide truly elementary proofs of Theorems 1.1 and 1.2 with ease. We also prove numerous arithmetic properties satisfied by $\overline{a}(n)$ modulo small powers of 2 which are new.

In order to complete the proofs in the next section, we require a few elementary tools. First, we recall Ramanujan's ψ and φ functions which are defined as

$$\psi(q) := \sum_{n \geq 0} q^{n(n+1)/2} \tag{4}$$

and

$$\varphi(q) := 1 + 2\sum_{n \geq 1} q^{n^2}. \tag{5}$$

The presence of the multiplier 2 in the representation of $\varphi(q)$ above will allow us to prove a number of arithmetic properties modulo small powers of 2 which are satisfied by $\overline{a}(n)$. Secondly, using Jacobi's Triple Product Identity [1, Theorem 2.8], we know that

$$\psi(q) = \frac{(q^2;q^2)_\infty^2}{(q;q)_\infty} \tag{6}$$

and

$$\varphi(q) = \frac{(q^2;q^2)_\infty^5}{(q;q)_\infty^2 (q^4;q^4)_\infty^2}. \tag{7}$$

Equations (6) and (7) are crucial for our proof of the functional equations for $A(q)$ and $\overline{A}(q)$. It is these functional equations which drive all of the other proofs in this note.

2 Arithmetic Properties

We begin this section by proving the following functional equations for $A(q)$ and $\overline{A}(q)$.

Theorem 2.1 *The following are true:*

$$A(q) = \psi(q)\ \psi(q^2)\, A(q^2)^2$$

and

$$\overline{A}(q) = \varphi(q)\, \varphi(q^2)\, \overline{A}(q^2)^2.$$

Proof. First, using (1) and (6), we have

$$\begin{aligned}\psi(q)\,\psi(q^2)\,A(q^2)^2 &= \frac{(q^2;q^2)_\infty^2}{(q;q)_\infty}\cdot\frac{(q^4;q^4)_\infty^2}{(q^2;q^2)_\infty}\cdot\frac{1}{(q^2;q^2)_\infty^2(q^4;q^4)_\infty^2}\\ &= \frac{1}{(q;q)_\infty(q^2;q^2)_\infty}\\ &= A(q).\end{aligned}$$

Similarly, using (2) and (7), we see that

$$\begin{aligned}\varphi(q)\,\varphi(q^2)\,\overline{A}(q^2)^2 &= \frac{(q^2;q^2)_\infty^5}{(q;q)_\infty^2(q^4;q^4)_\infty^2}\cdot\frac{(q^4;q^4)_\infty^5}{(q^2;q^2)_\infty^2(q^8;q^8)_\infty^2}\cdot\frac{(q^8;q^8)_\infty^2}{(q^2;q^2)_\infty^4(q^4;q^4)_\infty^2}\\ &= \frac{(q^4;q^4)_\infty}{(q;q)_\infty^2(q^2;q^2)_\infty}\\ &= \overline{A}(q).\end{aligned}$$

□

Iteration of the functional equations in Theorem 2.1 ad infinitum allows us to obtain extremely valuable representations for $A(q)$ and $\overline{A}(q)$ in terms of the ψ and φ functions, respectively.

Corollary 2.2

$$A(q) = \psi(q)\prod_{i\geq 1}\psi(q^{2^i})^{3\cdot 2^{i-1}}.$$

Corollary 2.3

$$\overline{A}(q) = \varphi(q)\prod_{i\geq 1}\varphi(q^{2^i})^{3\cdot 2^{i-1}}.$$

With Corollaries 2.2 and 2.3 in hand, we can prove a number of congruence properties satisfied by $a(n)$ and $\overline{a}(n)$ with relative ease. We begin by proving a number of arithmetic properties satisfied by $\overline{a}(n)$ modulo small powers of 2.

Theorem 2.4 *For all $n \geq 1$, $\overline{a}(n) \equiv 0 \pmod 2$.*

Proof. This result follows from Corollary 2.3 and the fact that $\varphi(q) \equiv 1 \pmod 2$ thanks to (5). □

Theorem 2.5 *For all $n \geq 1$,*

$$\overline{a}(n) \equiv \begin{cases} 2 \pmod 4 & \text{if } n = k^2 \text{ or } n = 2k^2 \text{ for some integer } k\\ 0 \pmod 4 & \text{otherwise.}\end{cases}$$

Proof. Thanks to the binomial theorem, it is clear that $\varphi(q^{2^i})^{3\cdot 2^{i-1}} \equiv 1 \pmod 4$ for each $i \geq 2$. Thus,

$$\begin{aligned}
\sum_{n\geq 0} \overline{a}(n)q^n &\equiv \varphi(q)\varphi(q^2)^3 \pmod 4 \\
&= \left(1 + 2\sum_{n\geq 1} q^{n^2}\right)\left(1 + 2\sum_{n\geq 1} q^{2n^2}\right)^3 \\
&\equiv \left(1 + 2\sum_{n\geq 1} q^{n^2}\right)\left(1 + 6\sum_{n\geq 1} q^{2n^2}\right) \pmod 4 \\
&\equiv 1 + 2\sum_{n\geq 1} q^{n^2} + 2\sum_{n\geq 1} q^{2n^2} \pmod 4.
\end{aligned}$$

The result follows. □

It is clear that infinitely many Ramanujan-like congruences follow as corollaries of Theorem 2.5. We mention one such family of congruences here.

Corollary 2.6 *For all $j \geq 0$ and $n \geq 0$, $\overline{a}(2^j(4n+3)) \equiv 0 \pmod 4$.*

We can certainly prove additional results similar to Theorem 2.5 for moduli which are larger powers of 2. Unfortunately, such results become less elegant as the modulus increases. Thus, we provide only one additional result of this type.

Theorem 2.7 *For all $n \geq 1$,*

$$\overline{a}(n) \equiv \begin{cases} 2 \pmod 8 & \text{if } n = k^2 \text{ or } n = 2(2k)^2 \text{ for some integer } k \geq 1 \\ 6 \pmod 8 & \text{if } n = 2(2k-1)^2 \text{ for some integer } k \geq 1 \\ 4 \pmod 8 & \text{if } n = k^2 + 2\ell^2 \text{ for some integer } k, \ell \geq 1 \\ 0 \pmod 4 & \text{otherwise.} \end{cases}$$

Proof. Thanks to the binomial theorem, it is clear that $\varphi(q^{2^i})^{3\cdot 2^{i-1}} \equiv 1 \pmod 8$ for each $i \geq 3$. Thus, modulo 8,

$$\begin{aligned}
\sum_{n\geq 0} \overline{a}(n)q^n &\equiv \varphi(q)\varphi(q^2)^3\varphi(q^4)^6 \\
&\equiv \left(1 + 2\sum_{n\geq 1} q^{n^2}\right)\left(1 + 4\sum_{n\geq 1} q^{4n^2} + 6\sum_{n\geq 1} q^{2n^2}\right) \\
&\qquad \times \left(1 + 4\sum_{n\geq 1} q^{4n^2} + 4\sum_{n\geq 1} q^{8n^2}\right) \\
&\equiv 1 + 2\sum_{n\geq 1} q^{n^2} + 6\sum_{n\geq 1} q^{2(2n-1)^2} + 2\sum_{n\geq 1} q^{2(2n)^2} + 4\sum_{m,n\geq 1} q^{n^2+2m^2}
\end{aligned}$$

after simplification. The result follows. □

Similar to Corollary 2.6, we state a straightforward corollary to Theorem 2.7.

Corollary 2.8 *For all $j \geq 0$ and $n \geq 0$,*

$$\begin{aligned}\overline{a}(2^j(8n+5)) &\equiv 0 \pmod 8,\\ \overline{a}(2^j(8n+7)) &\equiv 0 \pmod 8.\end{aligned}$$

We now close this paper by returning to Theorems 1.1 and 1.2. Thanks to Corollaries 2.2 and 2.3, these can be easily proven in elementary fashion.

Proof. (of Theorem 1.1) From Corollary 2.2, we know

$$A(q) = \psi(q) \prod_{i \geq 1} \psi(q^{2^i})^{3 \cdot 2^{i-1}}.$$

This implies that

$$A(q) \equiv \psi(q) \prod_{i \geq 1} \psi(q^{3 \cdot 2^i})^{2^{i-1}} \pmod 3.$$

Note that

$$\prod_{i \geq 1} \psi(q^{3 \cdot 2^i})^{2^{i-1}}$$

is a function of q^3, and we are considering the behavior of $a(3n+2) \pmod 3$. Thus, as long as every coefficient of q^{3n+2} in the power series representation of $\psi(q)$ (as given in (4)) is divisible by 3, then $a(3n+2) \equiv 0 \pmod 3$ for all n. With this said, our proof is complete because no triangular number is congruent to 2 modulo 3. Hence, for each $n \geq 0$, the coefficient of q^{3n+2} in the power series representation of $\psi(q)$ is identically 0. Our result follows. □

An extremely similar proof can be used to prove Theorem 1.2.

Proof. (of Theorem 1.2) First, thanks to Theorem 2.4, we only need to prove that, for all $n \geq 0$, $\overline{a}(3n+2) \equiv 0 \pmod 3$ to complete our proof. From Corollary 2.3, we know

$$\overline{A}(q) = \varphi(q) \prod_{i \geq 1} \varphi(q^{2^i})^{3 \cdot 2^{i-1}}.$$

This implies that

$$\overline{A}(q) \equiv \varphi(q) \prod_{i \geq 1} \varphi(q^{3 \cdot 2^i})^{2^{i-1}} \pmod 3.$$

Note that

$$\prod_{i \geq 1} \varphi(q^{3 \cdot 2^i})^{2^{i-1}}$$

is a function of q^3, and we are considering the behavior of $\overline{a}(3n+2) \pmod 3$. Thus, as long as every coefficient of q^{3n+2} in the power series representation of $\varphi(q)$ (as given in (5)) is divisible by 3, then $\overline{a}(3n+2) \equiv 0 \pmod 3$ for all n. With this said, our proof is complete because no square is congruent to 2 modulo 3 (i.e., 2 is a quadratic non–residue modulo 3). Hence, for each $n \geq 0$, the coefficient of q^{3n+2} in the power series representation of $\varphi(q)$ is identically 0. Our result follows. □

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