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An introduction to true-palindromic compositions

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Abstract

A *true-palindromic composition* is an integer composition whose digit-comma-sequence is the same whether read from left to right or right to left. Generating functions and asymptotic formulas are provided for several quantities related to the set of true-palindromic compositions of a given integer.

1 Introduction

A composition of n is a sequence of positive integers, called *parts*, that sum to n. A *palindromic composition* or *palindrome* is a composition whose part-sequence is the same whether it is read from left to right or right to left. For example, $(12, 6, 12)$ is a palindromic composition of 30. It is well-known that there are $2^{\lfloor \frac{n}{2} \rfloor}$ palindromic compositions of n, and numerous other results have been obtained for palindromic compositions, many in the last decade [1, 3, 4, 5, 8, 9, 12]. However, the digits of a palindromic composition of n containing at least one non-palindromic part do not actually form a palindrome in the traditional sense. The composition $(12, 6, 12)$ is by definition a palindromic composition of 30, yet reflecting the digit-comma-sequence we get $(21, 6, 21)$ which is neither equivalent to $(12, 6, 12)$ nor a composition of 30.

In this paper we concern ourselves with compositions that better match the traditional characteristics of a palindrome. We define a *true-palindromic composition* or *true-palindrome* to be a composition whose digit-comma-sequence is the same whether read from left to right or right to left. We provide a few examples and non-examples:

Example 1.
The sequence $(12, 6, 21)$ *is a true-palindromic composition of* 39.
The sequence $(126, 621)$ *is a true-palindromic composition of* 747.
The sequence $(12, 621)$ *is not a true-palindromic composition.*
The sequence $(120, 6, 21)$ *is not a true-palindromic composition.*

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Note that a palindromic composition is not always true-palindromic and that a true-palindromic composition is not always palindromic. For example, $(12, 6, 12)$ is not true-palindromic and $(12, 6, 21)$ is not palindromic. It is also important to note that true-palindromes do not contain parts that are congruent to $0 \pmod{10}$ because the ending zeros are lost when the digits are reversed, as in the last example above.

Our general approach to true-palindromes is to fix the center part of a true-palindromic composition (if there is one) and consider the compositions formed on either side, as is done in both [4] and [12] in studying palindromic compositions. However, in the case of true-palindromic compositions, the compositions formed on either side of the center part are structurally more complex: For each part λ on the left side, its reversal $R(\lambda)$ (formed by reversing the digits of λ) appears on the right, meaning that the sequences on either side of the center part often compose different integers. Therefore, an important part of understanding true-palindromes is understanding $R(\lambda)$. We will spend the first section of this paper deriving an exact formula for $R(\lambda)$.

Throughout this paper we will denote the coefficient of x^n in a formal power series $f(x)$ by $[x^n]f(x)$.

2 Integer reversals

In a true-palindromic composition, any non-center part λ will be paired with its reversal $R(\lambda)$. Therefore, we are interested in the quantity $\lambda + R(\lambda)$. (This quantity has been well-studied due to the popular yet widely disputed Palindromic Number Conjecture [10, 11, 13].) Lemma 1 serves the purposes of this paper by providing an easy way to compute $\lambda + R(\lambda)$ directly.

Lemma 1. *Let $x = a_d \cdots a_2 a_1$ be a positive integer with $d := d(x)$ digits, and define $R(x) := R(x, d) = a_1 a_2 \cdots a_d$ to be its reversal. Then*

$$R(x) = 10^{1-d}x + 99 \sum_{k=1}^{d-1} 10^{d-2k-1} x \bmod 10^k \ .$$

The number of digits d can be computed exactly using the well-known formula

$$d(x) = 1 + \lfloor \log_{10} x \rfloor \ .$$

Proof. Any integer x with d digits can be written as

$$x = a_d \cdots a_2 a_1 = 10^{d-1}a_d + 10^{d-2}a_{d-1} + \cdots + 10a_2 + a_1 \ .$$

Hence, the reversal of any integer x can be written as

$$R(x) = a_1 a_2 \cdots a_d = 10^{d-1}a_1 + 10^{d-2}a_2 + \cdots + 10a_{d-1} + a_d \ . \tag{1}$$

Furthermore, the digit a_k can be written as

$$a_k = \begin{cases} x \bmod 10 & k = 1 \\ 10^{1-k}(x \bmod 10^k - x \bmod 10^{k-1}) & k = 2, \ldots, d-1 \\ 10^{1-d}(x - x \bmod 10^{d-1}) & k = d \end{cases} \ . \tag{2}$$

Putting (2) back into (1), we have

$$
\begin{aligned}
R(x) &= 10^{d-1}x \bmod 10 + \sum_{k=2}^{d-1} 10^{d-2k+1}(x \bmod 10^k - x \bmod 10^{k-1}) \\
&\quad + 10^{1-d}(x - x \bmod 10^{d-1}) \\
&= 10^{d-1}x \bmod 10 + \sum_{k=2}^{d-1} 10^{d-2k+1}x \bmod 10^k - \sum_{k=1}^{d-2} 10^{d-2k-1}x \bmod 10^k \\
&\quad + 10^{1-d}x - 10^{1-d}x \bmod 10^{d-1} \\
&= 10^{1-d}x + \sum_{k=1}^{d-1} 10^{d-2k+1}x \bmod 10^k - \sum_{k=1}^{d-1} 10^{d-2k-1}x \bmod 10^k \\
&= 10^{1-d}x + \sum_{k=1}^{d-1} (10^{d-2k+1} - 10^{d-2k-1})x \bmod 10^k \\
&= 10^{1-d}x + 99\sum_{k=1}^{d-1} 10^{d-2k-1}x \bmod 10^k \ . \qquad \square
\end{aligned}
$$

3 Number of true-palindromic compositions

Theorem 1. *Let $\mathcal{P}$ denote the set of nonnegative integer palindromes, and let S denote $\mathbb{Z}_+\backslash 10\mathbb{Z}_+$. Let T_n denote the set of true-palindromic compositions of n. Then*

$$
|T_n| = [x^n]\frac{\sum\limits_{\lambda\in\mathcal{P}} x^\lambda}{1 - \sum\limits_{\lambda\in S} x^{\lambda+R(\lambda)}} \ .
$$

Proof. Define $F(x) = \sum\limits_{\lambda\in\mathcal{P}} x^\lambda$ and $G(x) = \sum\limits_{\lambda\in S} x^{\lambda+R(\lambda)}$. To build a true-palindrome, we begin by selecting a center part λ_0, which must itself be an integer palindrome. (If the composition has an even number of parts, there is no center part and we use the convention $\lambda_0 = 0$.) We then construct an ordered sequence $\lambda_1, \lambda_2, \ldots, \lambda_t$ by choosing integers from the set S under the constraint

$$
\lambda_0 + (\lambda_1 + R(\lambda_1)) + \cdots + (\lambda_t + R(\lambda_t)) = n \ .
$$

The sequence $\lambda_1, \lambda_2, \ldots, \lambda_t$ represents the parts that fall on one side of the center part λ_0. By this reasoning, the number of true-palindromes of n with t parts on each side of λ_0 is

$$
[x^n]F(x)G(x)^t \ .
$$

We next sum over all possible values of t, while noting that $\lambda + R(\lambda) \geq 2$, to get

$$
|T_n| = \sum_{t=0}^{\lfloor \frac{n}{2} \rfloor}[x^n]F(x)G(x)^t = [x^n]F(x)\sum_{t=0}^{\infty} G(x)^t = [x^n]\frac{F(x)}{1-G(x)} \ . \qquad \square
$$

We now analyze the equation $1 - \sum_{\lambda \in S} x^{\lambda+R(\lambda)} = 0$ and use the results to obtain an asymptotic estimate for the number of true-palindromes.

Lemma 2. *The function* $f(x) = 1 - \sum_{\lambda \in S} x^{\lambda+R(\lambda)}$ *has a unique positive real root* p *in the interval* $(0,1)$ *and it is the root of smallest magnitude.*

Proof. Observe that $f(0) = 1$ and $\lim_{x \to 1^-} f(x) = -\infty$, and therefore a root exists in the interval $(0,1)$. Call this root p and note that $\sum_{\lambda \in S} p^{\lambda+R(\lambda)} = 1$. Define $g(x) = 1$ and $\epsilon > 0$, and observe that for $|x| = p - \epsilon$,

$$|f(x) - g(x)| = \left| -\sum_{\lambda \in S} x^{\lambda+R(\lambda)} \right| \leq \sum_{\lambda \in S} |x|^{\lambda+R(\lambda)} < \sum_{\lambda \in S} p^{\lambda+R(\lambda)} = 1 = |g(x)| \ .$$

Therefore, by Rouché's theorem, $f(x)$ has no zeros with magnitude less than p. We next show that $f(x)$ has only one zero of magnitude p. Suppose $x = -p$ is a zero of f and therefore $f(-p) = 0$. Then, since $\lambda + R(\lambda)$ is odd for at least one element of S (for example, $\lambda = 12$), we have

$$f(-p) = 1 - \sum_{\lambda \in S} (-p)^{\lambda+R(\lambda)} > 1 - \sum_{\lambda \in S} p^{\lambda+R(\lambda)} = 0 \ .$$

Therefore, if there does exist a second root with magnitude p, it must have an imaginary part. Suppose $\hat{p}$ is such a root. Then $\hat{p} = p(\cos\theta + i \sin\theta)$ where $\theta \in (0,\pi) \cup (\pi, 2\pi)$. Since $f(\hat{p}) = 0$, the real part of $f(\hat{p})$ is also zero:

$$\begin{aligned} 0 &= Re(1 - \sum_{k \in S} \hat{p}^{\lambda+R(\lambda)}) \\ &= 1 - \sum_{\lambda \in S} p^{\lambda+R(\lambda)} \cos((\lambda + R(\lambda))\theta) \\ &= (1 - \sum_{\lambda \in S} p^{\lambda+R(\lambda)}) + \sum_{k \in S} p^{\lambda+R(\lambda)}(1 - \cos((\lambda + R(\lambda))\theta)) \\ &= 0 + \sum_{\lambda \in S} p^{\lambda+R(\lambda)}(1 - \cos((\lambda + R(\lambda))\theta)) \ . \end{aligned}$$

Because $1 - \cos((\lambda + R(\lambda))\theta) \geq 0$, each term in the sum must be zero. Therefore, for all $\lambda \in S$, there is an $\ell_\lambda \in \mathbb{Z}$ such that $(\lambda + R(\lambda))\theta = 2\pi\ell_\lambda$. However, if we choose λ to be any positive integer less than 9, then $R(\lambda + 1) = R(\lambda) + 1$ and

$$(\lambda + 1 + R(\lambda + 1))\theta - (\lambda + R(\lambda))\theta = (\lambda + 1 + R(\lambda) + 1 - \lambda - R(\lambda))\theta = 2\theta$$

whereas

$$(\lambda + 1 + R(\lambda + 1))\theta - (\lambda + R(\lambda))\theta = 2\pi\ell_{\lambda+1} - 2\pi\ell_\lambda = 2\pi(\ell_{\lambda+1} - \ell_\lambda) \ .$$

Therefore $\theta = \pi(\ell_{\lambda+1} - \ell_\lambda)$ and since $(\ell_{\lambda+1} - \ell_\lambda) \in \mathbb{Z}$, this contradicts the fact that $\theta \in (0,\pi) \cup (\pi, 2\pi)$. $\square$

Lemma 3. *Let $\bar{p}$ be the root of next smallest magnitude after p, and define r to be a real number such that $p < r < |\bar{p}|$. Define also the following constants:*

$$C_1 = \sum_{\lambda \in S} (\lambda + R(\lambda)) p^{\lambda + R(\lambda)}$$

$$C_2 = \sum_{\lambda \in S} (\lambda + R(\lambda))(\lambda + R(\lambda) - 1) p^{\lambda + R(\lambda)}$$

Then, for any positive integer k,

$$[x^k] \frac{1}{f(x)} = \frac{1}{p^k C_1} + O\left(\frac{1}{r^k}\right)$$

and

$$[x^k] \frac{1}{f(x)^2} = \frac{k+1}{p^k C_1^2} + \frac{C_2}{p^k C_1^3} + O\left(\frac{1}{r^k}\right) .$$

Proof. Define $E(x) = \frac{x-p}{f(x)}$. Since Lemma 2 tells us that $E(x)$ is analytic inside $|x| \leq r$, we can expand around p to obtain

$$[x^k] \frac{1}{f(x)} = [x^k] \frac{E(x)}{x-p} = [x^k] \frac{E(p)}{x-p} + [x^k] \frac{1}{x-p} \sum_{s=1}^{\infty} \frac{E^{(s)}(p)}{s!} (x-p)^s = \epsilon_1 + \epsilon_2 . \quad (3)$$

To evaluate ϵ_1, we note that

$$f'(x) = \frac{E(x) - (x-p)E'(x)}{E(x)^2}$$

and therefore

$$E(p) = \frac{1}{f'(p)} = \frac{1}{-\sum_{\lambda \in S} (\lambda + R(\lambda)) p^{\lambda + R(\lambda) - 1}} .$$

We then apply the generalized binomial theorem to obtain

$$\epsilon_1 = [x^k] \frac{E(p)}{x-p} = \frac{E(p)}{-p \cdot p^k} = \frac{1}{p^k \sum_{\lambda \in S} (\lambda + R(\lambda)) p^{\lambda + R(\lambda)}} . \quad (4)$$

To estimate ϵ_2, we note that

$$\frac{1}{x-p} \sum_{s=1}^{\infty} \frac{E^{(s)}(p)}{s!} (x-p)^s$$

is analytic inside $|x| \leq r$ with a removable singularity at $x = p$. Therefore we can apply Cauchy's inequality to obtain

$$|\epsilon_2| \leq \frac{\max \left| \frac{1}{x-p} \sum_{s=1}^{\infty} \frac{E^{(s)}(p)}{s!} (x-p)^s \right|}{r^k} = O\left(\frac{1}{r^k}\right) . \quad (5)$$

Putting (4) and (5) back into (3) gives the first half of the lemma. By similar reasoning, we obtain

$$\begin{aligned}
[x^k]\frac{1}{f(x)^2} &= [x^k]\frac{E(x)^2}{(x-p)^2} \\
&= [x^k]\frac{E(p)^2}{(x-p)^2} + [x^k]\frac{2E(p)E'(p)}{x-p} + [x^k]\frac{1}{(x-p)^2}\sum_{s=2}^{\infty}\frac{[E^2]^{(s)}(p)}{s!}(x-p)^s \\
&= \epsilon_1 + \epsilon_2 + \epsilon_3 \ . \qquad (6)
\end{aligned}$$

To evaluate ϵ_1, we note that

$$E(p)^2 = \left(\frac{1}{f'(p)}\right)^2 = \frac{1}{\left(\sum_{\lambda\in S}(\lambda+R(\lambda))p^{\lambda+R(\lambda)-1}\right)^2}$$

and apply the generalized binomial theorem to obtain

$$\epsilon_1 = [x^k]\frac{E(p)^2}{(x-p)^2} = \frac{E(p)^2(k+1)}{p^2\cdot p^k} = \frac{k+1}{p^k\left(\sum_{\lambda\in S}(\lambda+R(\lambda))p^{\lambda+R(\lambda)}\right)^2} \ . \qquad (7)$$

To evaluate ϵ_2, we note that

$$f''(x) = \frac{-(x-p)E(x)E''(x) - 2E(x)E'(x) + 2(x-p)E'(x)^2}{E(x)^3}$$

and therefore

$$2E(p)E'(p) = -f''(p)E(p)^3 = \frac{-f''(p)}{f'(p)^3} = \frac{-\sum_{\lambda\in S}(\lambda+R(\lambda))(\lambda+R(\lambda)-1)p^{\lambda+R(\lambda)-2}}{\left(\sum_{\lambda\in S}(\lambda+R(\lambda))p^{\lambda+R(\lambda)-1}\right)^3} \ .$$

We then apply the generalized binomial theorem to obtain

$$\epsilon_2 = [x^k]\frac{2E(p)E'(p)}{x-p} = \frac{2E(p)E'(p)}{-p\cdot p^k} = \frac{\sum_{\lambda\in S}(\lambda+R(\lambda))(\lambda+R(\lambda)-1)p^{\lambda+R(\lambda)}}{p^k\left(\sum_{\lambda\in S}(\lambda+R(\lambda))p^{\lambda+R(\lambda)}\right)^3} \ . \qquad (8)$$

To estimate ϵ_3, we note that

$$\frac{1}{(x-p)^2}\sum_{s=2}^{\infty}\frac{[E^2]^{(s)}(p)}{s!}(x-p)^s$$

is analytic inside $|x| \le r$ with a removable singularity at $x = p$. Therefore we can apply Cauchy's inequality to obtain

$$|\epsilon_3| \le \frac{\max\left|\frac{1}{(x-p)^2}\sum_{s=2}^{\infty}\frac{[E^2]^{(s)}(p)}{s!}(x-p)^s\right|}{r^k} = O\left(\frac{1}{r^k}\right) \ . \qquad (9)$$

Putting (7), (8), and (9) back into (6) gives the second half of the lemma. □

We record the following simple lemma without proof for easy reference during the proof of Theorem 2.

Lemma 4. *Define the following constants:*

$$D_1 = \sum_{j \in \mathcal{P}} p^j \qquad\qquad D_2 = \sum_{j \in \mathcal{P}} j p^j$$

Then

$$\sum_{\substack{j \in \mathcal{P} \\ j \leq n}} p^j = D_1 + O(p^n)$$

and

$$\sum_{\substack{j \in \mathcal{P} \\ j \leq n}} j p^j = D_2 + O(np^n) \ .$$

Theorem 2. *Let T_n denote the number of true-palindromic compositions of n. Then*

$$|T_n| = \frac{D_1}{p^n C_1} + O\left(\frac{1}{r^n}\right) \ .$$

Proof. It follows from Theorem 1 that the number of true-palindromes of n is

$$|T_n| = \sum_{j \in \mathcal{P},\, j \leq n} [x^{n-j}] \frac{1}{1 - \sum\limits_{\lambda \in S} x^{\lambda + R(\lambda)}} \ .$$

Combining this fact with Lemmas 3 and 4, we have

$$\begin{aligned} |T_n| &= \sum_{j \in \mathcal{P},\, j \leq n} \left(\frac{1}{p^{n-j} C_1} + O\left(\frac{1}{r^{n-j}}\right) \right) \\ &= \frac{1}{p^n C_1} \sum_{j \in \mathcal{P},\, j \leq n} p^j + O\left(\frac{1}{r^n} \sum_{j \in \mathcal{P},\, j \leq n} r^j \right) \\ &= \frac{1}{p^n C_1} (D_1 + O(p^n)) + O\left(\frac{1}{r^n} \sum_{j \in \mathcal{P},\, j \leq n} r^j \right) \ . \qquad (10) \end{aligned}$$

To make the final step of the proof, we observe that $|\bar{p}| < 1$, since $f(0) = 1$ and (using Lemma 1 to get an approximation) $f(-\frac{3}{4}) < 0$. Therefore f has a real root in the interval $(-1, 0)$ and r must be chosen to be less than 1. Hence

$$\frac{1}{r^n} \sum_{j \in \mathcal{P},\, j \leq n} r^j \leq \frac{1}{r^n(1-r)} = O\left(\frac{1}{r^n}\right) \ . \qquad (11)$$

Putting (11) back into (10) provides the statement of the theorem. □

4 Number of parts and size of the last part

Combinatorial structures are often studied from a probabilistic standpoint and there are numerous papers, for instance the series by Bender and Canfield [2], that concern probability distributions of quantities related to integer compositions. In this section we will consider a composition chosen uniform randomly from the set of all true-palindromic compositions of n. We will use $\mathbb{P}_n$ to denote the uniform probability measure on T_n and $\mathbb{E}_n$ to denote the expected value with respect to $\mathbb{P}_n$.

Theorem 3. *Let $N := N_n(\vec{\lambda})$ be the number of parts in a given composition $\vec{\lambda}$ of n. The average number of parts over all true-palindromic compositions of n is*

$$\mathbb{E}_n(N) = \frac{2n}{C_1} + \frac{2}{C_1} + \frac{2C_2}{C_1^2} - \frac{2D_2}{C_1 D_1} - \frac{1}{D_1} - 1 + O\left(\left(\frac{p}{r}\right)^n\right)$$

where p is again the unique positive real root of smallest magnitude of f, $\bar{p}$ is the root of next smallest magnitude after p, and r is a real number such that $p < r < |\bar{p}|$.

Proof. Define $s(A)$ to be the sum of the number of parts over all compositions in a set A. Recall that $F := F(x) = \sum_{\lambda \in \mathcal{P}} x^\lambda$ and $G := G(x) = \sum_{\lambda \in S} x^{\lambda + R(\lambda)}$. Then we have

$$\begin{aligned}
s(T_n) &= s(\{\vec{\lambda} \in T_n \mid N(\vec{\lambda}) = 1\}) + s(\{\vec{\lambda} \in T_n \mid N(\vec{\lambda}) \text{ is odd and } N(\vec{\lambda}) \geq 3\}) \\
&\qquad + s(\{\vec{\lambda} \in T_n \mid N(\vec{\lambda}) \text{ is even}\}) \\
&= [x^n](F-1) + \sum_{t=1}^{\lfloor \frac{n}{2} \rfloor} (2t+1)[x^n](F-1)G^t + \sum_{t=1}^{\lfloor \frac{n}{2} \rfloor} (2t)[x^n]G^t \\
&= [x^n](F-1) + [x^n]2(F-1)\sum_{t=1}^{\infty} tG^t + [x^n](F-1)\sum_{t=1}^{\infty} G^t + [x^n]2\sum_{t=1}^{\infty} tG^t \\
&= [x^n](F-1) + [x^n]\frac{2(F-1)G}{(1-G)^2} + [x^n]\frac{(F-1)G}{1-G} + [x^n]\frac{2G}{(1-G)^2} \\
&= [x^n]\left(\frac{F-1}{1-G} + \frac{2FG}{(1-G)^2}\right) .
\end{aligned}$$

Since $\mathbb{E}_n(N) = \frac{s(T_n)}{|T_n|}$, we use this generating function to obtain

$$\begin{aligned}
\mathbb{E}_n(N) &= \frac{1}{|T_n|}\left([x^n]\frac{F}{1-G} - [x^n]\frac{1}{1-G} + 2\sum_{\substack{j \in \mathcal{P} \\ j \leq n}} [x^{n-j}]\frac{G}{(1-G)^2} \right) \\
&= \frac{1}{|T_n|}\left([x^n]\frac{F}{1-G} - [x^n]\frac{1}{1-G} + 2\sum_{\substack{j \in \mathcal{P} \\ j \leq n}} [x^{n-j}]\frac{1}{(1-G)^2} - 2\sum_{\substack{j \in \mathcal{P} \\ j \leq n}} [x^{n-j}]\frac{1}{1-G} \right) \\
&= \frac{1}{|T_n|}(\epsilon_1 - \epsilon_2 + 2\epsilon_3 - 2\epsilon_4) .
\end{aligned}$$

The quantities ϵ_1 and ϵ_2 are given directly by Theorem 1 and Lemma 3. We use Lemma 3 to obtain estimates for ϵ_3 and ϵ_4:

$$\begin{aligned}\epsilon_3 &= \sum_{\substack{j\in\mathcal{P}\\ j\le n}}[x^{n-j}]\frac{1}{(1-G)^2} = \sum_{\substack{j\in\mathcal{P}\\ j\le n}}\left(\frac{n-j+1}{p^{n-j}C_1^2}+\frac{C_2}{p^{n-j}C_1^3}+O\left(\frac{1}{r^{n-j}}\right)\right)\\
&= \frac{n+1}{p^nC_1^2}\sum_{\substack{j\in\mathcal{P}\\ j\le n}}p^j - \frac{1}{p^nC_1^2}\sum_{\substack{j\in\mathcal{P}\\ j\le n}}jp^j + \frac{C_2}{p^nC_1^3}\sum_{\substack{j\in\mathcal{P}\\ j\le n}}p^j + O\left(\frac{1}{r^n}\sum_{\substack{j\in\mathcal{P}\\ j\le n}}r^j\right)\\
&= \left(\frac{n+1}{p^nC_1^2}+\frac{C_2}{p^nC_1^3}\right)(D_1+O(p^n)) - \frac{1}{p^nC_1^2}(D_2+O(np^n)) + O\left(\frac{1}{r^n}\right)\\
&= \frac{(n+1)D_1}{p^nC_1^2}+\frac{D_1C_2}{p^nC_1^3}-\frac{D_2}{p^nC_1^2}+O\left(\frac{1}{r^n}\right)\end{aligned}$$

$$\begin{aligned}\epsilon_4 &= \sum_{\substack{j\in\mathcal{P}\\ j\le n}}[x^{n-j}]\frac{1}{1-G} = \sum_{\substack{j\in\mathcal{P}\\ j\le n}}\left(\frac{1}{p^{n-j}C_1}+O\left(\frac{1}{r^{n-j}}\right)\right)\\
&= \frac{1}{p^nC_1}\sum_{\substack{j\in\mathcal{P}\\ j\le n}}p^j + O\left(\frac{1}{r^n}\sum_{\substack{j\in\mathcal{P}\\ j\le n}}r^j\right) = \frac{1}{p^nC_1}(D_1+O(p^n)) + O\left(\frac{1}{r^n}\right)\\
&= \frac{D_1}{p^nC_1}+O\left(\frac{1}{r^n}\right)\end{aligned}$$

Finally, we have

$$\begin{aligned}\mathbb{E}_n(N) &= \frac{1}{|T_n|}(\epsilon_1-\epsilon_2+2\epsilon_3-2\epsilon_4)\\
&= 1+\frac{p^nC_1}{D_1}\left(1+O\left(\left(\frac{p}{r}\right)^n\right)\right)(-\epsilon_2+2\epsilon_3-2\epsilon_4)\\
&= 1+\left(1+O\left(\left(\frac{p}{r}\right)^n\right)\right)\left(-\frac{1}{D_1}+\frac{2(n+1)}{C_1}+\frac{2C_2}{C_1^2}-\frac{2D_2}{D_1C_1}-2+O\left(\left(\frac{p}{r}\right)^n\right)\right)\\
&= \frac{2(n+1)}{C_1}-\frac{1}{D_1}+\frac{2C_2}{C_1^2}-\frac{2D_2}{D_1C_1}-1+O\left(\left(\frac{p}{r}\right)^n\right).\end{aligned}$$

□

Theorem 4. *Let $L := L_n(\vec{\lambda})$ be the size (or value) of the last part in a given composition $\vec{\lambda}$ of n. The average size of the last part over all true-palindromic compositions of n is*

$$\mathbb{E}_n(L) = \sum_{k\in S}kp^{k+R(k)} + O\left(\left(\frac{p}{r}\right)^n\right).$$

Proof. Define the following indicator functions:

$$\mathtt{I}_{\mathcal{P}} = \begin{cases} 1 & n \in \mathcal{P} \\ 0 & \text{else} \end{cases} \qquad \mathtt{I}_L = \begin{cases} 1 & \lambda + R(\lambda) < n \\ 0 & \text{else} \end{cases}$$

$$\mathtt{I}_E = \begin{cases} 1 & \lambda + R(\lambda) = n \\ 0 & \text{else} \end{cases} \qquad \mathtt{I}_G = \begin{cases} 1 & \lambda + R(\lambda) > n \\ 0 & \text{else} \end{cases}$$

Then the average value of L is given by

$$\begin{aligned}
\mathbb{E}_n(L) &= \sum_{k=1}^{n} k\mathbb{P}_n(L = k) \\
&= n\mathbb{P}_n(L = n) + \sum_{k=1}^{n-1} k\mathbb{P}_n(L = k) \\
&= \frac{n\mathtt{I}_{\mathcal{P}}}{|T_n|} + \sum_{\substack{k\in S \\ k<n}} \frac{k\mathtt{I}_E}{|T_n|} + \sum_{\substack{k\in S \\ k<n}} \frac{k|T_{n-k-R(k)}|\mathtt{I}_L}{|T_n|} \\
&= \epsilon_1 + \epsilon_2 + \epsilon_3 \ .
\end{aligned}$$

We first obtain asymptotic estimates for ϵ_1 and ϵ_2:

$$|\epsilon_1| = \frac{n\mathtt{I}_{\mathcal{P}}}{|T_n|} \le \frac{n}{\frac{D_1}{p^n C_1} + O\left(\frac{1}{r^n}\right)} = O\left(np^n\right) \tag{12}$$

$$|\epsilon_2| = \frac{1}{|T_n|} \sum_{\substack{k\in S \\ k<n}} k\mathtt{I}_E \le \frac{1}{|T_n|} \sum_{k=1}^{n-1} n \le \frac{(n-1)n}{\frac{D_1}{p^n C_1} + O\left(\frac{1}{r^n}\right)} = O\left(n^2 p^n\right) \tag{13}$$

Now we compute the dominant term ϵ_3 in steps:

$$\begin{aligned}
\epsilon_3 &= \sum_{\substack{k\in S \\ k<n}} \frac{k|T_{n-k-R(k)}|\mathtt{I}_L}{|T_n|} \\
&= \sum_{\substack{k\in S \\ k<n}} k \left(\frac{\frac{D_1}{p^{n-k-R(k)}C_1} + O\left(\frac{1}{r^{n-k-R(k)}}\right)}{\frac{D_1}{p^n C_1} + O\left(\frac{1}{r^n}\right)} \right) \mathtt{I}_L \\
&= \sum_{\substack{k\in S \\ k<n}} k \left(\frac{p^{k+R(k)} + O\left(\frac{p^n}{r^{n-k-R(k)}}\right)}{1 + O\left(\frac{p^n}{r^n}\right)} \right) \mathtt{I}_L \\
&= \left(1 + O\left(\left(\frac{p}{r}\right)^n\right)\right) \left(\sum_{\substack{k\in S \\ k<n}} kp^{k+R(k)}\mathtt{I}_L + O\left(\left(\frac{p}{r}\right)^n\right) \sum_{\substack{k\in S \\ k<n}} kr^{k+R(k)}\mathtt{I}_L \right) \\
&= \left(1 + O\left(\left(\frac{p}{r}\right)^n\right)\right) (\delta_1 + \delta_2)
\end{aligned}$$

$$\begin{aligned}
\delta_1 &= \sum_{\substack{k\in S\\ k<n}} kp^{k+R(k)}\mathtt{I}_L \\
&= \sum_{\substack{k\in S\\ k<n}} kp^{k+R(k)} - \sum_{\substack{k\in S\\ k<n}} kp^{k+R(k)}(\mathtt{I}_E+\mathtt{I}_G) \\
&= \sum_{k\in S} kp^{k+R(k)} - \sum_{\substack{k\in S\\ k\geq n}} kp^{k+R(k)} - \sum_{\substack{k\in S\\ k<n}} kp^{k+R(k)}(\mathtt{I}_E+\mathtt{I}_G) \\
&= \sum_{k\in S} kp^{k+R(k)} + \gamma_1 + \gamma_2
\end{aligned}$$

$$|\gamma_1| = \sum_{\substack{k\in S\\ k\geq n}} kp^{k+R(k)} \leq \sum_{k=n}^{\infty} kp^k = O\left(np^n\right)$$

$$|\gamma_2| = \sum_{\substack{k\in S\\ k<n}} kp^{k+R(k)}(\mathtt{I}_E+\mathtt{I}_G) \leq 2\sum_{k=1}^{n-1} kp^n \leq (n-1)np^n = O\left(n^2p^n\right)$$

$$|\delta_2| = O\left(\left(\frac{p}{r}\right)^n\right)\sum_{\substack{k\in S\\ k<n}} kr^{k+R(k)}\mathtt{I}_L \leq O\left(\left(\frac{p}{r}\right)^n\right)\sum_{k=1}^{\infty} kr^k = O\left(\left(\frac{p}{r}\right)^n\right)$$

Putting these back into ϵ_3, we have

$$\begin{aligned}
\epsilon_3 &= \left(1+O\left(\left(\frac{p}{r}\right)^n\right)\right)(\delta_1+\delta_2) \\
&= \left(1+O\left(\left(\frac{p}{r}\right)^n\right)\right)\left(\sum_{k\in S} kp^{k+R(k)} + O\left(\left(\frac{p}{r}\right)^n\right)\right) \\
&= \sum_{k\in S} kp^{k+R(k)} + O\left(\left(\frac{p}{r}\right)^n\right). \qquad (14)
\end{aligned}$$

Finally, putting (12), (13), and (14) together gives the statement of the theorem. □

5 Hairpin compositions

A *gapped palindrome* is a word that is palindromic except for a sequence of letters in the center. For example, the word ABCDBA is a gapped palindrome. Hence, we define a *gapped-palindromic composition with t parts* to be a composition such as $(12, 35, 1, 2, 3, 35, 12)$ that is palindromic except for a part-sequence in the center of length at least 2 but no more than $t-2$. (Note that t must be at least 4.) While gapped palindromes have been studied a great deal from an algorithmic and computer science theoretic perspective [6, 7], gapped-palindromic compositions have not.

It is not difficult to count the number of gapped-palindromic compositions of n. Let P_n be the set of palindromic compositions of n and recall that $|P_n| = 2^{\lfloor \frac{n}{2} \rfloor}$. Let Λ_n be the set of compositions of n whose first and last parts are the same (including the case when there is only one part of size n). Then, trivially, the number of gapped-palindromic compositions is equivalent to $|\Lambda_n| - |P_n|$. Therefore, the following calculation allows for an exact enumeration of the set of gapped-palindromic compositions of n:

If n is odd, then

$$|\Lambda_n| = 1 + \sum_{k=1}^{\frac{n-1}{2}} 2^{n-2k-1} = 1 + 2^{n-1} \sum_{k=1}^{\frac{n-1}{2}} \frac{1}{4^k} = \frac{2^{n-1}+2}{3} \ . \tag{15}$$

If n is even, then

$$|\Lambda_n| = 1 + 1 + \sum_{k=1}^{\frac{n}{2}-1} 2^{n-2k-1} = 2 + 2^{n-1} \sum_{k=1}^{\frac{n}{2}-1} \frac{1}{4^k} = \frac{2^{n-1}+4}{3} \ . \tag{16}$$

We can analogously define a *gapped true-palindromic composition* or *hairpin composition with t parts* to be a composition that is true-palindromic except for a part-sequence in the center of length at least 1 but no more than $t-2$. (Note that t must be at least 3.) The word "hairpin" is reminiscent of a structure that arises in DNA computing. A single strand of DNA is a string over the alphabet {A,T,C,G}, and two single strands can bind to each other if they are Watson-Crick complementary (A is complementary to T, and C is complementary to G). A DNA "stem-loop" or "hairpin" occurs when two regions of the same strand bind to each other leaving an unpaired loop in the center. For example, the sequence CCTGATCTTGGGTCAGG might bind in the following way:

$$\begin{array}{ccccccccc} & & & & & & \mathrm{C} & & \\ \mathrm{C} & \mathrm{C} & \mathrm{T} & \mathrm{G} & \mathrm{A} & \mathrm{T} & & \mathrm{T} & \\ & & & & & & & & \mathrm{T} \\ \mathrm{G} & \mathrm{G} & \mathrm{A} & \mathrm{C} & \mathrm{T} & \mathrm{G} & & \mathrm{G} & \\ & & & & & & \mathrm{G} & & \end{array}$$

This idea extends naturally to true-palindromic compositions since the parts on each end complement each other and yet are not the same. To count the number of hairpin compositions, we define Ω_n to be the set of compositions of n whose first and last part are reversals (including the case when there is one part of size n which is a palindrome), i.e. if the first part is λ, then the last part is $R(\lambda)$. Then the number of hairpin compositions is $|\Omega_n| - |T_n|$.

Theorem 5. *Let H_n denote the set of hairpin compositions of n. Then*

$$|H_n| \sim 2^{n-1} \sum_{\lambda \in S} \frac{1}{2^{\lambda + R(\lambda)}} \ .$$

Proof. A composition in Ω_n with t parts is subject to the restrictions $\lambda_1 \in S$ and

$$\lambda_1 + R(\lambda_1) + \lambda_2 + \cdots + \lambda_{t-1} = n \ .$$

Hence,

$$\begin{aligned}|\Omega_n| &= |\Omega_n \cap \{t=1\}| + |\Omega_n \cap \{t \geq 2\}| \\ &= \mathtt{I}_{\mathcal{P}} + \sum_{t=2}^{n} [x^n] G(x) \left(\frac{x}{1-x}\right)^{t-2} \ .\end{aligned} \tag{17}$$

The second term on the right hand side can be simplified to obtain

$$\sum_{t=2}^{n} [x^n] G(x) \left(\frac{x}{1-x}\right)^{t-2} = [x^n] G(x) \sum_{t=0}^{\infty} \left(\frac{x}{1-x}\right)^{t} = [x^n] \frac{G(x)}{1-\frac{x}{1-x}} \ .$$

Then

$$\begin{aligned}[x^n] \frac{G(x)}{1-\frac{x}{1-x}} &= [x^n] \frac{\sum\limits_{\lambda \in S} (\mathtt{I}_L + \mathtt{I}_E) x^{\lambda + R(\lambda)}}{1-\frac{x}{1-x}} \\ &= \sum_{\lambda \in S} \mathtt{I}_E + \sum_{\lambda \in S} \mathtt{I}_L [x^{n-\lambda-R(\lambda)}] \frac{1}{1-\frac{x}{1-x}} \\ &= \sum_{\lambda \in S} \mathtt{I}_E + \sum_{\lambda \in S} \mathtt{I}_L 2^{n-\lambda-R(\lambda)-1} \ .\end{aligned} \tag{18}$$

Combining (17) and (18), we get

$$\begin{aligned}|\Omega_n| &= \mathtt{I}_{\mathcal{P}} + \sum_{\lambda \in S} \mathtt{I}_E + 2^{n-1} \sum_{\lambda \in S} \frac{\mathtt{I}_L}{2^{\lambda + R(\lambda)}} \\ &= \mathtt{I}_{\mathcal{P}} + \sum_{\lambda \in S} \mathtt{I}_E + 2^{n-1} \sum_{\lambda \in S} \frac{1}{2^{\lambda + R(\lambda)}} - 2^{n-1} \sum_{\lambda \in S} \frac{\mathtt{I}_E + \mathtt{I}_G}{2^{\lambda + R(\lambda)}} \ .\end{aligned}$$

The first term, $\mathtt{I}_{\mathcal{P}}$ is either equal to 0 or 1. The second term can be crudely bounded above by n, which we will see is sufficient for analyzing the asymptotics of H_n. The constant in the third term can be approximated using Lemma 1 to be slightly less than $\frac{1}{3}$, a figure that seems to parallel the formulas for the number of gapped-palindromic compositions of n given in (15) and (16). The fourth term is bounded as follows: Let $a(n)$ be the value of the smallest λ in S such that $\lambda + R(\lambda) \geq n$. Then

$$\left| 2^{n-1} \sum_{\lambda \in S} \frac{\mathtt{I}_E + \mathtt{I}_G}{2^{\lambda + R(\lambda)}} \right| = 2^{n-1} \sum_{\substack{\lambda \in S \\ \lambda \geq a(n)}} \frac{\mathtt{I}_E + \mathtt{I}_G}{2^{\lambda + R(\lambda)}} \leq 2^{n-1} \sum_{\lambda \geq a(n)} \frac{2}{2^{\lambda}} = 2^{n-a(n)+1} \ .$$

We get a crude bound on $a(n)$ through the following argument: Suppose $10^k \leq \lambda < 10^{k+1}$ for some $k \in \mathbb{Z}_+$. Then it is also true that $R(\lambda) < 10^{k+1}$. Furthermore, $d(\lambda)$, the number of digits of λ, is equal to $k+1$. Hence

$$R(\lambda) < 10^{k+1} = 10^{d(\lambda)} = 10^{1+\lfloor \log_{10} \lambda \rfloor} \leq 10^{1+\log_{10} \lambda} = 10\lambda \ .$$

This is a crude bound in that it is often very bad (consider $R(91)$); however, it is sometimes quite good (consider $R(109)$). Now we have

$$\lambda + R(\lambda) < 11\lambda < n$$

whenever $\lambda < \frac{n}{11}$. Therefore, $a(n) \geq \frac{n}{11}$ and

$$\left| 2^{n-1} \sum_{\lambda \in S} \frac{\mathtt{I}_E + \mathtt{I}_G}{2^{\lambda+R(\lambda)}} \right| \leq 2^{n-\frac{n}{11}+1} = 2^{\frac{10n}{11}+1} \ .$$

Finally we put these estimates together to obtain

$$\begin{aligned} |H_n| &= |\Omega_n| - |T_n| \\ &= \mathtt{I}_{\mathcal{P}} + \sum_{\lambda \in S} \mathtt{I}_E + 2^{n-1} \sum_{\lambda \in S} \frac{1}{2^{\lambda+R(\lambda)}} - 2^{\frac{10n}{11}+1} - \frac{D_1}{p^n C_1} + O\left(\frac{1}{r^n}\right) \\ &\sim 2^{n-1} \sum_{\lambda \in S} \frac{1}{2^{\lambda+R(\lambda)}} \ . \end{aligned}$$

□

6 Remarks

An interesting quantity that we do not fully understand is c_k, the number of positive integers λ such that λ is not congruent to $0 \pmod{10}$ and $\lambda + R(\lambda) = k$. Notice that

$$\sum_{\lambda \in S} x^{\lambda+R(\lambda)} = \sum_{k \in \mathbb{Z}_+} c_k \, x^k$$

Therefore, if c_k were known exactly, Theorems 2 and 5 could be exact rather than asymptotic formulas. However, preliminary calculations show c_k to be an unpredictable sequence (more so if multiples of 10 are included) with many terms equal to zero. Until c_k is better understood, Lemma 1 enables the constants in Theorems 2, 3, 4 and 5 to be numerically approximated as follows:

$$|T_n| \sim (0.8380008..)(1.4137001..)^n$$

$$|H_n| \sim \ = (0.1666661..)2^n$$

$$\mathbb{E}_n(N) = (0.5028927..)n + (0.5710118..) + o(1)$$

$$\mathbb{E}_n(L) = (1.9884956..) + o(1)$$

The beginning terms of the actual sequences $|T_n|$ and $|H_n|$, beginning at $n = 1$, are:

$$\{|T_n|\} = \{1, 2, 2, 4, 4, 8, 8, 16, 16, 31, 32, 62, 63, 124, 126, 248, 252, 496, 504, 991 \dots\}$$

$$\{|H_n|\} = \{0, 0, 0, 0, 2, 4, 14, 28, 70, 140, 310, 621, 1302, 2607, 5335, 10675, 21593, \dots\}$$

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