ON FACETS OF THE THREE-INDEX ASSIGNMENT POLYTOPE*

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Abstract

Two new classes of facet-defining inequalities for the three-index assignment polytope are identified in this paper. According to the shapes of their support index sets, we call these facets bull facets and comb facets respectively. The bull facet has Chvàtal rank 1, while the comb facet has Chvàtal rank 2. For a comb facet-defining inequality, the right-hand-side coefficient is a positive integer, and the left-hand-side coefficients equal to 0 or 1. For a bull facet-defining inequality, the right-hand-side coefficient is a positive even integer, and the left-hand-side coefficients equal to 0, 1 or 2. Furthermore, we give an $O(n^3)$ procedure for finding a bull facet with the right-hand-side coefficient 2, violated by a given noninteger solution to the linear programming relaxation of the three-index assignment problem, or showing that no such facet exists. Such an algorithm is called a separation algorithm. Since the number of variables is n^3 and one needs to check through all the variables in such a separation algorithm, this algorithm is linear-time and the order of its complexity is the best possible.

Keywords: three-index assignment, facet, rank, separation algorithm.

1. Introduction

Several classes of facets of the three-index assignment polytope have been identified by Balas and Saltzman in [3], and Balas and Qi in [2]. Among these facet classes, four subclasses of facets have been found to possess linear-time, i.e., $O(n^3)$, separation algorithms [1] [2]. Recall that a separation algorithm for a facet class of a combinatorial optimization problem is a procedure for finding a facet-defining inequality in this facet class, violated by a given noninteger solution to the linear programming relaxation of this combinatorial optimization problem, or showing that no such inequality exists. Such an algorithm is an important component of

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the polyhedral method for solving NP-hard combinatorial optimization problems [8] [10].

In this paper, we identify two more facet classes of the three-index assignment polytope and one more facet subclass with linear-time separation algorithm.

The three-index assignment problem of order n can be stated as a 0-1 programming problem as follows:

$$
\begin{aligned}
&\min \sum \{c_{ijk}x_{ijk} : i \in I, j \in J, k \in K\}, \\
&\text{s.t.} \sum \{x_{ijk} : j \in J, k \in K\} = 1, \ \ \forall i \in I, \\
&\sum \{x_{ijk} : i \in I, k \in K\} = 1, \ \ \forall j \in J, \\
&\sum \{x_{ijk} : i \in I, j \in J\} = 1, \ \ \forall k \in K, \\
&x_{ijk} \in \{0,1\}, \quad \forall i,j,k,
\end{aligned}
\tag{1.1}
$$

where I, J and K are disjoint sets with $|I| = |J| = |K| = n$ and c_{ijk} are cost coefficients. Let A be the coefficient matrix of the constraint set of (1.1). Then $R = I \cup J \cup K$ is the row index set of A. Let S be the column index set of A. Let

$$P = \{x \in \Re^{n^3} : Ax = e, x \geq 0\},$$

where $e = (1, ..., 1)^T \in \Re^{3n}$. Then

$$P_I = \text{conv}\{x \in \{0,1\}^{n^3} : x \in P\}$$

is the three-index assignment polytope of order n.

We use the same symbols as [1] [2] [3]. In particular, a^s means a column of A associated with $s \in S$. We may specify $s = \{i, j, k\}$ for $i \in I$, $j \in J$, $k \in K$. For a set $Q \subseteq R$, we use Q_I, Q_J and Q_K to denote the parts of Q in I, J and K respectively. Furthermore, we also regard $s \in S$ as a three-element subset of R. So we may write $s \cap Q$ for $s \in S$ and $Q \subseteq R$. For $x \in \Re^{n^3}$ and $S' \subseteq S$, let

$$x(S') = \sum \{x_s : s \in S'\}.$$

The new facets we shall identify in this paper are called *bull facets* and *comb facets* respectively.

Let $D \subset R$ have the same cardinalities in I, J and K, i.e., $|D_I| = |D_J| = |D_K| = r$, where $1 \leq r \leq n-4$. Let $H \subseteq R$, $H \cap D = \emptyset$, $|H| = 2$, $|H_L| \leq 1$ for $L = I, J, K$. Let $Q = D \cup H$. If $|H_I| = |H_J| = 1$, then D is just like the body of a fat bull and H is like the head of the bull. Let

$$B(D) = \{t \in S : t \subseteq D\},$$

$$F(Q, D) = \{t \in S : |t \cap Q| \geq 2, 2 \geq |t \cap D| \geq 1\}.$$

In Section 2, we show that

$$2x(B(D)) + x(F(Q, D)) \leq 2r \tag{1.2}$$

defines a facet of P_I. We call such a facet a bull facet. We also show that all the bull facets have Chvátal rank 1 (see [10] or the appendix of this paper for the definition of Chvàtal rank). Among bull facets, we pay special attention to the case with $r = 1$. In Section 3, we give an $O(n^3)$ separation algorithm for this subclass of bull facets. Since the number of variables of the three-index assignment polytope is n^3, this algorithm is linear-time and its complexity is the best possible. As we mentioned in the beginning, this will be the fifth subclass of facets with linear-time separation algorithms identified until now.

In [2], Balas and Qi identified another class of facets. Suppose that $D, Q \subseteq R$, $D \cap Q = \emptyset$, $|(D \cap Q)_L| = p + 1$ and $1 \leq |Q_L| \leq r$ for $L = I, J, K$, $|Q| = 2r + 1$, $1 \leq r \leq p \leq n - 3$. Let

$$C_1(D) = \{t \in S : t \subseteq D\},$$

$$C_2(D, Q) = \{t \in S : |t \cap D| = 1, |t \cap Q| = 2\},$$

$$C(D, Q) = C_1(D) \cup C_2(D, Q).$$

Then

$$x(C(D, Q)) \leq p \tag{1.3}$$

defines a facet of P_I for $n \geq 4$ [2]. We may think that D is a "handle" of a "comb" and Q is the sole "tooth" of this comb. One may think to extend the above case to more general cases, i.e., with several "teeth". The name "comb facet" is borrowed from the traveling salesman polytope [8]. However, the formation rule of a general comb facet is somewhat complicated. Thus, we first discuss two-tooth facets in Section 4 and then extend the discussion to more general comb facets in Section 5. In any case, the cardinalities of the support index sets of a comb facet at I, J and K are the same. A tooth may have a common index with the handle. We call such a tooth a *linked tooth* and call a tooth an *unlinked tooth* if it has no common indices with the handle. The number of unlinked teeth is $3\eta+1$, where η is a nonnegative integer. Thus, at least one tooth is unlinked. Some other rules on the formation of comb facets will be specified in Section 5. We conclude this paper in Section 6 with some final remarks and further questions.

For applications and literature on the three-index assignment problem, also see [4] [5] [6] [7] [9] [11] [12].

2. Bull Facets

We now show that (1.2) defines a facet of P_I. Recall that an inequality $\pi x \leq \pi_0$ is said to define a *face* of P if it is satisfied by every $x \in P$; it is said to define a *improper face* (*proper face*) of P if the equality $\pi x = \pi_0$ is (not) satisfied by every $x \in P$; it is said to define a *facet* of P if the polyhedron $P^\pi := \{x \in P | \pi x = \pi_0\}$ has dimension dim $P - 1$ [10].

Without loss of generality, we may assume that $D_I = D_J = D_K = \{1, 2, ..., r\}$, $H_I = H_J = \{r+1\}$, $H_K = \emptyset$. Let

$$\bar{S} = S \setminus (B(D) \cup F(Q, D))$$

and

$$P_I^C = \{x \in P_I : 2x(B(D)) + x(F(Q, D)) = 2r\}.$$

We first show:

Theorem 2.1. *The inequality (1.2) is valid and of Chvàtal rank* 1.

Proof: The inequality (1.2) can be obtained by multiplying all the equations of $Ax = e$ indexed $i \in D_I, j \in D_J, k \in D_K$ by $\frac{2}{3}$, and multiplying all the equations of $Ax = e$ indexed $i \in H_I, j \in H_J$ by $\frac{1}{3}$, and adding them together, dividing the resulted inequality by 2 and rounding down all coefficients to their nearest integers. Thus, (1.2) is a valid inequality of Chvátal rank at most 1.

We now prove that it is of Chvátal rank 1 and it induces a proper face of P_I. Let $x_{iii} = 1$ for $i = 1, ...r-1, r+3, ...n$ and $x_{r,r+2,r+2} = x_{r+1,r+1,r} = x_{r+2,r,r+1} = 1$, and $x_t = 0$ for all other t. Then $x \in P$ and

$$2x(B(D)) + x(F(Q, D)) = 2r - 1 < 2r.$$

So, (1.2) defines a proper face and is of Chvátal rank 1. ∎

Lemma 2.1. *For any $s = (i,j,k) \in \bar{S}$ and $(a,b,c) \in S$, $i \neq a \geq r+1, j \neq b \geq r+1$ and $k \neq c \geq r+1$, there exists an $x \in P_I^C$ such that $x_{ijk} = x_{abc} = 1$.*

Proof: Without loss of generality, assume $(a,b,c) = (n,n,n)$. There are four cases:

(i) $|s \cap Q| = 0$. Without loss of generality, assume $s = (r+2, r+2, r+2)$. Let $x_{iii} = 1, i = 1, \ldots, n$, and $x_t = 0$ for all other t. Then $x \in P_I^C$ and satisfies the requirements.

(ii) $|s \cap Q| = 1$, the common index is in D. Without loss of generality, assume $s = (r, r+2, r+2)$. Let $x_{iii} = 1, i = 1, \ldots, r-1, r+3, \ldots, n$ and $x_{r,r+2,r+2} = x_{r+2,r+1,r} = x_{r+1,r,r+1} = 1$, and $x_t = 0$ for all other t. Then $x \in P_I^C$ and satisfies the requirements.

(iii) $|s \cap Q| = 1$, the common index is in H. Without loss of generality, assume $s = (r+1, r+2, r+2)$. Let $x_{iii} = 1, i = 1, \ldots, r-1, r+3, \ldots, n$ and $x_{r,r,r+1} = x_{r+2,r+1,r} = x_{r+1,r+2,r+2} = 1$, and $x_t = 0$ for all other t. Then $x \in P_I^C$ and satisfies the requirements.

(iv) $i = j = r+1$. Without loss of generality, assume $k = r+1$. Using the same x in (i), we also get the conclusion. ■

Theorem 2.2. *Assume $n \geq r+4$. then (1.2) defines a facet of P_I.*

Proof: According to Theorem 2.1, (1.2) is of Chvátal rank 1, and does not induce an improper face of P_I. To show that (1.2) defines a facet of P_I, i.e. dim $(P_I^C) =$ dim $(P_I) - 1$, it suffices to show that if $\alpha x = \alpha_0$ for all $x \in P_I^C$, then it is a linear combination of the $3n$ equality constraints in (1.1) and the equality indexed by (1.2), i.e. there exist scalars $\lambda_i, i \in I, \mu_j, j \in J, \nu_k, k \in K$ and π such that

$$\alpha_{ijk} = \begin{cases} \lambda_i + \mu_j + \nu_k, & \text{if}(i,j,k) \in \bar{S}, \\ \lambda_i + \mu_j + \nu_k + \pi, & \text{if } (i,j,k) \in F(Q,D), \\ \lambda_i + \mu_j + \nu_k + 2\pi, & \text{if } (i,j,k) \in B(D), \end{cases} \tag{2.1}$$

and

$$\alpha_0 = \sum\{\lambda_i : i \in I\} + \sum\{\mu_j : j \in J\} + \sum\{\nu_k : k \in K\} + 2r\pi. \tag{2.2}$$

Define

$$\begin{aligned} \lambda_i &= \alpha_{inn} - \alpha_{nnn}, & i \in I, \\ \mu_j &= \alpha_{njn} - \alpha_{nnn}, & j \in J, \\ \nu_k &= \alpha_{nnk}, & k \in K. \end{aligned}$$

Then we have to show that for $(i, j, k) \in \bar{S}$,

$$\alpha_{ijk} = \lambda_i + \mu_j + \nu_k = \alpha_{inn} + \alpha_{njn} + \alpha_{nnk} - 2\alpha_{nnn}. \tag{2.3}$$

If at least two of the indices i, j, k are equal to n, (2.3) obviously holds. Consider $x \in P_I$ such that $x_{ijk} = x_{i'j'k'} = 1$. Define x' by $x'_{ijk} = x'_{i'j'k'} = 0$, $x'_{i'jk} = x'_{ij'k'} = 1$ and $x'_t = x_t$ otherwise. Then $x' \in P_I$. As in [3], we call the construction of x' from x a first index interchange on the triplets (i, j, k) and (i', j', k') (second and third index interchanges are defined analogously).

Now show (2.3) for $i = n, j \neq n, k \neq n$. Let $h = n - 1, l \neq k$, and $r + 2 \leq l \leq n - 1$. Then $(h, j, l) \in \bar{S}$. By Lemma 2.1, there is an $x \in P_I^C$, such that $x_{nnn} = x_{hjl} = 1$. Performing a second index interchange on (n, n, n) and (h, j, l), we get x' which is still in P_I^C. By $\alpha x = \alpha x'$, we have

$$\alpha_{nnn} + \alpha_{hjl} = \alpha_{njn} + \alpha_{hnl}. \tag{2.4}$$

Similarly, there is an $x \in P_I^C$ such that $x_{njk} = x_{hnl} = 1$. Performing a second index interchange on (n, j, k) and (h, n, l), we get x' which is still in P_I^C. By $\alpha x = \alpha x'$, we have

$$\alpha_{njk} + \alpha_{hnl} = \alpha_{nnk} + \alpha_{hjl}. \tag{2.5}$$

Summing (2.4) and (2.5), we have

$$\alpha_{njk} = \alpha_{njn} + \alpha_{nnk} - \alpha_{nnn}. \tag{2.6}$$

This proves (2.3) for $i = n$, $j \neq n$ and $k \neq n$. By symmetry, (2.3) holds when one of i, j and k is n.

Suppose now that $i \neq n$, $j \neq n$ and $k \neq n$. By Lemma 2.1, We have an $x \in P_I^C$ such that $x_{nnn} = x_{ijk} = 1$. Performing a first index interchange on (n, n, n) and (i, j, k), we get $x' \in P_I^C$. Thus, $\alpha x = \alpha x'$. This yields

$$\alpha_{nnn} + \alpha_{ijk} = \alpha_{inn} + \alpha_{njk},$$

i.e.,

$$\alpha_{ijk} = \alpha_{inn} + \alpha_{njk} - \alpha_{nnn}. \tag{2.7}$$

Combining (2.7) and (2.6), we get (2.3). This exhausts the cases $(i, j, k) \in \bar{S}$.

Next consider any $(i, j, k) \in F(Q, D)$. Define

$$\pi_{ijk} = \alpha_{ijk} - \lambda_i - \mu_j - \nu_k. \tag{2.8}$$

To prove the second equality of (2.1), we have to show that all π_{ijk} are equal for $(i, j, k) \in F(Q, D)$.

Let $u = (i_u, j_u, k_u), t = (i_t, j_t, k_t) \in S$, such that $u \subseteq Q$, $|u \cap D| \leq 2, t \subseteq R \setminus Q$, then it is easy to see that there exists an $x \in P_I^C$ such that $x_u = x_t = 1$. Define x' from x by first index interchange on u and t. We have $x'_{u'} = x'_{t'} = 1, x'_u = x'_t = 0$, where $u' = (i_t, j_u, k_u), t' = (i_u, j_t, k_t)$. Thus, $u, u' \in F(Q, D)$, $t, t' \in \bar{S}$. Hence, $x' \in P_I^C$ and $\alpha x = \alpha x'$. We obtain

$$\alpha_u + \alpha_t = \alpha_{u'} + \alpha_{t'}. \tag{2.9}$$

By (2.3),

$$\alpha_t = \lambda_{i_t} + \mu_{j_t} + \nu_{k_t}, \tag{2.10}$$

$$\alpha_{t'} = \lambda_{i_u} + \mu_{j_t} + \nu_{k_t}. \tag{2.11}$$

By (2.8),

$$\alpha_u = \pi_u + \lambda_{i_u} + \mu_{j_u} + \nu_{k_u}, \tag{2.12}$$

$$\alpha_{u'} = \pi_{u'} + \lambda_{i_t} + \mu_{j_u} + \nu_{k_u}. \tag{2.13}$$

Combining (2.9)-(2.13), we have

$$\pi_u = \pi_{u'}. \tag{2.14}$$

If we perform a second index exchange, we still have (2.14). If $|u \cap D| = 2$, we may also perform a third index exchange to get (2.14). This shows that π_{ijk} are equal for all $(i, j, k) \in F(Q, D)$ except the case of $i \in D_I, j \in D_J, k \in K \setminus D_K$. Denote this number by π, then

$$\alpha_{ijk} = \lambda_i + \mu_j + \nu_k + \pi \tag{2.15}$$

holds for these cases. Assume now $i \in D_I, j \in D_J, k \in K \setminus D_K$. It is easy to see that there exists an $x \in P_I^C$ such that $x_{ijk} = x_{r+1,r+1,r} = 1$. Performing a second index exchange on (i, j, k) and $(r+1, r+1, r)$, we have an $x' \in P_I^C$ such that $\alpha x = \alpha x'$. So,

$$\alpha_{ijk} + \alpha_{r+1,r+1,r} = \alpha_{i,r+1,k} + \alpha_{r+1,j,r}. \tag{2.16}$$

By (2.15) we have:

$$\alpha_{r+1,r+1,r} = \lambda_{r+1} + \mu_{r+1} + \nu_r + \pi, \tag{2.17}$$

$$\alpha_{i,r+1,k} = \lambda_i + \mu_{r+1} + \nu_k + \pi, \tag{2.18}$$

and

$$\alpha_{r+1,j,r} = \lambda_{r+1} + \mu_j + \nu_r + \pi. \tag{2.19}$$

Combining (2.16) to (2.19), we prove (2.15) also holds for $i \in D_I, j \in D_J, k \in K \setminus D_K$. This exhausts the cases of $(i,j,k) \in F(Q,D)$.

Now, let $u = (i,j,k) \subseteq D$, then there exists an $x \in P_I^C$ such that $x_{ijk} = x_{r+1,r+1,r+1} = 1$. Performing a third index exchange on (i,j,k) and $(r+1,r+1,r+1)$, we still get an $x' \in P_I^C$. Because $\alpha x = \alpha x'$, we have

$$\alpha_{ijk} + \alpha_{r+1,r+1,r+1} = \alpha_{i,j,r+1} + \alpha_{r+1,r+1,k}. \tag{2.20}$$

Since $(r+1,r+1,r+1) \in \bar{S}$, $(r+1,r+1,k)$ and $(i,j,r+1) \in F(Q,D)$, by the first two equations of (2.1), we have

$$\alpha_{r+1,r+1,r+1} = \lambda_{r+1} + \mu_{r+1} + \nu_{r+1}, \tag{2.21}$$

$$\alpha_{r+1,r+1,k} = \lambda_{r+1} + \mu_{r+1} + \nu_k + \pi, \tag{2.22}$$

and

$$\alpha_{i,j,r+1} = \lambda_i + \mu_j + \nu_{r+1} + \pi. \tag{2.23}$$

Combining (2.20) to (2.23), we have

$$\alpha_{ijk} = \lambda_i + \mu_j + \nu_k + 2\pi,$$

i.e. the third equality of (2.1) holds for any $(i,j,k) \in B(D)$.

Finally, let x be defined by $x_{iii} = 1$ for $i = 1, ..., n$, and $x_t = 0$ for all other t, then $x \in P_I^C$. Hence $\alpha x = \alpha_0$. This leads to (2.2) and completes the proof of the theorem. ■

3. A Linear-Time Separation Algorithm

Now we discuss the subclass of bull facets with $r = 1$.

We denote this subclass by $\mathcal{C}$.

Remark 3.1. In fact, $\mathcal{C}$ is also a subclass of the facet class identified by Balas and Saltzman in Theorem 6.9 of [3] with $p = 2$. In the proof of Theorem 6.9 of [3], it was mentioned that those facets are of Chvátal rank 2. This claim is incorrect for this subclass. As we proved in Theorem 2.1, the Chvátal rank of a bull facet is actually 1. In Theorem 6.9 of [3], actually, it was proved only that those facets have at most Chvátal rank 2. ■

Without loss of generality, we may assume a facet defining inequality in $\mathcal{C}$ has the form

$$2x_s + x(F(Q,s)) \leq 2, \tag{3.1}$$

where $s = (i_s, j_s, k_s) \in S, Q_I = \{i_s, i_q\}, Q_J = \{j_s, j_q\}, Q_K = \{k_s\}, i_s \neq i_q, j_s \neq j_q$, and

$$F(Q,s) = \{t \in S : |t \cap Q| \geq 2, 2 \geq |t \cap s| \geq 1\}.$$

For $i \in I, j \in J, k \in K$, define

$$x(i,j,K) = \sum \{x_{ijk'} : k' \in K\},$$

$$x(i,J,k) = \sum \{x_{ij'k} : j' \in J\},$$

and

$$x(I,j,k) = \sum \{x_{i'jk} : i' \in I\}.$$

Since each of the above three definitions sums up n components of x, we call each of them an *n-sum*. Also in each n-sum, only one index is summed up. So we call the other two indices *fixed indices* of that n-sum. We call i, j fixed indices of $x(i,j,K)$. Two n-sums are called *unrelated* if they do not have common fixed indices in between. Whereas two n-sums are called *related* if they have one common fixed index in between.

There are seven n-sums associated with (3.1). They are: $A_1 = x(i_s, j_s, K)$, $A_2 = x(i_s, j_q, K)$, $A_3 = x(i_q, j_s, K)$, $A_4 = x(i_q, J, k_s)$, $A_5 = x(i_s, J, k_s)$, $A_6 = x(I, j_q, k_s)$, and $A_7 = x(I, j_s, k_s)$.

Proposition 3.1 *Suppose that Q and s are given as above, x is a given non-integer point in P and x violates (3.1), i.e.,*

$$2x_s + x(F(Q,s)) > 2. \tag{3.2}$$

Then there are two distinct $A_l, l = 1, ..., 7$ with values greater than $1/4$.

Proof: In fact,

$$2x_s + x(F(Q,s)) = \sum_{i=1}^{7} A_i - x_s - 2x_{i_s j_q k_s} - 2x_{i_q j_s k_s} - x_{i_q j_q k_s}. \tag{3.3}$$

However,

$$A_4 + A_5 + A_6 + A_7 - x_s - x_{i_s j_q k_s} - x_{i_q j_s - k_s} - x_{i_q j_q k_s} \le \sum_{i,j} x_{ijk_s} = 1, \qquad (3.4)$$

where the equality is one of the constraints of (1.1). By (3.2), (3.3) and (3.4), $A_1 + A_2 + A_3 > 1$. Thus, at least one of A_1, A_2, A_3 is greater than $\frac{1}{3}$.

Similarly,

$$A_1 + A_2 + A_5 - x_s - x_{i_s j_q k_s} \le \sum_{j,k} x_{i_s jk} = 1,$$

and

$$A_1 + A_3 + A_7 - x_s - x_{i_q j_s k_s} \le \sum_{i,k} x_{ij_s k} = 1.$$

We have $A_3 + A_4 + A_6 + A_7 > 1$ and $A_2 + A_4 + A_5 + A_6 > 1$. Hence, at least one of A_3, A_4, A_6, A_7 is greater than $\frac{1}{4}$ and at least one of A_2, A_4, A_5, A_6 is greater than $\frac{1}{4}$. Since there is no common element among the three groups $\{A_1, A_2, A_3\}$, $\{A_3, A_4, A_6, A_7\}$ and $\{A_2, A_4, A_5, A_6\}$, the conclusion follows. ∎

We call an n-sum a *big n-sum* if its value is greater than $\frac{1}{4}$.

Proposition 3.2 *There are at most $O(n)$ big n-sums.*

Proof: We first consider n-sum $x(i, j, K)$. For any $x \in P$, since

$$\sum\{x_{ijk} : j \in J, k \in K\} = 1$$

and

$$\sum\{x_{ijk} : j \in J, k \in K\} = \sum\{x(i, j, K) : j \in J\}, \qquad (3.5)$$

for any fixed i_0, there are at most three $x(i_0, j, K)$ are big n-sums. Otherwise (3.5) will exceed 1. There are n such i_0's. So for all $i \in I$, there are no more than $3n$ $x(i, j, K)$ are big n-sums.

Analogously, for n-sums with the forms of $x(i, J, k)$ and $x(I, j, k)$, the same conclusion holds.

So, for all possible forms of n-sums, there are at most $O(n)$ of them are big n-sums. ∎

Proposition 3.3 *There are at most $O(n^2)$ pairs of unrelated big n-sums and $O(n)$ pairs of related big n-sums.*

Proof: The conclusion on unrelated big n-sum pairs is a direct corollary of Proposition 3.2.

Consider now related big n-sum pairs. As we showed in the proof of Proposition 3.2, for any fixed i_0, there are at most three $x(i_0, j, K)$ and three $x(i_0, J, k)$ big n-sums. So among them, there are at most $O(1)$ selections can be made to form a related big n-sum pair with a common fixed index i_0. There are n such i_0's. The same argument also works for related big n-sum pairs with common fixed indices in J and K. So, there are at most $O(n)$ related big n-sum pairs. ■

By Propositions 3.1 and 3.2, a linear-time separation algorithm for $\mathcal{C}$ follows in a straightforward manner.

We give a linear-time separation algorithm for the case with $Q_I = \{i_s, i_q\}, Q_J = \{j_s, j_q\}, Q_K = \{k_s\}$. There are two other cases: $Q_I = \{i_s, i_q\}, Q_J = \{j_s\}, Q_K = \{k_s, k_q\}$ and $Q_I = \{i_s\}, Q_J = \{j_s, j_q\}, Q_K = \{k_s, k_q\}$. Since they are symmetric, the algorithm can be applied to them without any difficulties.

We now describe the algorithm.

Algorithm 3.1. Suppose that x is a noninteger point in P.

Step 1. For all $i \in I$, $j \in J$ and $k \in K$, calculate all the n-sums $x(i, j, K)$, $x(i, J, k)$ and $x(I, j, k)$.

Step 2. Check all pairs of big n-sums. For each pair, if the pair is an unrelated one, add another adequate index in R to form Q and s; if the pair is a related one, add two other adequate indices in R to form Q and s. Consider all possible ways to form Q and s. Then use (3.3) to check whether (3.1) is violated in each case. ■

Theorem 3.1. *Algorithm 3.1 determines in $O(n^3)$ steps whether a given $x \in P$ violates a facet defining inequality (3.1).*

Proof: By Proposition 3.1, Algorithm 3.1 checks all possible situations when (3.1) may be violated. We now discuss the complexity of the procedure. Clearly, the complexity of Step 1 is $O(n^3)$.

According to Proposition 3.3, there are $O(n^2)$ unrelated and $O(n)$ related big n-sum pairs. For each unrelated big n-sum pair, there are four distinct fixed indices. There are $O(n)$ ways to select the additional index in R to form Q and s. So for the "unrelated" case, we have $O(n^3)$ ways to form Q and s. For each related big n-sum pair, there are three distinct fixed indices. There are $O(n^2)$ ways to select the two additional indices in R to form Q and s. So we also have $O(n^3)$ ways to form Q and s in the "related" case.

The testing time for (3.1) by using (3.3) is $O(1)$ since the values of A_i have already been calculated in Step 1. So, for Step 2, the complexity is also $O(n^3)$.

Therefore, the overall complexity of the algorithm is $O(n^3)$. This completes the proof. ■

4. Two-Tooth Comb Facets

In this section, we discuss a special case of comb facets, i.e. comb facets with only two teeth.

Let $s, t_1, t_2 \in S$ and assume that $s \cap t_1 = \emptyset, s \cap t_2 = \emptyset, t_1 \cap t_2 = \emptyset$. let $l \in t_1, D = s \cup \{l\}$,

$$C_1(D) = \{v \in S : v \subset D\},$$

$$C_2(D, t_m) = \{v \in S : |v \cap D| = 1, |v \cap t_m| = 2\},$$

$$S(t_1, l, L) = \{v \in S : |v \cap t_1| \geq 2, l \in v\},$$

where $m = 1, 2, L = I, J, K$, and $l \in L$. Let

$$C(D, t_1, t_2) = C_1(D) \cup C_2(D, t_1) \cup C_2(D, t_2) \cup S(t_1, l, L). \tag{4.1}$$

Then

$$x(C(D, t_1, t_2)) \leq 2 \tag{4.2}$$

is a comb inequality with two teeth.

We prove it is a facet of P_I. Define

$$P_I^C = \{x \in P_I : x(C(D, t_1, t_2)) = 2\}. \tag{4.3}$$

Certainly, this P_I^C is different from the one we used in Section 2. Since no confusion will occur, we use the same symbol. Without loss of generality, we may assume that $D_I = \{1, 2\}, D_J = D_K = \{1\}, t_1 = (2, 2, 2), t_2 = (3, 3, 3)$.

Theorem 4.1. *The inequality (4.2) is valid and of Chvàtal rank* 2.

Proof: For any $t \in S$, define

$$S(t) = \{t' \in S : |t \cap t'| \geq 2\}.$$

Then by [3], $x(S(t)) \le 1$ defines a facet of P_I with Chvátal rank 1. The inequality (4.2) can be obtained by adding the equations of $Ax = e$ indexed by $i \in D_I$, $j \in D_J$, $k \in D_K$, and twice the inequalities $x(S(t_1)) \le 1$ and $x(S(t_2)) \le 1$; dividing the resulting inequality by 3 and rounding down all coefficients to the nearest integers. Thus (4.2) is a valid inequality of Chvátal rank at most 2.

It suffices now to prove that it is not of rank 1. If (4.2) is of rank 1, then there exists ϵ, $0 < \epsilon < 1$, such that every solution to the linear system $Ax = e$, $x \ge 0$ satisfies

$$x(C(D, t_1, t_2)) \le 2 + 1 - \epsilon.$$

We now show that there is $x \in P$ such that

$$x(C(D, t_1, t_2)) = 2 + 1, \tag{4.4}$$

which proves the theorem.

Let $x_{iii} = 1$ for $i = 4, \ldots, n$. Let $x_{1,2,2} = x_{1,3,3} = x_{2,1,2} = x_{3,1,3} = x_{2,2,1} = x_{3,3,1} = \frac{1}{2}$, and $x_t = 0$ for all other t's. Then $x \in P$ and (4.4) holds. ■

Theorem 4.2. *Assume $n \ge 6$. If $C(D, t_1, t_2)$ is defined by (4.1) , then (4.2) defines a facet of P_I.*

Proof: According to Theorem 4.1, (4.2) is of Chvátal rank 2, so (4.2) is linearly independent from the constraints of $Ax = e, x \ge 0$. By (4.3) we can see that dim (P_I^C) is less than dim (P_I). Thus the inequality (4.2) does not induce an improper facet of P_I.

To show that (4.2) defines a facet of P_I, i.e. that dim (P_I^C) = dim $(P_I) - 1$, we use the same approach as Section 2 of this paper, i.e. we introduced scalars $\lambda_i, i \in I, \mu_j, j \in J, \nu_k, k \in K$ and π such that if $\alpha x = \alpha_0$ for all $x \in P_I^C$, then

$$\alpha_{ijk} = \begin{cases} \lambda_i + \mu_j + \nu_k, & \text{if } (i,j,k) \in S \setminus C(D, t_1, t_2), \\ \lambda_i + \mu_j + \nu_k + \pi, & \text{if } (i,j,k) \in C(D, t_1, t_2), \end{cases} \tag{4.5}$$

and

$$\alpha_0 = \sum\{\lambda_i : i \in I\} + \sum\{\mu_j : j \in J\} + \sum\{\nu_k : k \in K\} + p\pi. \tag{4.6}$$

Define

$$\begin{aligned} \lambda_i &= \alpha_{inn} - \alpha_{nnn}, && i \in I, \\ \mu_j &= \alpha_{njn} - \alpha_{nnn}, && j \in J, \\ \nu_k &= \alpha_{nnk}, && k \in K. \end{aligned}$$

The first equality of (4.5) can be proved in the same way as the proof of Theorem 2.2. The existence for relevant x and x' in P_I^C can be observed directly or from general lemmas in the next section. Hence we omit the proof here.

Consider any $(i,j,k) \in C(D,t_1,t_2)$. Define

$$\pi_{ijk} = \alpha_{ijk} - \lambda_i - \mu_j - \nu_k. \tag{4.7}$$

To prove the second equality of (4.5), we have to show that all π_{ijk} are equal for $(i,j,k) \in C(D,t_1,t_2)$.

Let $x \in P_I^C$ be such that $x_u = x_t = 1$, where $u = (i_u,j_u,k_u) \subseteq D, t = (i_t,j_t,k_t) = t_2$. It is easy to see this can be realized. Define x' from x by an index interchange on u and t. We have $x'_{u'} = x'_{t'} = 1, x'_u = x'_t = 0$, where $u' = (i_t,j_u,k_u), t' = (i_u,j_t,k_t)$. Thus, $u,t' \in C(D,t_1,t_2)$, $u',t \notin C(D,t_1,t_2)$. Hence, $x' \in P_I^C$ and $\alpha x = \alpha x'$. With the same arguments as (2.9)-(2.14), we have

$$\pi_u = \pi_{t'}. \tag{4.8}$$

Thus, all π_{ijk} are equal for $(i,j,k) \in C_1(D) \cup C_2(D,t_2)$. For $u \in C_2(D,t_1) \cup S(t_1,2,I)$, $t = t_2$, $u \cap t_2 = \emptyset$, $k_u \neq 3$ and we perform a first index interchange, (4.8) holds. We may also use $u = (2,3,2)$ and $t = (3,2,1)$, or $u = (2,2,3)$ and $t = (3,1,2)$, and perform a first index interchange. Then (4.8) still holds. Combining all these cases, we see that all π_{ijk} are equal for $(i,j,k) \in C(D,t_1,t_2)$. Then the second equality of (4.5) holds by (4.7).

Finally, let $x \in P_I^C$, $x \in \{0,1\}^{n^3}$. By $\alpha x = \alpha_0$, (4.3) and (4.5), we obtain (4.6). ■

Remark 4.1. We may regard D as the handle of a comb and t_1 and t_2 as two teeth. We also may regard the common index of D and t_1 as the link between them. The link is necessary. If we replace $D_I = \{1,2\}$ by $D_I = \{1,4\}$, we shall have no integer point x in P_I^C such that $x_{332} = 1$. Thus, the dimension of P_I^C will be less that that of P_I by more than 1 and the resultant inequality does not define a facet. ■

We shall discuss the separation algorithm for (4.2) elsewhere.

5. General Comb Facets

We now discuss the formulas of general comb facets. A subset of R is called a *uniform subset* of R if the cardinalities of the intersections of this set with I, J and K are the same.

Suppose that $D, T^1, ..., T^\nu \subseteq R$. Denote $T = \{T^1, ..., T^\nu\}$ and $T^0 = \cup_{\mu=1}^{\nu} T^\mu$. Here, we regard D as the handle and $T^1, ..., T^\nu$ as the teeth of the comb. Several restrictions are needed on the handle and the teeth.

(a). Each tooth is an odd set, i.e., for $\mu = 1, ..., \nu$ and $L = I, J, K$, $|T^\mu| = 2r_\mu + 1$, $1 \leq |(T^\mu)_L| \leq r_\mu$.

Let $r = \sum_{\mu=1}^{\nu} r_\mu$.

(b). Any pair of teeth are disjoint, i.e., $T^\mu \cup T^\rho = \emptyset$ for all $\mu \neq \rho$.

(c). The whole index set $D \cup T^0$ is a uniform subset of R, and $\sigma = |(D \cup T^0)_I| = |(D \cup T^0)_J| = |(D \cup T^0)_K| \leq n - 2$.

(d). The cardinality of the handle satisfies $|D| = 3p - 2r + 2$ for some $p \geq r$.

(e). There is at most one link, i.e., one common index of a tooth and the handle, i.e., for $\mu = 1, ..., \nu$, $|D \cap T^\mu| \leq 1$.

(f). If a tooth T^μ is linked, assume l_μ is the link and $l_\mu \in L_\mu$ where L_μ is one of I, J and K. Then $T^\mu_{L_\mu} = \{l_\mu\}$, i.e., l_μ is the sole index in $T^\mu_{L_\mu}$.

Lemma 5.1. *Let $D^0 = D \setminus T^0$. Under the above conditions, the number of unlinked teeth is $3\eta + 1$, where η is a nonnegative integer and*

$$\eta = \sigma - p - 1. \tag{5.1}$$

Thus at least one tooth is unlinked. Furthermore,

$$|(D^0)_L| \geq p - r + \eta + 1 \tag{5.2}$$

for $L = I, J, K$.

Proof: Let ζ be the number of linked teeth. By (c), (d) and (e),

$$3p - 2r + 2 = |D| = |D^0| + \zeta = 3\sigma - (2r + \nu) + \zeta.$$

Hence, the number of unlinked teeth is

$$\nu - \zeta = 3(\sigma - p - 1) + 1.$$

This proves (5.1). Thus, at least one tooth is unlinked. By (d),

$$|D^0| = |D| - \zeta \geq 3(p - r) + r + 2 - \zeta. \tag{5.3}$$

By (a), if we expand an odd set T^μ to a uniform subset of R without expanding the largest of $(T^\mu)_I$, $(T^\mu)_J$ and $(T^\mu)_K$, than we add at most $r_\mu - 1$ indices. So we can expand T^0 to a uniform subset of R by adding no more than $r - \nu$ indices. By (5.3), there are still at least $3(p-r)+\nu-\zeta+2 = 3(p-r+\eta+1)$ indices in $|D^0|$. By (c), these indices are uniformly distributed in I, J and K. This proves (5.2). ■

Without loss of generality, assume that the first ζ teeth are linked. Assume that the link for T^μ is l_μ and $l_\mu \in L_\mu$, where L_μ is I or J or K. Let

$$C_1(D) = \{t \in S : t \subseteq D\},$$

$$C_2(D, T_\mu) = \{t \in S : |t \cap D| = 1, |t \cap T^\mu| = 2\}$$

for $\mu = 1, ..., \nu$, and

$$S(T^\mu, l_\mu, L_\mu) = \{t \in S : |t \cap T^\mu| \geq 2, l_\mu \in t\}$$

for $\mu = 1, ..., \zeta$. Let

$$C(D,T) = C_1(D) \cup (\cup_{\mu=1}^{\nu} C_2(D, T^\mu)) \cup (\cup_{\mu=1}^{\zeta} S(T^\mu, l_\mu, L_\mu)).$$

Theorem 5.1. *Under the above conditions,*

$$x(C(D,T)) \leq p \tag{5.4}$$

defines a facet of P_I. Its Chvátal rank is 2.

Before proving this theorem, we prove several lemmas. Let

$$P_I^C = \{x \in P_I : x(C(D,T)) = p\}.$$

Without loss of generality, assume that $(D \cap T^0)_L = \{1, 2, ..., \sigma\}$ for $L = I, J, K$.

Lemma 5.2. *Under the above conditions, for any $s = (i,j,k) \in S \setminus C(D,T)$, where $i \neq n, j \neq n, k \neq n$, there exists an $x \in P_I^C$ such that $x_s = x_{nnn} = 1$.*

Proof: Call a tooth *s-free* if it is disjoint from s. Otherwise, call it *s-joint*. We may reduce σ in the following ways.

Suppose that $r_\mu \geq 2$ for some μ. Without loss of generality, assume that $|T_I^\mu| \geq |T_J^\mu| \geq |T_K^\mu|$. Choose $u = (i_u, j_u, k_u) \in S$ such that u is disjoint from s and any links, $i_u \in T_I^\mu$, $j_u \in T_J^\mu$ and $k_u \in D_K^0$. According to our assumption, this is possible. Add $x_u = 1$ to the constraints defines P and P_I. Then the situation is the same as if r_μ and thus σ have been reduced by 1. Hence it suffices to prove the claim for all $r_\mu = 1$.

If there is a linked, s-free tooth T^μ, by above assumption, we have $r_\mu = 1$. Assume $T^\mu = u \in S$. Add $x_u = 1$ to the constraints. Then the situation is the same but ν and σ have been reduced by 1. Thus, we may assume that all linked teeth are s-joint. There are at most three such teeth, i.e., $\zeta \leq 3$.

If $p > r$, by (5.2), we may choose $u \in S$ such that u is disjoint from s and $u \subset D^0$. As above, adding $x_u = 1$ to the constraints, we reduce p and thus σ by 1. Hence we may assume that $p = r$.

If $\eta \geq 1$, there are at least four unlinked teeth. At least there are three of them, say $T^{\zeta+1}$, $T^{\zeta+2}$ and $T^{\zeta+3}$, such that

$$|s \cap T^{\zeta+\mu}| \leq 1$$

for $\mu = 1, 2, 3$. We now can find u_0, u_1, u_2 and u_3 in S such that $u_0 \in D^0$, $|u_0 \cap u_\mu| = 1$, $|u_\mu \cap T^\mu| = 2$ for $\mu = 1, 2, 3$. Add $x_{u_\mu} = 1$, $\mu = 1, 2, 3$, to the constraints and remove the remaining three indices out of the comb. Then conditions (a)-(f) still hold but η has been reduced by 1, p has been reduced by 3 and σ has been reduced by 4. Thus, we may assume that $\eta = 0$.

If $\zeta = 0$, then we have the situation (1.3). Thus we may assume that $\zeta \geq 1$.

If $n \geq \sigma + 3$, we may pick $u = (i_u, j_u, k_u) \in S$ such that u is disjoint from s and $\sigma < i_u, j_u, k_u < n$. Again, adding $x_u = 1$ to the constraints, we reduce n by 1.

Now, we have only limited cases to analyze: $\zeta = 1, 2, 3$ with $p = r = \nu = \zeta + 1$, $\sigma = p + 1$ and $n = \sigma + 2$. By direct observation, we may draw the conclusion. ■

Lemma 5.3. *Under the above conditions, for any $s = (n, j, k) \in S \setminus C(D, T)$, where $j \neq n, k \neq n$. Let $h = n - 1$. Let $l = n - 1$ if $k \neq n - 1$ and $l \in (T^\nu)_K$ if $k = n - 1$. Then there exists an $x \in P_I^C$ such that $x_s = x_{hnl} = 1$ and an $\bar{x} \in P_I^C$ such that $\bar{x}_{nnn} = \bar{x}_{hjl} = 1$.*

Proof: Notice that T^ν is unlinked. Thus, $(h, j, l) \notin C(D, T)$. Thus, the conclusion on $\bar{x}$ is merely a corollary of Lemma 5.2. For x, if $k \neq n - 1$, then $(h, n, l) = (n-1, n, n-1)$. Since $n-1$ and n are equivalent with respect to $C(D, T)$, we may also draw the conclusion on x. If $k = n-1$, then $(h, n, k) = (n-1, n, n-1)$. Thus there exists $\hat{x} \in P_I^C$ such that $\hat{x}_{njl} = \hat{x}_{hnk} = 1$. Performing a third index interchange on $\hat{x}$, we get x. Since $(n, j, k), (n, j, l), (h, n, k), (h, n, l) \notin C(D, T)$, x is also in P_I^C. ■

Lemma 5.4. *Under the above conditions, there exists $x \in P$ such that*

$$x(C(D, T)) = p + 1.$$

■

To prove Lemma 5.4, we may use a reduction technique similar to that in proving Lemma 5.2 to reduce the case to a low-dimensional situation. Then we may construct x. We omit the proof detail here.

We are now ready to prove Theorem 5.1.

Proof of Theorem 5.1: For $\mu = 1, ..., \nu$, define

$$S(T^{\mu}) = \{t \in S : |t \cap T^{\mu}| \geq 2\}.$$

By [3], $x(S(T^{\mu})) \leq r_{\mu}$ defines a facet of P_I with Chvátal rank 1. The inequality (5.4) can be obtained by adding the equations of $Ax = e$ indexed by $i \in D_I$, $j \in D_J$, $k \in D_K$, and twice the inequalities $x(S(T^{\mu})) \leq r_{\mu}$ for $\mu = 1, ..., \nu$; dividing the resulting inequality by 3 and rounding down all coefficients to the nearest integer. Thus, (5.4) is a valid inequality of Chvátal rank at most 2.

By Lemma 5.4 and an argument similar to the proof of Theorem 4.1, we may show that the Chvátal rank of (5.4) is exactly 2.

To prove (5.4) defines a facet of P_I, we may follow the proof of Theorem 4.2 by invoking Lemmas 5.2 and 5.3 and assigning the role of t_1 to any linked tooth and the role of t_2 to any unlinked tooth. ■

Remark 5.1. The condition (d) is from the Chvátal procedure in the proof of Theorem 5.1. The conditions (a), (b), (c) and (f) are due to the need of the proof of Lemma 5.2. Lemma 5.2 is important. If for some $s \in S$, there is no $x \in P_I^C$ such that $x_s = 1$, then P_I^C is in the superplane $x_s = 0$. Then dim (P_I^C) will be less that dim (P_I) - 1, and (5.4) does not define a facet in this case. Remark 4.1 gives an example that if we relax the uniformity assumption in (c), then (5.4) does not define a facet. If we relax the condition (f), the following example may illustrate the situation. Let $D_I = \{1,2\}$, $D_J = \{1,2\}$ and $D_K = \{1\}$. Let $T_I^1 = \{2,3\}$, $T_J^1 = \{3\}$, $T_K^1 = \{2,3\}$. Let $T_L^2 = \{4\}$ for $L = I, J, K$. Then the conditions (a)-(e) are satisfied but not (f). (The link is $i = 2$.) Then there is no $x \in P_I^C$ such that $x_{344} = 1$. Thus, the condition (f) cannot be relaxed. Perhaps the condition (e) can be relaxed to allow more links for a tooth. We do not go into the details of this extension since the value of p will be too high to have efficient separation algorithm.

■

6. Final Remarks and Further Questions

In this paper, we have identified two more facet classes of the three-index assignment polytope, namely, bull facets and comb facets. The bull facet has Chvàtal rank 1, while the comb facet has Chvàtal rank 2. For a comb facet-defining inequality, the right-hand-side coefficient is a positive integer, and the left-hand-side coefficients equal to 0 or 1. For a bull facet-defining inequality, the right-hand-side coefficient is a positive even integer, and the left-hand-side coefficients equal to 0, 1 or 2. Furthermore, we give an $O(n^3)$ (linear-time) separation algorithm for the subclass of bull facets with the right-hand-side coefficient 2. Combining the results in [1] and [2], we now know five subclasses of facets of the three-index assignment plytope, which possess linear-time ($O(n^3)$) separation algorithms. Among these five subclasses of facets, two of them are clique facets, i.e., their defining inequalities have coefficients 0 and 1 only; three of them have right-hand-side coefficients 2 in their defining inequalities.

These raise a question: Do linear-time separation algorithms exist for any facet subclasses with right-hand-side coefficients $p = 2$? (The answer is "yes" for $p = 1$ by [1] and [3].) For example, the two-tooth comb facet subclass discussed in Section 4 is such a facet subclass. Furthermore, are there any other unknown facet subclasses with right-hand-side coefficients $p = 2$? Are there any other unknown facet classes with Chvàtal ranks 1 and 2? A further task is to apply the five facet subclasses with linear-time separation algorithms to a practical solution procedure for the three-index assignment problem. These need computational experiments.

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APPENDIX

One may find the following definition of *Chvàtal rank* on pages 210 and 226 of [10].

For $S = \{x \in Z_+^n : Ax \leq b\}$ where $A = (a_1, ..., a_n)$, we do followings to $Ax \leq b$:

1.

$$\sum_{j=1}^{n} (ua_j)x_j \leq ub, \forall u \in R_+^n;$$

2.

$$\sum_{j=1}^{n}(\lfloor ua_j \rfloor x_j \leq ub,$$

since x is nonnegative;

3.

$$\sum_{j=1}^{n}(\lfloor ua_j \rfloor x_j \leq \lfloor ub \rfloor,$$

since x is an integer vector.

The crucial step is step 3, where we invoke integrality to round down the right-hand side.

The valid inequality

$$\sum_{j=1}^{n}(\lfloor ua_j \rfloor x_j \leq \lfloor ub \rfloor$$

can be added to $Ax \leq b$, and then the procedure can be repeated by combining generated inequalities and/or original ones. This procedure is called the *Chvátal-Gomory rounding method*, and the inequalities it produces are called *C-G inequalities.*

We say that a valid inequality $\pi x \leq \pi_0$ for S is of *Chvátal rank* k if $\pi x \leq \pi_0$ is not equivalent to or dominated by any nonnegative linear combination of C-G inequalities, each of which can be determined by no more than $k-1$ applications of the C-G procedure, but is equivalent to or dominated by a nonnegative linear combination of some C-G inequalities that require no more than k applications of the procedure.

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