ON THE ACHROMATIC NUMBER OF THE CARTESIAN PRODUCT $G_1 \times G_2$

NAM-PO CHIANG AND HUNG-LIN FU[1]

Department of Applied Mathematics
National Chiao Tung University
Hsin-Chu, Taiwan, Republic of China

ABSTRACT. We study the achromatic number of the Cartesian product of graphs G_1 and G_2 and obtain the following results:

(i) $\max_{1\le t\le m} \min\{\lfloor \frac{mn}{t} \rfloor, t(m+n-1)-t^2+1\}$

$\ge \Psi(K_m \times K_n)$

$\ge \begin{cases} m+n-1 & \text{if } n > m = 2 \text{ or } m = n > 2 \text{ ; and} \\ 2n - \lceil \frac{n}{m-1} \rceil & \text{if } n > m > 2. \end{cases}$

Moreover, for $m = 2, 3$, the bounds give the exact achromatic numbers $\Psi(K_m \times K_n)$ if not both m and n are equal to 2.

(ii) $\Psi(G_1 \times G_2) \ge \Psi(K_m \times K_n)$ if $\Psi(G_1) = m$ and $\Psi(G_2) = n$.

(iii) $\Psi(P_\ell \times K_m) \le (\frac{m(m+1)}{2})^{1/2}(\Psi(P_\ell)+3)+1$ and

$\Psi(C_\ell \times K_m) \le (\frac{m(m+1)}{2})^{1/2}(\Psi(C_\ell)+3)+1$

where P_k, C_k and K_k are the path, the cycle and the complete graph of order k respectively.

1. Introduction

Let $G = (V, E)$ be a simple graph. A **k-coloring** of G is a surjection from V to the set $\{1, 2, ..., k\}$ (which represents colors) so that any two adjacent vertices in V receive different colors. Moreover, if for each pair of colors c_1 and c_2 there are adjacent vertices v_1 and v_2 so that v_i is colored with c_i, $i = 1, 2$, then the coloring is **complete**. The largest k such that there exists a complete k-coloring of G is the **achromatic number** $\Psi(G)$ of G.

Let $G_i = (V_i, E_i)$, $i = 1, 2$, be simple graphs. The **Cartesian product** $G_1 \times G_2$ is the graph with $V_1 \times V_2$ as vertex set, and the two vertices $v = (v_1, v_2)$ and $w = (w_1, w_2)$ are adjacent in $G_1 \times G_2$ whenever $v_1 = w_1$ and v_2 is adjacent to w_2 in G_2 or symmetrically if $v_2 = w_2$ and v_1 is adjacent to w_1 in G_1.

Suppose that $G = (V, E)$ is a graph where $V = \{v_1, v_2, ..., v_p\}$. Let $\Gamma(m) = \{\alpha_{r,s} : 1 \le r < s \le m\}$ be a set of $\binom{m}{2}$ permutations $\alpha_{r,s}$ on the set $\{1, 2, ..., p\}$.

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Then the **multipermutation graph**, $P_{\Gamma(m)}(G)$, is defined to be the graph consisting of m disjoint copies of G, say G^1, G^2, ..., G^m, together with $p \cdot \binom{m}{2}$ additional edges $e_t^{r,s}$, $t = 1, 2, ..., p$, where $e_t^{r,s}$ joins the vertex v_t of G^r with the vertex $v_{\alpha_{r,s}(t)}$ of G^s. It is clear that $P_{\Gamma(m)}(G)$ is isomorphic to $G \times K_m$ if all the $\alpha_{r,s}$ are the identity permutation on $\{1, 2, ...p\}$. If there is a $\Psi(G)$-coloring of G such that $v_{\alpha_{r,s}(i)}$ and v_i are in the same color class, $i = 1, 2, ..., p$, for each pair of r and s, then we say the multipermutation graph $P_{\Gamma(m)}(G)$ is **class-invariant**. For example, if each $\alpha_{r,s}$ is the identity permutation on $\{1, 2, ..., p\}$ then $P_{\Gamma(m)}(G)$ is a class-invariant multipermutation graph.

In [1,3,4], Bhave, Geller and Kronk, Harary and Hedetniemi gave some excellent results for the achromatic number of general graphs, but to determine the exact achromatic number, even for simple structures such as trees, is quite difficult.[2,5] Milazoo and Vacirca studied, in [6,7], the achromatic numbers of permutation graphs and $G \times K_m$ and obtained some results. In this paper, we study the achromatic number of the Cartesian product of graphs G_1 and G_2 and obtain the following results:

(i) $\max_{1 \le t \le m} \min\{\lfloor \frac{mn}{t} \rfloor, t(m+n-1) - t^2 + 1\}$

$\ge \Psi(K_m \times K_n)$

$$\ge \begin{cases} m+n-1 & \text{if } n > m = 2 \text{ or } m = n > 2 \text{ ; and} \\ 2n - \lceil \frac{n}{m-1} \rceil & \text{if } n > m > 2. \end{cases}$$

Moreover, for $m = 2, 3$, the bounds give the exact achromatic numbers $\Psi(K_m \times K_n)$ if not both m and n are equal to 2.

(ii) $\Psi(G_1 \times G_2) \ge \Psi(K_m \times K_n)$ if $\Psi(G_1) = m$ and $\Psi(G_2) = n$.

(iii) $\Psi(P_\ell \times K_m) \le (\frac{m(m+1)}{2})^{1/2}(\Psi(P_\ell) + 3) + 1$ and

$\Psi(C_\ell \times K_m) \le (\frac{m(m+1)}{2})^{1/2}(\Psi(C_\ell) + 3) + 1$

where P_k, C_k and K_k are the path, the cycle and the complete graph of order k respectively. These results improve the works of Milazoo and Vacirca appeared in [6,7].

2. The main results

Throughout this section, we assume that $m \le n$ and the vertex set of $K_m \times K_n$ is $\{(i, j) : 1 \le i \le m \text{ and } 1 \le j \le n\}$.

Lemma 2.1. $\Psi(K_m \times K_n) \le$

$$\max_{1 \le t \le m} \min\{\lfloor \frac{mn}{t} \rfloor, t(m+n-1) - t^2 + 1\}$$

Proof. Consider any complete $\Psi(K_m \times K_n)$-coloring of $K_m \times K_n$. Suppose that the number of vertices in the color class S with the least number of vertices is t. Since the independence number of $K_m \times K_n$ is m, we have $1 \le t \le m$. Every vertex in S is adjacent to $m + n - 2$ vertices not in S and each pair of vertices in S have exactly two adjacent vertices in common. Hence the number of vertices in $K_m \times K_n$ not in S but adjacent to a vertex in S is $t(m+n-2) - 2\cdot\binom{t}{2} = t(m+n-1) - t^2$. It follows that $\Psi(K_m \times K_n) \le t(m+n-1) - t^2 + 1$.

On the other hand, since each color class consists of at least t vertices, we have $\Psi(K_m \times K_n) \le \lfloor \frac{mn}{t} \rfloor$. Thus $\Psi(K_m \times K_n) \le \text{Min } \{\lfloor \frac{mn}{t} \rfloor \text{ , } t(m+n-1) - t^2 + 1\}$ and hence

$\Psi(K_m \times K_n) \le \max_{1 \le t \le m} \min\{\lfloor \frac{mn}{t} \rfloor, t(m+n-1) - t^2 + 1\}$

To see that the upper bound in Lemma 2.1 is best possible, let us consider the achromatic number of $K_m \times K_n$ for $m = 2, 3$. By Lemma 2.1, it is easy to see that $\Psi(K_2 \times K_n) \le n + 1$ if $n \ge 3$ and $\Psi(K_3 \times K_n) \le \begin{cases} 5 & \text{if } n = 3 \text{ ; and} \\ \lfloor \frac{3n}{2} \rfloor & \text{if } n > 3. \end{cases}$

Theorem 2.1.

(i) $\Psi(K_2 \times K_n) = n + 1$ if $n \ge 3$; and

(ii) $\Psi(K_3 \times K_n) = \begin{cases} 5 & \text{if } n = 3 \text{ ; and} \\ \lfloor \frac{3n}{2} \rfloor & \text{if } n > 3. \end{cases}$

Proof. For the proof, we need only give a complete $(n+1)$-coloring and a complete $\lfloor \frac{3n}{2} \rfloor$-coloring of $K_m \times K_n$ for $m = 2, 3$, respectively.

(i) Suppose $m = 2$. Let

$$f(i,j) = \begin{cases} i & \text{if } i = 1,2 \text{ and } j = 1; \text{ and} \\ 2 + k & \text{if } i = 1,2 \text{ and } j = 2,3,...,n. \end{cases}$$

where $k \equiv i + j - 2 \pmod{(n-1)}$ and $1 \le k \le n - 1$. By the definition of f, it is a routine matter to check that f is a complete $(n+1)$-coloring of $K_2 \times K_n$.

(ii) Suppose $m = 3$.

If $n = 3$, then let f be defined by
$f(1,1) = 1$, $f(2,1) = 2$, $f(3,1) = 3$,
$f(1,2) = 4$, $f(2,2) = 3$, $f(3,2) = 5$,
$f(1,3) = 5$, $f(2,3) = 4$, $f(3,3) = 2$.

It is clear that f is a complete 5-coloring.

For $n > 3$, we give a complete $\lfloor \frac{3n}{2} \rfloor$-coloring for each of the following two cases.

(a) If n is even, say $n = 2r$, then

$$f(i,j) = \begin{cases} i + 3s & \text{if } i = 1,2,3 \text{ and } j = 2s+1, s = 0,1,...,r-1; \text{ and} \\ k + 3s & \text{if } i = 1,2,3 \text{ and } j = 2s+2, s = 0,1,...,r-1. \end{cases}$$

where $k \equiv i + 1 \pmod 3$ and $1 \le k \le 3$.

(b) If n is odd, say $n = 2r + 1$, then $\lfloor \frac{3n}{2} \rfloor = 3r + 1$. Let

$$f(i,j)=\begin{cases} k+3s & \text{if } i=1,2,3 \text{ and } j=2s+2, s=0,1,...,r-1, \text{ and} \\ & (i,j)\neq(3,2); \\ i+3s & \text{if } i=1,2,3 \text{ and } j=2s+1, s=0,1,...,r-1, \text{ and} \\ & (i,j)\neq(2,1); \\ 3r+1 & \text{if } (i,j)=(2,1),(3,2) \text{ or } (1,n); \\ 2 & \text{if } (i,j)=(2,n); \text{ and} \\ 1 & \text{if } (i,j)=(3,n). \end{cases}$$

where $k\equiv i+1 \pmod 3$ and $1\leq k\leq 3$.

Since in both cases (a) and (b), each color class consists of at least two independent vertices (i.e. vertices not in the same row and not in the same column), it is clear that f is a complete $\lfloor\frac{3n}{2}\rfloor$ -coloring.

For $m\geq 4$, we can also get a lower bound for $\Psi(K_m\times K_n)$.

Theorem 2.2. Let $m\geq 4$. Then

$$\Psi(K_m\times K_n)\geq\begin{cases} m+n-1 & \text{if } m=n; \text{ and} \\ 2n-\lceil\frac{n}{m-1}\rceil & \text{otherwise.} \end{cases}$$

Proof. We give complete colorings corresponding to the two cases.

(i) Suppose $m=n$. Let

$$f(i,j)=\begin{cases} i & \text{if } i=1,2,...,m \text{ and } j=1; \\ m+k & \text{if } i=1,2,...,m-1 \text{ and } j=2,3,...,n \text{ except} \\ & j=n-i+1; \\ m & \text{if } i=2,3,...,m-1 \text{ and } j=n-i+1; \\ m+n-1 & \text{if } i=m \text{ and } j=2; \text{ and} \\ m-j+2 & \text{if } i=m \text{ and } j=3,4,...,n. \end{cases}$$

where $k\equiv i+j-2 \pmod{(n-1)}$ and $1\leq k\leq n-1$. By the definition of f, we can check that the given coloring is a complete $(m+n-1)$-coloring.

(ii) Suppose $m\neq n$.

(a) If $(m-1)|n$, say $n=q\cdot(m-1)$, then

$$f(i,j)=\begin{cases} j & \text{if } i=1 \text{ and } j=1,2,...,n; \\ j+1 & \text{if } i=2 \text{ and } j=1,2,...,q-1; \\ 1 & \text{if } (i,j)=(2,q); \\ (i-2)q+j & \text{if } i=3,...,m \text{ and } j=1,...,q; \text{ and} \\ n+k & \text{if } i=2,...,m \text{ and } j=q+1,...,n. \end{cases}$$

where $k\equiv i+j-q-2 \pmod{(n-q)}$ and $1\leq k\leq n-q$.

(b) If $(m-1)\not|n$, say $n=q\cdot(m-1)+r$ where $1\leq r<(m-1)$, then

$$f(i,j) = \begin{cases} j & \text{if } i = 1 \text{ and } j = 1, 2, ..., n; \\ j+1 & \text{if } i = 2 \text{ and } j = 1, 2, ..., q; \\ 1 & \text{if } (i,j) = (2, q+1); \\ (i-2)(q+1)+j & \text{if } i = 3, ..., r+1 \text{ and } j = 1, ..., q+1; \\ (i-2)q+r+j-1 & \text{if } i = r+2, ..., m \text{ and } j = 1, ..., q+1; \text{ and} \\ n+k & \text{if } i = 2, ..., m \text{ and } j = q+2, ..., n. \end{cases}$$

where $k \equiv i + j - q - 2 \pmod{(n-q)}$ and $1 \leq k \leq n - q$.

In both cases (a) and (b), f is a complete $(2n - \lceil \frac{n}{m-1} \rceil)$-coloring.

Theorem 2.3. If $\Psi(G_1) = m$ and $\Psi(G_2) = n$, then $\Psi(G_1 \times G_2) \geq \Psi(K_m \times K_n)$.

Proof. Consider a complete m-coloring and a complete n-coloring of G_1 and G_2 respectively. Let the color classes of G_1 and G_2 be $\sum_1 = \{S_1, S_2, ..., S_m\}$ and $\sum_2 = \{S'_1, S'_2, ..., S'_n\}$ respectively. Then the vertex set of $G_1 \times G_2$ is partitioned into independent sets $S_1 \times S'_1, ..., S_1 \times S'_n, ..., S_m \times S'_1, ..., S_m \times S'_n$.

Consider a complete $\Psi(K_m \times K_n)$-coloring f of $K_m \times K_n$. If we color all the vertices in $S_i \times S'_j$ with the color $f(i,j)$, $1 \leq i \leq m$ and $1 \leq j \leq n$, then we get a complete $\Psi(K_m \times K_n)$-coloring of $G_1 \times G_2$. This concludes the proof.

By Theorem 2.2 and Theorem 2.3, we can easily get the following

Corollary 2.1. If $\Psi(G_1) = m$ and $\Psi(G_2) = n$, then

$$\Psi(G_1 \times G_2) \geq \begin{cases} m+n-1 & \text{if } m = n > 2 \text{ or } n > m = 2; \text{ and} \\ 2n - \lceil \frac{n}{m-1} \rceil & \text{if } n > m > 2. \end{cases}$$

In [7], Milazzo and Vacirca got a lower bound for the achromatic number of $G \times K_m$.

Theorem 2.4. For every graph G and for every $m \geq 2$,

$$\lceil \frac{m}{\Psi(G)} \rceil \cdot \Psi(G) \leq \Psi(G \times K_m),$$

where the bound is best possible.(i.e. When $G = K_2$ and m is odd, $\Psi(G \times K_m)$ attains the bound.)

Comparing it with our result, we find that our bound improves their bound except when $\Psi(G) = 2$ and m is odd or $\Psi(G) = 3$ and $m = 4$, in which cases the bounds are equal.

As for a class-invariant multipermutation graph $P_{\Gamma(m)}(G)$, since the edges between different copies G^r and G^s do not join the vertices in different color classes, the coloring given above is still a proper and complete coloring. So we have the following

Corollary 2.2. Let G be any graph with $\Psi(G) = n \geq 2$ and $m \geq 2$. If $P_{\Gamma(m)}(G)$ is class invariant, then

$$\Psi(P_{\Gamma(m)}(G)) \geq \begin{cases} m+n-1 & \text{if } m=n>2 \text{ or either } m \text{ or } n \text{ is equal to 2 but not both;} \\ 2n-\lceil \frac{n}{m-1} \rceil & \text{if } n>m>2; \text{ and} \\ 2m-\lceil \frac{m}{n-1} \rceil & \text{if } m>n>2. \end{cases}$$

As was indicated by Milazoo and Vacirca in [6,7], there are some graphs G for which even for fixed $m \geq 2$ there does not exist a positive real number r such that $\Psi(G \times K_m) \leq r \cdot \Psi(G)$. However, they gave such number for $G = P_\ell$ and C_ℓ(P_ℓ and C_ℓ are a path and a cycle of order ℓ respectively).

Theorem 2.5. For $m \geq 2$, we have
(i) $\Psi(P_\ell \times K_m) \leq m \cdot \Psi(P_\ell)$, and
(ii) $\Psi(C_\ell \times K_m) \leq m \cdot \Psi(C_\ell)$.
Moreover, these bounds are attainable.

In [3] and [6], Geller and Kronk and Milazoo and Vacirca determined $\Psi(P_\ell)$ and $\Psi(C_\ell)$ independently.

Theorem 2.6. Let $M = \max\{n : \lceil \frac{n-1}{2} \rceil n \leq \ell\}$. Then

(i) For $\ell \geq 2$, $\Psi(P_\ell) = \begin{cases} M-1 & \text{if } M \text{ is odd and } \ell = \lceil \frac{M-1}{2} \rceil M; \\ M & \text{otherwise.} \end{cases}$

(ii) For $\ell \geq 3$, $\Psi(C_\ell) = \begin{cases} M-1 & \text{if } M \text{ is odd and } \ell = \lceil \frac{M-1}{2} \rceil M+1; \\ M & \text{otherwise.} \end{cases}$

In [1], Bhave gave an upper bound for the achromatic number.

Theorem 2.7. Let G be a graph of order p with maximum degree $\Delta(G)$. Then $\lceil \frac{\Psi(G)-1}{\Delta(G)} \rceil \cdot \Psi(G) \leq p$.

Now, we are ready to state and prove our other results.

Theorem 2.8. $\Psi(P_\ell \times K_m) \leq (\frac{m(m+1)}{2})^{1/2}(\Psi(P_\ell)+3)+1$ for $\ell \geq 3$.

Proof. It is clear that $P_\ell \times K_m$ is a graph of order $m\ell$ with maximum degree $m+1$. If $k((m+1)(k-1)+2) \leq m\ell < (k+1)((m+1)k+2)$, then
$\lceil \frac{((m+1)k+2)-1}{m+1} \rceil ((m+1)k+2) = (k+1)((m+1)k+2) > m\ell.$
Hence by Theorem 2.2., $\Psi(P_\ell \times K_m) \leq (m+1)k+1$,

But in this case, $\frac{k((m+1)(k-1)+2)}{m} \leq \ell < \frac{(k+1)((m+1)k+2)}{m}$. So,

$$\ell \geq \frac{(2(m+1)/m)^{1/2}k((2(m+1)/m)^{1/2}(k-1)+2(2/m(m+1))^{1/2})}{2}$$
$$= \frac{(2(m+1)/m)^{1/2}k((2(m+1)/m)^{1/2}k-(2/m(m+1))^{1/2}(m-1))}{2}.$$

Since $(2/m(m+1))^{1/2}(m-1) = ((2m^2-4m+2)/(m^2+m))^{1/2} < 2$,
$$\ell > \frac{(\lfloor(2(m+1)/m)^{1/2}k\rfloor - 1)(\lfloor(2(m+1)/m)^{1/2}k\rfloor - 2)}{2}.$$
Hence $\Psi(P_\ell) \geq \lfloor(2(m+1)/m)^{1/2}k\rfloor - 2 \geq (2(m+1)/m)^{1/2}k - 3$ and $\Psi(P_\ell \times K_m) \leq (m+1)k + 1 \leq ((m+1)m/2)^{1/2}(\Psi(P_\ell)+3)+1$.

For the same reason, we have

Theorem 2.9. $\Psi(C_\ell \times K_m) \leq ((m+1)m/2)^{1/2}(\Psi(C_\ell)+3)+1$.

The best upper bounds that we knew before for $P_\ell \times K_m$ and $C_\ell \times K_m$ are the bounds in Theorem 2.5. Comparing them with ours, we find that our bounds improve them asymptotically over $\frac{7m}{100} \cdot \Psi(G)$ for $\ell \geq 50$.

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