A COMPOSITION THEOREM FOR GENERALIZED BHASKAR RAO DESIGNS

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Abstract: Let H be a normal subgroup of a finite group G. We show that:
If a GBRD($v, k, \lambda; G/H$) exists and a GBRD($k, j, \mu; H$) exists then a GBRD($v, j, \lambda\mu; G$) exists. We apply this result to show that:

i) If k does not exceed the least prime factor of $|G|$, then a GBRD($k, k, \lambda; G$) exists for all $\lambda \equiv 0 \pmod{|G|}$;
ii) If G is of order $|G| \equiv 1$ or $5 \pmod 6$ then a GBRD($v, 3, \lambda = t|G|; G$), $v>3$, exists if and only if a BIBD($v, 3, t$) exists;
iii) If G is a nilpotent group of odd order then the necessary conditions are sufficient for the existence of a GBRD($v, 3, \lambda; G$); and,
iv) If G is a p-group, $p \neq 2$, then the necessary conditions are sufficient for the existence of a GBRD($v, 3, \lambda; G$).

1. Introduction

A *balanced incomplete block design*, BIBD(v, b, r, k, λ), is a design $(X, \mathcal{B})$ with v points and b blocks such that:

i) each point appears in exactly r blocks;
ii) each block contains exactly k $(<v)$ points; and
iii) each pair of distinct points appear together in exactly λ blocks.

As $r(k-1) = \lambda(v-1)$ and $vr = bk$ are well known necessary conditions for the existence of a BIBD(v, b, r, k, λ) we denote this design by BIBD(v, k, λ).

Throughout this paper we denote the identity matrix of size n by I_n and a $m \times n$ matrix in which each entry is 1 by $J_{m,n}$. The square matrix of size n whose entries are 1's is denoted by J_n.

Let $G = \{h_1 = e, h_2, ..., h_g\}$ be the finite group (with identity element e) of order $|G| = g$. Now form the $v \times b$ matrix W,

$$W = \sum_{i=1}^{g} h_i A_i,$$

with entries taken from G U{0}, and the $v \times b$ (0,1) - matrix N,

$$N = \sum_{i=1}^{g} A_i$$

where $A_1, \ldots, A_g$ are $v \times b$ (0,1) - matrices such that the Hadamard product $A_i * A_j = 0$ for any $i \neq j$. Let

$$W^+ = \left(\sum_{i=1}^{g} h_i^{-1} A_i \right)^T$$

If each column and row of N contains k and r 1's respectively and the matrices W and N satisfy the conditions

$$WW^+ = rI_v + \frac{\lambda \sum_{i=1}^{g} h_i}{g} (J_v - I_v)$$

$$NN^+ = (r - \lambda) I_v + \lambda J_v ,$$

then we say that W is a *generalized Bhaskar Rao design over* the group G *based* on the matrix N. We denote W by GBRD(v, b, r, k, λ; G). In W, we call the group element entries *non-zero* entries and the other entries *zero* entries.

If $k = v$ then $b = r$ and N is a $v \times b$ matrix whose entries are all 1's. If $k < v$, the second condition requires that N be the incidence matrix of BIBD(v, b, r, k, λ). In either case, we can use the shorter notation GBRD(v, k, λ; G) for a generalized Bhaskar Rao design over G based on N.

A GBRD(v, k, λ; G) with $v = b$ is a symmetric generalized Bhaskar Rao design or a *generalized weighing* matrix. A symmetric generalized Bhaskar Rao design which has no zero entries is also known as a *generalized Hadamard* matrix. A *Bhaskar Rao* design is a generalized Bhaskar Rao over Z_2.

The concept of a generalized Bhaskar Rao design has been extended to cover the case where N is the incidence matrix of a group divisible design (see Palmer (1989), (199)).

In this paper we give a new construction for generalized Bhaskar Rao designs and then use this construction to establish necessary and sufficient conditions for the existence of generalized Bhaskar Rao designs over large classes groups of odd order. We then show how to construct group divisible designs from these new generalized Bhaskar Rao designs.

2. The Construction

Let G be a finite group. Also let $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$, where x_i and y_i are any elements of G ∪ {0}, be row vectors of length n. For each vector, the group element entries are called *non-zero* entries and the other entries are called *zero* entries.

In the following: any product of elements of G ∪ {0} which involve a zero entry is a zero entry and if x is zero entry then x^{-1} is taken to be a zero entry.

If a belongs to G we write $a\mathbf{x}$ as $(ax_1, \dots, ax_n)$, $\mathbf{x}a$ as $(x_1a, \dots, x_na)$ and we define $\mathbf{y}^{-1}$ to be $(y_1^{-1}, \dots, y_n^{-1})$. The product $\mathbf{x}\mathbf{y}^T$ is defined to be the usual matrix product, $x_1y_1 + \dots + x_ny_n$, where the sum is taken to be in the group ring, $Z(G)$ of G over the integers, Z.

Theorem 2.1. *Let H be a normal subgroup of a finite group G. If a* GBRD$(v, b, r, k, \lambda; K{=}G/H)$, *$A$, exists and a* GBRD$(k, c, s, j, \mu; H)$, *$B$, exists, then a* GBRD$(v, bc, rs, j, \lambda\mu; G)$, *$C$, exists.*

Proof: Let t be the index of H in G and suppose that $S = \{ a_1 = e, a_2, \dots, a_t \}$ is a set of coset representatives in G. We observe that the non-zero entries of the matrix A are the cosets of a_lH, $l = 1, \dots, t$ and the non-zero entries of B are elements of the subgroup H. We denote the k rows of B by $\mathbf{a}_l$, $l = 1, \dots, k$.

We now form the matrix C from the matrices A and B. This is achieved by replacing each entry of A by a row vector of length c. In the first column of A replace the first non-zero entry, say a_lH, by the row vector $a_l\mathbf{a}_1$, the second non-zero entry, say a_mH, by the row vector $a_m\mathbf{a}_2$ and so on. This process is repeated for the remaining $b-1$ columns of A. Finally, we replace each zero entry of A by the row vector $(0, \dots, 0)$ consisting of c zero entries.

As A is based on a BIBD(v, b, r, k, λ) and B is based on a BIBD(k, c, s, j, μ) we observe that C is based on a BIBD$(v, bc, rs, j, \lambda\mu)$. We claim that C is a GBRD$(v, bc, rs, j, \lambda\mu; G)$.

Example 2.2. Take G to be the dicyclic group $Q_6 = \{1, x, \dots, x^5, y, xy, \dots, x^5y\}$ generated by x and y subject to the relations $x^6 = 1$, $y^2 = x^3$, $yxy^{-1} = x^{-1}$.

We now take H as $\langle x \rangle$, a normal subgroup of Q_6. Also Z_2 is isomorphic to G/H. From de Launey (1984) we have the GBRD $(7, 4, 2; Z_2)$

$$\begin{bmatrix} -1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & -1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & -1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & -1 \end{bmatrix}$$

where the zero entries are denoted by 0 and -1 and $+1$ are elements of Z_2. If we take 1 and y as coset representatives in G we obtain the GBRD$(7, 4, 2; G/H)$

$$A = \begin{bmatrix} yH & H & H & 0 & H & 0 & 0 \\ 0 & yH & H & H & 0 & H & 0 \\ 0 & 0 & yH & H & H & 0 & H \\ H & 0 & 0 & yH & H & H & 0 \\ 0 & H & 0 & 0 & yH & H & H \\ H & 0 & H & 0 & 0 & yH & H \\ H & H & 0 & H & 0 & 0 & yH \end{bmatrix}$$

As $Z_6 = Z_2 \times Z_3$ and consequently Z_2 is normal in Z_6, a GBRD(4, 3, 6; Z_6) can be obtained by combining a GBRD(4, 3, 2; Z_2) with a GBRD(3, 3, 3; Z_3) on application of Theorem 2.1. On noting that Z_6 is isomorphic to $<x>$ we exhibit a GBRD(4, 3, 6; $<x>$)

$$B = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & x^4 & x^2 & x^3 & x & x^5 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & x^2 & x^4 & 0 & 0 & 0 & x^3 & x & x^5 & x^3 & x & x^5 \\ 0 & 0 & 0 & 1 & x^2 & x^4 & x^3 & x^5 & x & 1 & x^2 & x^4 \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \\ \mathbf{a}_4 \end{bmatrix}$$

Finally, we apply Theorem 2.1. to obtain a GBRD(4, 3, 12; Q_6), C, which is shown below:

$$C = \begin{bmatrix} y\mathbf{a}_1 & 1\mathbf{a}_1 & 1\mathbf{a}_1 & 0 & 1\mathbf{a}_1 & 0 & 0 \\ 0 & y\mathbf{a}_2 & 1\mathbf{a}_2 & 1\mathbf{a}_1 & 0 & 1\mathbf{a}_1 & 0 \\ 0 & 0 & y\mathbf{a}_3 & 1\mathbf{a}_2 & 1\mathbf{a}_2 & 0 & 1\mathbf{a}_1 \\ 1\mathbf{a}_2 & 0 & 0 & y\mathbf{a}_3 & 1\mathbf{a}_3 & 1\mathbf{a}_2 & 0 \\ 0 & 1\mathbf{a}_3 & 0 & 0 & y\mathbf{a}_4 & 1\mathbf{a}_3 & 1\mathbf{a}_2 \\ 1\mathbf{a}_3 & 0 & 1\mathbf{a}_4 & 0 & 0 & y\mathbf{a}_4 & 1\mathbf{a}_3 \\ 1\mathbf{a}_4 & 1\mathbf{a}_4 & 0 & 1\mathbf{a}_4 & 0 & 0 & y\mathbf{a}_4 \end{bmatrix}$$

As H is a normal subgroup in the finite group $G = H \times K$ we see that Theorem 2.2 (and its Corollary 2.3) of Lam and Seberry (1984) are consequences of Theorem 2.1. Indeed the "direct product" construction contained in Theorem 2.2. of Lam and Seberry(1984) was the motivation for our "coset" construction contained in our Theorem 2.2.

Similarly, we have

Corollary 2.3. *Suppose that G is the semidirect product of subgroups H and K (with H normal in G). If the designs* GBRD(v, k, λ; K) *and* GBRD(k, j, μ; H) *exist then there exists a* GBRD(v, j, $\lambda\mu$; G).

Proof: As G is a semidirect product of H and K, H is normal in G, $HK = G$, $H \cap K = \{1\}$. Thus, by the Second Isomorphism Theorem, K is isomorphic to G/H. We now apply the construction contained in Theorem 2.1.

3. Existence Results

Theorem 3.1. *Suppose that G is a finite solvable group. Let p be the smallest prime factor of $|G|$. Then, if $k \leq p$, there exists a* GBRD(k, k, λ; G) *for all* $\lambda \equiv 0 \pmod{|G|}$.

Proof: Let G have a composition series $\{1\} = H_s < H_{s-1} < ... < H_0 = G$, where the factor groups $F_i = H_i / H_{i+1}$, $i = 0, \ldots, s-1$, are cyclic (and of prime order). It is well known (see, for example, Seberry (1979)) that, when q is prime, a GBRD(q, q, q; Z_q) exists.

Thus, for each i, $i = 0, \ldots, s\text{-}1$, we can construct a GBRD(k, k, $|F_i|$; F_i) by selecting $k \leq p$ distinct rows from a GBRD ($|F_i|$, $|F_i|$, $|F_i|$; F_i). We now apply Theorem 2.1. s times to the composition series $\{1\} = H_s < H_{s-1} < \ldots < H_0 = G$ to obtain a GBRD(k, k, $|G|$; G). Finally, we construct a GBRD(k, k, λ; G) by taking copies of a GBRD(k, k, $|G|$; G).

Corollary 3.2. *Suppose that G is a group of odd order. Then there exists a* GBRD(3, 3, λ; G) *for all* $\lambda \equiv 0 \pmod{|G|}$.

Proof: By the Feit-Thompson Theorem G is solvable and hence the result follows from Theorem 3.1.

Remark 3.3. An alternative derivation of Corollary 3.2. is found in Denes and Keedwell ((1974), Theorem 1.4.3.) wherein it was shown that:

If $G = \{ h_1 = e, h_2, \ldots, h_g\}$ is of odd order then

$$\begin{bmatrix} e & e \cdots e \\ e & h_2 \cdots h_g \\ e & h_2^2 \cdots h_g^2 \end{bmatrix}$$

is a GBRD(3, 3, $|G|$; G). The proof drew upon the well known fact that in G, $|G|$ odd, every element has a unique square root.

A generalized Bhaskar Rao design, GBRD(v, 3, λ; G), $v > 3$, is based on the incidence matrix of a BIBD(v, 3, λ; G) and so

$$\lambda(v-1) \equiv 0 \pmod 2 \qquad (3.1)$$

$$\lambda v(v-1) \equiv 0 \pmod 6 \qquad (3.2)$$

are necessary conditions of a GBRD(v, 3, λ; G), $v > 3$. Indeed, the conditions (3.1) and (3.2) are sufficient conditions for the existence of a BIBD(v, 3, λ; G) (Hall (1967), Theorem 15.4.5).

When $v = 3$, the GBRD(v, 3, λ; G) is based on the matrix $J_{3,\lambda}$; and (3.1) and (3.2) are satisfied. Thus,

$$v \geq 3 \qquad (3.3)$$

is a necessary condition for the existence of a GBRD(v, 3, λ; G).

The inner product of two distinct rows of a GBRD(v, 3, λ; G) is a multiple of group ring sum of the elements of G so we have the necessary condition:

$$\lambda \equiv 0 \pmod{|G|}.$$

We now establish necessary and sufficient conditions for the existence of GBRDs over two classes of groups of odd order.

Theorem 3.4. *Let G be a group of order* $|G| \equiv 1$ *or* $5 \pmod 6$. *If* $v > 3$, *then a* GBRD(v, 3, $t|G|$; G) *exists whenever a* BIBD(v, 3, t) *exists.*

Proof: As $|G| \equiv 1$ or $5 \pmod 6$ the necessary conditions for the existence of a GBRD(v, 3, $t|G|$; G) can be written as

$$v \geq 3$$
$$t(v-1) \equiv 0 \pmod 2$$
$$tv(v-1) \equiv 0 \pmod 6.$$

By Hanani's Theorem (Hall (1967), Theorem 15.4.5) these are the necessary and sufficient conditions for the existence of a BIBD(v, 3, t). Also a GBRD (3,3, $|G|$; G) exists as $|H|$ is odd (Corollary 3.2). As a BIBD(v, 3, t) and a GBRD (3,3, $|G|$; G) can be combined to construct a GBRD(v, 3, $t|G|$; G) on application of Corollary 2.3 of Lam and Seberry (1984) we have the result.

We now direct our attention to finite nilpotent groups. We first prove

Lemma 3.5. *Let F be a 3-group. Then the necessary conditions are sufficient for the existence of a* GBRD(v, 3, λ; F).

Proof: Consider a 3-group, F, of order 3^{t+1}, $t \geq 1$. If the parameters of a GBRD(v, 3, λ; F) satisfy the necessary conditions

$$v \geq 3$$
$$\lambda(v-1) \equiv 0 \pmod 2$$
$$\lambda v(v-1) \equiv 0 \pmod 6,$$

then the parameters of a GBRD(v, 3, $\lambda/3^t$; Z_3) satisfy the corresponding set of necessary conditions and hence exists (Seberry((1982), Theorem 5)).

But the 3 - group of order 3^t, H say, is normal in F; and, as $|H|$ is odd, we see, from Corollary 3.2, that a GBRD (3, 3, 3^t; H) exists.

Noting that Z_3 is isomorphic to F/H we apply Theorem 2.1 with the designs GBRD(v, 3,$\lambda/3^t$; F/H) and a GBRD(3, 3, 3^t; H) to construct a GBRD(v, 3, λ; F).

Theorem 3.6. *If G is a nilpotent group of odd order, then the necessary conditions are sufficient for the existence of a* GBRD(v, 3, λ; G).

Proof: The case $|G| \equiv 1$ or 5 (mod 6) is covered by Theorem 3.3. We direct our attention to the case $|G| \equiv 3$ (mod 6). As G is nilpotent then G is the direct product of its Sylow subgroups (Rotman (1965), Theorem 6.26). Thus $G = S \times C$ where S is the Sylow 3- subgroup of G and C, $|C| \equiv 1$ or 5 (mod 6), is a 3-complement of G.

If the parameters of a GBRD(v, 3, λ; G) satisfy the necessary conditions:

$$v \geq 3$$
$$\lambda \equiv 0 \pmod{|G|}$$
$$\lambda(v-1) \equiv 0 \pmod 2$$
$$\lambda v(v-1) \equiv 0 \pmod 6$$

then the parameters of a GBRD(v, 3, $\lambda / |C|$; S) satisfy the corresponding necessary conditions and hence exists by Lemma 3.5.

Now, as $|C| \equiv 1$ or 5 (mod 6) a GBRD(3,3,$|C|$; C) exists (Theorem 3.1). Thus, a GBRD(v, 3, λ; G) can be constructed on application of Theorem 2.1.

4. Application

As in Street and Rodger(1980) and Seberry (1982), we can construct the incidence matrix of a group divisible design from a GBRD(v, b, r, k, λ;G). Our existence results for generalized Bhaskar Rao designs over non-abelian groups gives rise to possibly new group divisible designs.

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