AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **58(2)** (2014), Pages 249–263

A construction of a class of graphs with depression three*

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Abstract

An *edge ordering* of a graph G is an injection $f : E \to \mathbb{R}$, the set of real numbers. A path in G for which the edge ordering f increases along its edge sequence is called an f-*ascent*; an f-ascent is *maximal* if it is not contained in a longer f-ascent. The *depression* of G is the smallest integer k such that any edge ordering f has a maximal f-ascent of length at most k. We provide a construction of a large class of graphs with depression three.

1 Introduction

An *edge ordering* of a graph G is an injection $f : E(G) \to \mathbb{R}$, the set of real numbers. Denote the set of all edge orderings of G by $\mathcal{F}(G)$. A path λ in G for which $f \in \mathcal{F}(G)$ increases along its edge sequence is called an f-*ascent*; an f-ascent is *maximal* if it is not contained in a longer f-ascent. The *flatness* of an edge ordering f, denoted by $h(f)$, is the length of a shortest maximal f-ascent of G. In [9] it was shown that for a given edge-ordering f of a graph G the problem of determining the value of $h(f)$ is NP-hard.

The *depression* of G was defined in [6] as $\varepsilon(G) = \max_{f\in\mathcal{F}(G)} \{h(f)\}$. The interpretation of the depression of a graph G is that any edge ordering f has a maximal f-ascent of length at most $\varepsilon(G)$, and $\varepsilon(G)$ is the smallest integer for which this statement is true.

Clearly, $\varepsilon(G) = 1$ if and only if K_2 is a component of G. Graphs with depression two were characterized in [6], while trees with depression three were characterized

* This paper is based on part of Mark Schurch's doctoral dissertation.
† Supported by an NSERC discovery grant.

in [10]. Graphs with depression three and no adjacent vertices of degree three or higher were characterized in [13]. In this paper we further investigate graphs with depression three and describe a construction of a large class of graphs with depression three, which includes cyclic graphs and graphs with adjacent vertices of high degree. This paper is based on part of the second author's dissertation [15].

2 Definitions and Background

We consider simple, finite graphs $G = (V(G), E(G))$. For basic graph theoretic definitions we refer the reader to the book [4] or any of its predecessors. The *open neighbourhood* of a vertex v of G is the set of all vertices adjacent to v and is denoted by $N_G(v)$, or just $N(v)$, and its *closed neighbourhood* is $N_G[v] = N[v] = N(v) \cup \{v\}$.

Consider two disjoint graphs G_1 and G_2 and vertices $v_i \in V(G_i)$. The *vertex-coalescence of* G_1 *and* G_2 *via* v_1 *and* v_2 is the graph obtained by identifying v_1 and v_2 to form a new vertex v, and is denoted $(G_1 \cdot G_2)(v_1, v_2 : v)$. In forming $G = (G_1 \cdot G_2)(v_1, v_2 : v)$, if v_2 is unimportant we also say we *attach* G_1 *to* G_2 *at* v_1, and if G is the resulting graph, we say that G contains G_1 *as an attachment at* v_1.

A *branch vertex* of a tree is a vertex with degree at least three. Let $B(T)$ and $L(T)$ respectively denote the sets of all branch vertices and all leaves of the tree T. For $v \in V(T)$ and $l \in L(T)$, a (v, l)-*endpath*, or v-*endpath* if l is unimportant, or *endpath* if neither v nor l is important, is a path P from v to l such that each internal vertex of P has degree two in T. A *spider* $S(a_1, a_2, ..., a_r)$ is a tree with exactly one branch vertex v and v-endpaths of lengths $1 \leq a_1 \leq a_2 \leq \cdots \leq a_r$, where $r = \deg v$.

Given an edge ordering f of the graph G, an f-ascent λ is simply called an *ascent* if the ordering is clear, and if λ has length k, it is also called a (k, f)-*ascent*. If the path λ with vertex sequence $v_0, v_1, ..., v_k$ or edge sequence $e_1, e_2, ..., e_k$ forms an f-ascent, we denote this fact by writing λ as $v_0v_1...v_k$ or $e_1e_2...e_k$. which $f \in \mathcal{F}(G)$ increases along the edges of P, is called a *u-v direct f-ascent*, or a *direct f-ascent* if u and v are clear, or simply a *direct ascent* if u, v, and f are clear.

We emphasize that to show that $\varepsilon(G) = k$, we must show that

(a) each edge ordering of G has a maximal ascent of length at most k – this shows that $\varepsilon(G) \leq k$,

(b) there exists an edge ordering f of G with no maximal ascents of length less than k, i.e. for which each (l, f)-ascent, where $l < k$, can be extended to a (k, f)-ascent – this shows that $\varepsilon(G) \geq k$.

The *height* of an edge ordering f, denoted $H(f)$, is the length of a longest f-ascent of G. In [2] the *altitude* of G was defined as $\alpha(G) = \min_{f \in \mathcal{F}(G)} \{H(f)\}$. The interpretation of the altitude of a graph G is that any edge ordering $f \in \mathcal{F}(G)$ has an f-ascent of length at least $\lambda(G)$, and $\lambda(G)$ is the largest integer for which this statement is true.

The study of lengths of increasing paths was initiated by Chvátal and Komlós [5] who posed the problem of determining the altitude of the complete graph. This is a difficult problem and $\alpha(K_n)$ is known only for $1 \leq n \leq 8$ (see [2, 5]). The altitude of graphs was also investigated in e.g. [1, 2, 3, 8, 9, 11, 14, 16].

3 Known Results

Let $\tau(G)$ denote the length of a longest path in G, called the *detour length* in G. If we assume that G is connected and of size at least two, then

$$2 \leq \varepsilon(G), \alpha(G) \leq \tau(G).$$

By taking the edge ordering f for the path P_n, $n \geq 3$, to increase along its edge sequence we see that $\varepsilon(P_n) = \tau(P_n) = n-1$. On the other hand, by taking the edge ordering for the path P_n, $n \geq 3$, as $1, n-1, 2, n-2, ..., \lceil \frac{n}{2} \rceil$ along its edge sequence, we see that $\alpha(P_n) = 2$.

If a connected graph G has a vertex v that is adjacent to u, w, where u, w are end-vertices or adjacent vertices of degree two, then in any edge ordering f of G, either u, v, w or w, v, u is a maximal $(2, f)$-ascent, hence $\varepsilon(G) = 2$. In [6] it was shown that the converse of this statement is also true, which gives the following characterization of graphs with depression two.

Theorem 1. [6] *If G is connected, then $\varepsilon(G) = 2$ if and only if G has a vertex adjacent to two end-vertices or to two adjacent vertices of degree two.*

It is reasonable to expect a link between the depression of a graph and the diameter of its line graph, and indeed the following result appeared in [6].

Theorem 2. [6] *If* $\operatorname{diam} L(G) = 2$, *then* $\varepsilon(G) \leq 3$.

However, the difference $\operatorname{diam} L(G) - \varepsilon(G)$ can be arbitrarily large, a result that easily follows from Theorem 1. Much harder to see is that the difference $\varepsilon(G) - \operatorname{diam} L(G)$ can also be arbitrarily large, as shown by Gaber-Rosenblum and Roditty in [7].

We see from Theorem 1 that if v is the central vertex of P_3 or any vertex of K_3, and G is any connected graph containing P_3 or K_3 as an attachment at v, then $\varepsilon(G) = 2$.

An interesting question arises from this result.

- If H is a graph with $\varepsilon(H) = k$ and $v \in V(H)$, what properties should H and v satisfy so that if we attach an arbitrary graph to H at v, the resulting graph has depression at most k?

To help answer this question, a k-*kernel* of a graph G is defined in [10] as a set $U \subseteq V(G)$ such that for any edge ordering f of G there exists a maximal (l, f)-ascent

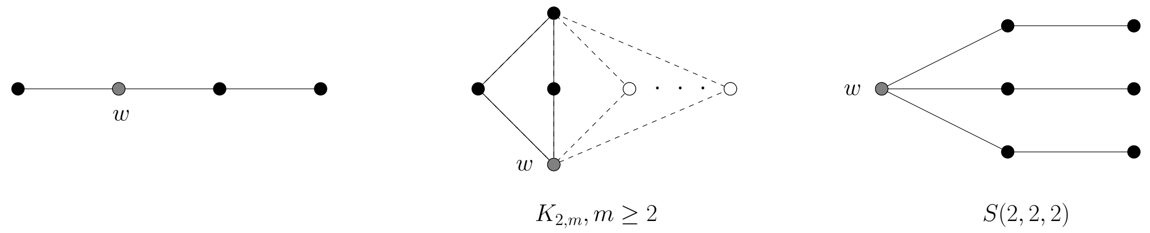

Figure 1: The set of graphs $\mathcal{H}$.

for some $l \leq k$ that neither starts nor ends at a vertex in U and k is the smallest value for which this is true. For example, it is easy to verify that any vertex of P_4 with degree two is a 3-kernel of P_4. If an f-ascent λ neither starts nor ends in a set $A \subset V(G)$, we say that λ is an A-*avoiding* (*maximal*) f-ascent or an a-*avoiding* (*maximal*) f-ascent if A contains a single vertex a (and λ is not contained in a longer f-ascent). The following theorem relates the concept of kernels to the question above.

Theorem 3. [10] *Let H be an arbitrary graph and let U be a k-kernel of H. Form a graph G by adding any set A of new vertices and arbitrary edges joining vertices in $U \cup A$. Then $\varepsilon(G) \leq k$.*

Therefore, if G has a non-empty k-kernel, Theorem 3 provides us with a method of forming a family of graphs with depression at most k. For example, if v is a vertex of P_4 with degree 2 and G is any graph that contains P_4 as an attachment at v, then by Theorem 3, $\varepsilon(G) \leq \varepsilon(P_4) = 3$.

The following theorem describes a necessary condition for a vertex v to be a k-kernel of a graph G with $\text{diam}(L(G)) = 2$, where $k \in \{2, 3\}$.

Theorem 4. [12] *Let G be a graph with* $\text{diam}(L(G)) = 2$. *If v is a vertex such that $N[v]$ is a vertex cover of G, then v is a k-kernel of G for some $k \in \{2, 3\}$.*

Theorem 4 allows one to construct a large class of graphs with depression three. For example, the line graph of any complete graph K_n with $n \geq 4$ has diameter two, and for any vertex $v \in K_n$, $N[v]$ is a vertex cover of K_n. Therefore, it follows from Theorem 4 that any graph G with an end-block $B \cong K_n$, where $n \geq 4$, has depression at most three.

Graphs with depression three and no adjacent vertices of degree three or more were characterized in [13].

Let $\mathcal{H}$ be the set of graphs consisting of P_4, $K_{2,m}$ for $m \geq 2$, and the spider $S(2, 2, 2)$– see Figure 1. For each graph in Figure 1 the vertex labelled w is a 3-kernel of its associated graph.

Theorem 5. [13] *Let G be a connected graph with* $\text{diam}(L(G)) \geq 3$, *no vertex adjacent to two end-vertices or to two adjacent vertices of degree two, and no adjacent vertices of degree three or more. Then $\varepsilon(G) = 3$ if and only if $G = S(2, 2, 2)$, or for some $H \in \mathcal{H}$, G contains H as an attachment at a vertex which is a 3-kernel of H.*

The following characterization of trees with depression three was given in [10].

Let $\mathcal{S}_k$ be the class of trees S_k, $k \geq 1$, that can be constructed recursively as follows. Let $S_0 = K_2$ with $V(S_0) = \{\alpha, \alpha'\}$. Define $U_0 = \emptyset$ and $Y_0 = \{\alpha\}$. Once S_i has been constructed, construct S_{i+1} by performing one of the following two operations.

O1: For any $y \in Y_i$, join y to the vertex u of a new edge ux; let $U_{i+1} = U_i \cup \{u\}$ and $Y_{i+1} = Y_i$.

O2: For any $y \in Y_i$, join y to the central vertex w of a new $P_5 : s, r, w, t, z$; let $U_{i+1} = U_i \cup \{w\}$ and $Y_{i+1} = Y_i \cup \{r, t\}$.

Let $\mathcal{S} = \bigcup_{k=1} \mathcal{S}_k$. Note that $S_0 = K_2$ is not in $\mathcal{S}$. For a tree $S \in \mathcal{S}$, define $U_S = U_k$. Let $\mathcal{G}$ be the class of all graphs G_S constructed as follows.

O3: Add any set $A = A(G_S)$ of new vertices to a tree $S \in \mathcal{S}$ and arbitrary edges between vertices in $A \cup U_S$.

Let $\mathcal{T} = \{T \in \mathcal{G} : T \text{ is a tree}\}$.

Theorem 6. [10] *For any tree T, $\varepsilon(T) = 3$ if and only if $T \in \mathcal{T}$ and no vertex of T is adjacent to two leaves.*

The main result of this paper is a generalization of this characterization of trees with depression three.

4 Main Result

In this section we provide a construction of a large class of graphs with depression three which includes acyclic graphs and graphs with adjacent vertices of high degree. The construction is a generalization of the construction used in [10] to characterize trees with depression three.

Let $\mathcal{S}'_k$ be the class of graphs S_k, $k \geq 1$, that can be constructed recursively in k steps as follows. Let $S_0 = K_2$ with $V(S_0) = \{x_0, y_0\}$. Define $U_0 = \emptyset$ and $Y_0 = \{y_0\}$. Once S_i has been constructed, construct S_{i+1} by performing one of the following five operations.

O1: For any $y \in Y_i$, join y to the vertex u_1 of a new edge u_1x_1; let $U_{i+1} = U_i \cup \{u_1\}$ and $Y_{i+1} = Y_i$.

O2: For any $y \in Y_i$, join y to the central vertex u_2 of a new $P_5 : x_2, y_2, u_2, y'_2, x'_2$; let $U_{i+1} = U_i \cup \{u_2\}$ and $Y_{i+1} = Y_i \cup \{y_2, y'_2\}$.

O3: For any $y \in Y_i$, join y to the vertices u_3 and v_3 of a new edge u_3v_3; let $U_{i+1} = U_i \cup \{u_3\}$ and $Y_{i+1} = Y_i$.

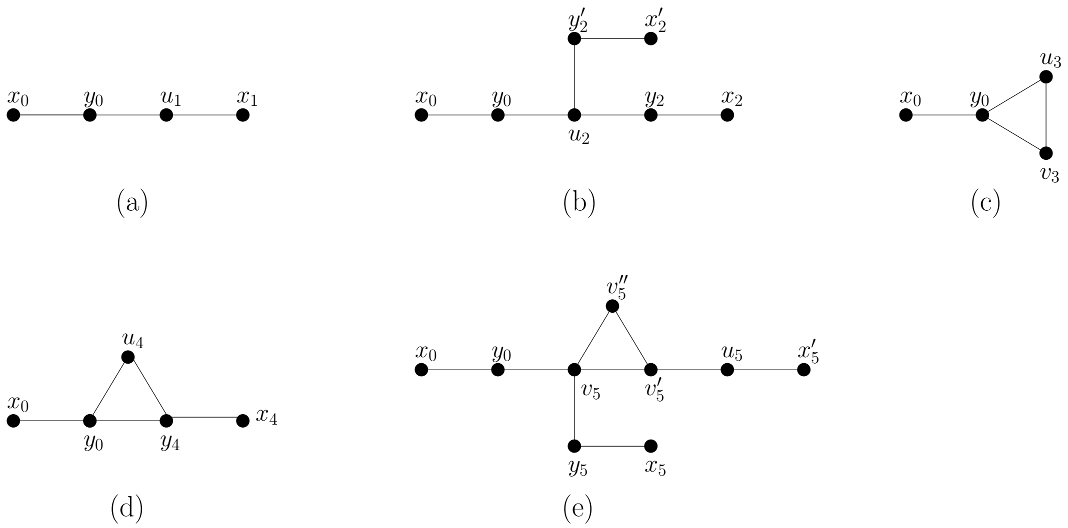

Figure 2: S_1 for each of the five operations **O1-O5**.

O4: For any $y \in Y_i$, join y to the central vertex y_4 and an end vertex u_4 of a new $P_3 : u_4, y_4, x_4$; let $U_{i+1} = U_i \cup \{u_4\}$ and $Y_{i+1} = Y_i$.

O5: For any $y \in Y_i$, join y to the vertex v_5 of the graph $G_5 = (\{x_5, x_5', v_5, v_5', v_5'', u_5, y_5\}, \{v_5y_5, y_5x_5, v_5v_5', v_5v_5'', v_5'v_5'', v_5'u_5, u_5x_5'\})$; let $U_{i+1} = U_i \cup \{u_5\}$ and $Y_{i+1} = Y_i \cup \{y_5\}$.

The operations **O1-O5** performed on S_0 are illustrated in Figure 2.

Let $\mathcal{S}_k$ be the family of graphs such that $S_k \in \mathcal{S}_k$ whenever $S_k \in \mathcal{S}_k'$ and in the construction of S_k, any vertex $y \in Y_k$ is involved in **O3** at most once. Define $\mathcal{S} = \bigcup_{k \geq 1} \mathcal{S}_k$. Note that $S_0 = K_2$ is not in $\mathcal{S}$. For a graph $S = S_k \in \mathcal{S}$, define $U_S = U_k$ and $Y_S = Y_k$. Let $\mathcal{G}$ be the class of all graphs G_S formed by performing the following two operations.

O6: Add any set $A = A(G_S)$ of new vertices to a graph $S \in \mathcal{S}$ and arbitrary edges between vertices in $A \cup U_S$.

O7: Add any arbitrary edges between vertices in Y_S.

Remark 7. *Let $S \in \mathcal{S}$. The operations **O1-O5** show that if $y \in Y_S$, then y is adjacent to exactly one vertex of degree one.*

We define the following property for a graph G.

P1: A graph G has property **P1** with respect to an edge ordering f and sets $U_G, Y_G \subseteq V(G)$, if for each $y \in Y_G$ for which a U_G-avoiding maximal $(2, f)$- or $(3, f)$-ascent ends (starts) at y, there exists a U_G-avoiding maximal $(2, f)$- or $(3, f)$-ascent for which its last (first) edge is assigned the largest (smallest) value under f over all edges incident with y.

Lemma 8. *If $S \in \mathcal{S}$ and f is an edge ordering of S for which there exists a U_S-avoiding maximal f-ascent of length at most three and all such ascents start or end in Y_S, then S has property **P1** with respect to f, U_S and Y_S.*

Proof. Let $y \in Y_S$ be a vertex for which a U_S-avoiding maximal $(2, f)$- or $(3, f)$-ascent ends at y, A_y be the set of all such f-ascents, and $\lambda = aby$ or $\lambda = acby$, where λ is the maximal f-ascent such that its last edge by is assigned the largest value over all edges of ascents in A_y. Let x be the end vertex adjacent to y. Clearly, $f(by) > f(yx)$.

Suppose to the contrary that $f(by) \neq \max_{v \in N(y)}\{f(vy)\}$. Then there exists an edge $wy \in E(S)$ such that $w \neq b$ and $f(wy) = \max_{v \in N(y)}\{f(vy)\}$. Since λ is a maximal f-ascent, w is a vertex of λ. By the construction of graphs in $\mathcal{S}$, all cycles of S have length three and we may assume that wby is a 3-cycle. If the cycle was introduced by **O3**, then $\lambda = wby$, $b \in U_S$, $w \notin U_S \cup Y_S$, and both w and b have degree 2. But since $f(yw) > f(wb)$ and $\deg(w) = 2$, xyw is a $U_S \cup Y_S$-avoiding maximal f-ascent, a contradiction.

Suppose then that the cycle wby was introduced by **O4**. Then $w \in Y_S$ and there exists an end vertex x' adjacent to w. If $f(x'w) < f(wy)$, then $x'wy$ is a maximal f-ascent, which contradicts our choice of λ. Now if $f(x'w) > f(wy)$, then $xywx'$ is a maximal f-ascent which is also a contradiction.

A similar argument may be used to show that if a U_S-avoiding maximal f-ascent of length at most three starts at y, then there exists a U_S-avoiding maximal $(2, f)$- or $(3, f)$-ascent λ such that for the initial edge yb of λ, $f(yb) = \min_{v \in N(y)}\{f(yv)\}$. □

Theorem 9. *For each $S \in \mathcal{S}$, $\varepsilon(S) \leq 3$ and U_S is a k-kernel of S for some $k \in \{2, 3\}$.*

Proof. The proof is by induction on k, the number of steps used to construct $S = S_k$ from $K_2 = S_0$. To prove the result we must show that for any edge ordering f of S there exists a U_S-avoiding maximal $(2, f)$- or $(3, f)$-ascent.

If $k = 1$, then S was constructed by performing one of the operations **O1**-**O5** on $K_2 = S_0$

Case 1 **O1** is performed. Then $S = P_4$ and $U_S = \{u_1\}$. Since $\mathrm{diam}(L(S)) = 2$ and $N[u_1]$ is a vertex cover of S, the result follows from Theorem 4.

Case 2 **O2** is performed. Then $S = S(2, 2, 2)$ and $U_S = \{u_2\}$. Consider any edge ordering f of S. Without loss of generality we may assume $f(x_0y_0) < f(y_0u_2)$. If $f(y_0u_2) > y(u_2y_2)$, then either $x_2y_2u_2y_0$ (if $f(x_2y_2) < f(y_2u_2)$) or $y_2u_2y_0$ (if $f(x_2y_2) > f(y_2u_2)$) are u_2-avoiding maximal f-ascents of S with length at most three. The same argument applies if $f(y_0u_2) > f(u_2'y_2')$. Suppose then that $f(y_0u_2) < f(u_2y_2)$ and $f(y_0u_2) < f(u_2'y_2')$. To avoid a u_2-avoiding maximal f-ascents of length at most three, both $x_0y_0u_2x_2y_2$ and $x_0y_0u_2x_2'y_2'$ are maximal $(4, f)$-ascents of S. This implies either $y_2u_2y_2'x_2'$ (if $f(y_2u_2) < f(u_2y_2')$) or $y_2'u_2y_2x_2$ (if $f(y_2u_2) > f(u_2y_2')$) is a u_2-avoiding maximal f-ascent of the required length.

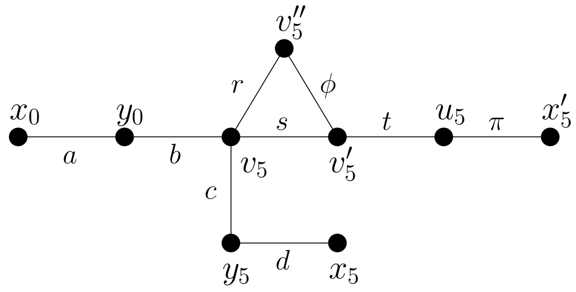

Figure 3: Operation O5 is performed, and the paths $abcd$ and rst are f-ascents of S.

Case 3 O3 is performed. Then $U_S = \{u_3\}$. Since $\mathrm{diam}(L(S)) = 2$ and $N[u_3]$ is a vertex cover of S, the result follows from Theorem 4.

Case 4 O4 is performed. Then $U_S = \{u_4\}$. Since $\mathrm{diam}(L(S)) = 2$ and $N[u_4]$ is a vertex cover of S, once again, the result follows from Theorem 4.

Case 5 O5 is performed. Then $U_S = \{u_5\}$. Suppose to contrary that u_5 is not a 3-kernel of S. Let f be an edge ordering f of S for which all maximal $(2, f)$- and $(3, f)$-ascents either start or end at u_5. Necessarily, either $x_0y_0v_5y_5x_5$ or its reverse is a $(4, f)$-ascent of S, and without loss of generality we assume the former. Furthermore, by our assumption, neither $v_5''v_5v_5'$ nor its reverse is a maximal $(2, f)$-ascent of S, which implies either $v_5''v_5v_5'u_5$, $v_5''v_5v_5'u_5x_5'$, or the reverse of one of these paths is a maximal f-ascent. We need only consider the former two of these cases since for any f-ascent present in an edge ordering extended from these cases, its reverse will be present in one of the latter cases–with the roles of x_0 and y_0 switched with x_5 and y_5 respectively. These cases are shown in Figure 3 where the paths labelled $abcd$ and rst are f-ascents of S. Moving forward we will refer to the labels in this figure to simplify notation.

Firstly, suppose rst is a maximal f-ascent. Then $t > \pi$ and, since u_5 is not a 3-kernel of S, $\pi t \phi r$ is a $(4, f)$-ascent. But then $t < \phi < r < s < t$, which is a contradiction.

Secondly, suppose that $rst\pi$ is an f-ascent of S. If $r < b$, then since $t > r$, either rb (if $\phi > r$) or ϕrb (if $\phi < r$) is a maximal f-ascent, which in either case is a contradiction. Therefore we may assume $r > b$. We may also assume that $\phi > r$, or else abr is a u_5-avoiding maximal f-ascent. Furthermore, if $c > r$, then rcd is a maximal f-ascent, so we may assume $c < r$. Now if $\phi < s$, then ϕs is a u_5-avoiding maximal f-ascent, which is a contradiction. Thus we may assume $\phi > s$. Since $r < s$ by assumption, we now have $c < r < s < \phi$, which implies that $cs\phi$ is a maximal f-ascent, and again we have a contradiction.

This case completes the basis step of the proof.

Assume the result to be true for graphs in $\mathcal{S}$ constructed from K_2 in fewer than $k \geq 2$ steps. Consider any graph $S = S_k$ constructed from K_2 in k steps, and any edge ordering f of S.

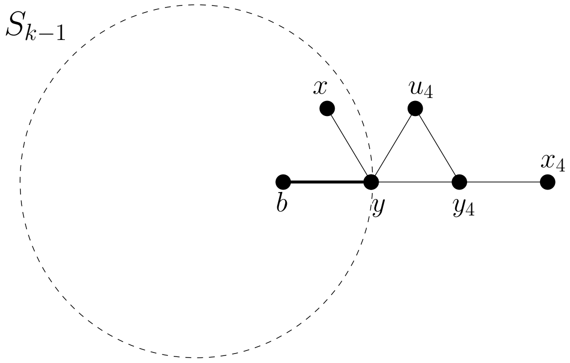

Figure 4: S is constructed by joining y to y_4 and u_4 of a new $P_3 : u_4, y_4, x_4$.

Suppose that in the construction of S one of **O1**, **O2** or **O5** was performed at least once. Then S contains $y \in Y_S$ such that y was joined to a new vertex in step $i \geq 2$ and such that y is incident with at least two bridges. Let $y \in Y_S$ be incident to at least two bridges, and x be the vertex of degree one adjacent to y. Note that one of the bridges incident with y is xy. Let $G_1, G_2, ..., G_m$ be the components of $S - y$ which consist of at least two vertices. For each $1 \leq i \leq m$, let G'_i be the subgraph induced by $\{x, y\} \cup V(G_i)$. Then each $G'_i \in \mathcal{S}_j$ for some $1 \leq j < k$. If $G'_i \cong S_j \in \mathcal{S}_j$, then let $U_{G'_i} = U_j$ and f'_i be the edge ordering of G'_i induced by f.

Since y is incident with a bridge other than xy, there exists an i, say $i = 1$, such that $\deg_{G'_1}(y) = 2$. Let $H = S - G_1$ and f_H be the edge ordering of H induced by f. Then $H \cong S_j \in \mathcal{S}_j$ for some $1 \leq j < k$. Let $U_H = U_j$. By the induction hypothesis there exists at least one U_H-avoiding maximal $(2, f_H)$- or $(3, f_H)$-ascent and we may assume that all such maximal f_H-ascents start or end at y, or else there exists a U_S-avoiding maximal f-ascent of length at most three in S and we are done. Without loss of generality assume that there exists a U_H-avoiding maximal f_H-ascent of length at most three which ends at y. Then by Lemma 8 there exists a maximal f_H-ascent $\lambda = aby$ or $\lambda = acby$ such that $f_H(by) = \max_{v \in N(y)}\{f_H(vy)\}$ and $a \in V(H) - U_H$.

Let b_1 be the neighbour of y in G_1. By the induction hypothesis, there exists at least one $U_{G'_1}$-avoiding maximal $(2, f'_1)$- or $(3, f'_1)$-ascent and we may assume that all such maximal f'_1-ascents start or end at y, or else we are done. Thus either b_1y is the initial or final edge of a $U_{G'_1}$-avoiding maximal f'_1-ascent α of length at most three. If α starts at y, then $f'_1(b_1y) < f(xy) < f(by)$ and λ is a U_S-avoiding maximal f-ascent of length at most three. If α ends at y, then in S either α (if $f'_1(b_1y) > f_H(by)$) or λ (if $f'_1(b_1y) < f_H(by)$) is a U_S-avoiding maximal f-ascent of length at most three.

Suppose then that only **O3** and **O4** are used in the construction of S.

Firstly, suppose that S is constructed from S_{k-1} by joining y to y_4 and u_4 of a new $P_3 : u_4, y_4, x_4$ (see Figure 4). Then $U_S = U_{k-1} \cup \{u_4\}$. Let f' be the edge ordering of S_{k-1} induced by f, and x the end vertex adjacent to y. By the induction hypothesis, in S_{k-1} there exists a U_{k-1}-avoiding maximal f'-ascent of length at most three. We may assume that all such f'-ascents start or end at y or else we are done. Without loss of generality assume that there exists a U_{k-1}-avoiding maximal f'-ascent of length at most three which ends at y. By Lemma 8 there exists a maximal f'-ascent $\lambda = aby$ or $\lambda = acby$ such that $f'(by) = \max_{v\in N(y)}\{f'(vy)\}$ and $a \in V(S_{k-1}) - U_{k-1}$. If λ is a maximal f-ascent, then we are done so we may assume that either

$$f(yu_4) > f(by) \text{ or } f(yy_4) > f(by). \tag{1}$$

- Suppose $f(yu_4) > f(by)$. Then $f(yu_4) = \max_{v\in N(y)-y_4}\{f(vy)\}$.
    - If $f(y_4u_4) < f(u_4y)$, then either y_4u_4y or $x_4y_4u_4y$ is a U_S-avoiding maximal f-ascent.
    - Suppose $f(y_4u_4) > f(u_4y)$. Then $f(x_4y_4) > f(y_4u_4)$, or else $x_4y_4u_4$ is a U_S-avoiding maximal f-ascent.
        - If $f(yy_4) > f(y_4x_4)$, then $f(yy_4) = \max_{v\in N(y)}\{f(vy)\}$ and x_4y_4y is a U_S-avoiding a maximal f-ascent.
        - If $f(yy_4) < f(y_4x_4)$, then either xyy_4x_4 (if $f(xy) < f(yy_4)$) or y_4yx (if $f(xy) > f(yy_4)$) is a U_S-avoiding maximal f-ascent.
- Suppose then that $f(yu_4) < f(by)$. Then by (1), $f(yy_4) > f(by)$ and $f(yy_4) = \max_{v\in N(y)}\{f(vy)\}$. This implies either xyy_4x_4 (if $f(yy_4) < f(y_4x_4)$) or x_4y_4y (if $f(yy_4) > f(y_4x_4)$) is a maximal f-ascent, neither of which starts or ends in U_S.

Secondly, suppose that S is constructed from S_{k-1} by joining $y \in Y_{k-1}$ to the vertices v_3 and u_3 of a new edge u_3v_3. Then $U_S = U_{k-1} \cup \{u_3\}$. Let S' be the subgraph of S induced by $\{x, y, v_3, u_3\}$, f' the edge ordering of S' induced by f, and f'' the edge ordering of S_{k-1} induced by f. Note that $S' \cong S_1 \in \mathcal{S}_1$. Let $U_{S'} = \{u_3\}$. By the induction hypothesis, there exists a u_3-avoiding maximal f'-ascent α of length at most three. We may assume that α either starts or ends at y, or else we are done. Without loss of generality assume that α starts at y. Necessarily, $f'(yx) > f'(yu_3)$ and $\alpha = yu_3v_3$. Furthermore, we may assume that $f'(yv_3) > f'(yu_3)$, or else $f'(yv_3) < f(yu_3) < f(u_3v_3)$ and v_3yx is a U_S-avoiding maximal f-ascent of length two and we are done. Thus $f'(yu_3) = \min_{v\in N(y)}\{f'(vy)\}$.

By the induction hypothesis, there exists a U_{k-1}-avoiding maximal f''-ascent λ of length at most three in S_{k-1}. We may assume that λ starts or ends at y or else we are done. If λ starts at y, then by Lemma 8 there exists a maximal f''-ascent $\lambda' = aby$ or $\lambda' = acby$ such that $f''(by) = \min_{v\in N(y)}\{f''(vy)\}$ and $a \in V(S_{k-1}) - U_{k-1}$. This implies either λ' or α is a U_S-avoiding maximal f-ascent of length at most three. Assume then that λ ends at y, and furthermore, that all U_{k-1}-avoiding maximal f''-ascents of length at most three end at y. Then there exists an edge $vy \in E(S_{k-1})$ such that $f''(vy) < f'(yu_3)$ otherwise α is a U_S-avoiding maximal f-ascent of length two and we are done. Let wy be the edge in S_{k-1} such that $f''(wy) = \min_{v\in N(y)}\{f''(vy)\}$. Then $f''(wy) < f'(yu_3) < f'(yv_3)$ which implies $f(wy) = \min_{v\in N(y)}\{f(vy)\}$. Recall

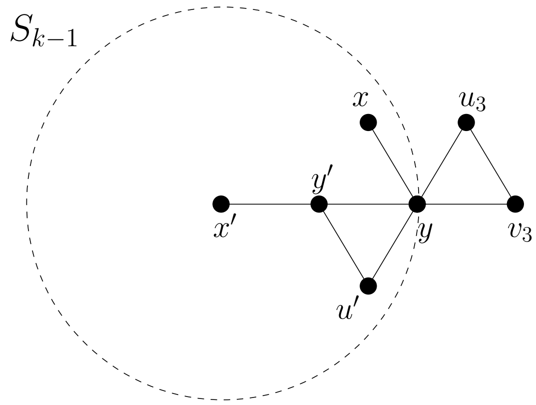

Figure 5: S is constructed from S_{k-1} by joining y to u_3 and v_3 of a new edge $\{u_3, v_3\}$.

that we have assumed S is constructed using only **O3** and **O4**, and that for any graph in $\mathcal{S}$, each vertex in $y \in Y_S$ is involved in **O3** at most once. Thus the edge wy was introduced by **O4**, which implies either $w = u' \in U_{k-1}$ and is adjacent to a vertex $y' \in Y_{k-1}$, or $w = y' \in Y_{k-1}$ and is adjacent to a vertex $u' \in U_{k-1}$. In either case, let x' be the vertex of degree one adjacent to y' – see Figure 5.

Suppose $w = y'$. If $f(x'y') < f(y'y)$, then, since $f(y'y) < f(xy)$, $x'y'yx$ is a U_S-avoiding maximal f-ascent of length three. If $f(x'y') > f(y'y)$, then, since $f(y'y) = \min_{v \in N(y)}\{f(vy)\}$, $yy'x'$ is a U_S-avoiding maximal f-ascent of length two.

Suppose then that $w = u'$. Let G_1 be the component of $S - y$ containing w, and G_1' the subgraph of S induced by $V(G_1) \cup \{y, x\}$. Then $G_1' \cong S_j \in \mathcal{S}_j$ for some $1 \leq j < k$. Let $U_{G_1'} = U_{S_j}$ and f_1' be the edge ordering of G_1' induced by f. By the induction hypothesis, there exists a $U_{G_1'}$-avoiding maximal f_1' ascent of length at most three in G_1'. Necessarily all $U_{G_1'}$-avoiding maximal f_1' ascent of length at most three start or end at y or else we are done. Suppose there exists such an ascent which starts at y. By Lemma 8 there exists a $U_{G_1'}$-avoiding maximal f_1' ascent λ of length at most three whose initial edge is $yw = yu'$. But since $f(yu') = \min_{v \in N(y)}\{f(yv)\}$, λ is also a U_S-avoiding maximal f-ascent which is a contradiction. Hence we may assume that there exists a $U_{G_1'}$-avoiding maximal f_1'-ascent λ of length at most three which ends at y. Since $f_1'(u'y) = \min_{v \in N(y)}\{f_1'(vy)\}$, $f_1'(u'y) > f_1'(xy)$ and the last edge of λ is $y'y$. This implies $f_1'(yy') > f_1'(xy)$ or equivalently, $f(y'y) > f(yx)$. Necessarily, $f(x'y') < f(y'y)$, or else $xyy'x'$ is a U_S-avoiding maximal f-ascent of length at most three. Now we look at three cases for the value of $f(y'u')$. In these cases we assume that $\deg_S(y') > 3$ or else either xyy' (if $f(y'u') < f(yy')$) or $yu'y'$ (if $f(y'u') > f(yy')$) is a U_S-avoiding maximal f-ascent.

Case 1 $f(yu') < f(y'u') < f(x'y')$. Then $yu'y'x'$ is a U_S-avoiding maximal f-ascent.

We define the following to aid us in the next two cases. Let H_1 be the component of $S_{k-1} - y'$ containing w, H_1' the the subgraph of S_{k-1} induced by $V(H_1) \cup \{y', x'\}$, and H_2' the subgraph of S_{k-1} induced by $V(S_{k-1}) - V(H_1)$. Then each $H_i \in \mathcal{S}_\ell$ for some $1 \leq \ell < k$. If $H_i' \cong S_\ell \in \mathcal{S}_\ell$, then let $U_{H_i'} = U_\ell$ and f_i be the edge ordering of

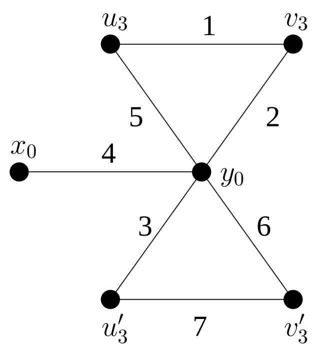

Figure 6: A graph G constructed from S_0 by performing **O3** twice at y_0, and an edge labelling f of G for which every maximal f-ascent of length at most three starts or ends in $U_G = \{u_3, u_3'\}$.

H_i' induced by f.

Case 2 $f(y'u') < f(x'y')$ and $f(y'u') < f(u'y)$. Then, in H_1', $y'u'yx$ is a $U_{H_1'}$-avoiding maximal f_1-ascent starting at y' and xyy' is a $U_{H_1'}$-avoiding maximal f_1-ascent ending at y. By the induction hypothesis, in H_2, there exists a $U_{H_2'}$-avoiding maximal f_2-ascent of length at most three. We may assume that all such f_2-ascents start or end at y'. Without loss of generality suppose there exists a $U_{H_2'}$-avoiding maximal f_2-ascent of length at most three that ends at y'. By Lemma 8, there exists a $U_{H_2'}$-avoiding maximal f_2-ascent $\lambda = aby'$ or $\lambda = acby'$ such that $f_2(by') = \max_{v \in N(y')}\{f_2(vy')\}$. Thus, in S, either λ or xyy' is a U_S-avoiding maximal f-ascent of length at most three.

Case 3 $f(y'u') > f(x'y')$. Then either xyy' (if $f(y'u') < f(yy')$) or $yu'y'$ (if $f(y'u') > f(yy')$) is a $U_{H_1'}$-avoiding maximal f_1-ascent which ends at y'. Again, by the induction hypothesis, in H_2, there exists a $U_{H_2'}$-avoiding maximal f_2-ascent of length at most three and we assume that all such f_2-ascents start or end at y'. Suppose there exists a $U_{H_2'}$-avoiding maximal f_2-ascent of length at most three that ends at y'. By Lemma 8, there exists a $U_{H_2'}$-avoiding maximal f_2-ascent $\lambda = aby'$ or $\lambda = acby'$ such that $f_2(by') = \max_{v \in N(y')}\{f_2(vy')\}$. Therefore, in S, either λ, xyy', or $xu'y'$ is a U_S-avoiding maximal f-ascent of length at most three. Suppose then that there exists a $U_{H_2'}$-avoiding maximal f_2-ascent of length at most three that starts at y'. By Lemma 8, there exists a $U_{H_2'}$-avoiding maximal f_2-ascent $\lambda = aby'$ or $\lambda = acby'$ such that $f_2(by') = \min_{v \in N(y')}\{f_2(vy')\}$. Necessarily, $f(by') < f(y'x')$, and since $f(y'y) > f(y'x')$ and $f(y'u') > f(y'x')$, λ is a U_S-avoiding maximal f-ascent of length at most three. □

In the construction of $S_k \in \mathcal{S}_k$, any vertex $y \in Y_k$ is involved in **O3** at most once. If not, then U_k is no longer a 3-kernel of S_k. Consider the graph G shown in Figure 6, which is constructed from S_0 by performing **O3** twice at y_0. Let $U_G = \{u_3, u_3'\}$. For the edge labelling f of G shown in the figure, any maximal f-ascent of length at most three starts or ends in U_G.

Recall that the graphs $G_S \in \mathcal{G}$ are obtained from a graph $S \in \mathcal{S}$ by performing operations **O6** and **O7**. We now show that these graphs also have depression at most three.

Theorem 10. *For each $G_S \in \mathcal{G}$, $\varepsilon(G) \leq 3$.*

Proof. Let G'_S be constructed from $S \in \mathcal{S}$ by adding $n \geq 0$ edges between vertices in $Y_{G'_S} = Y_S$ and let $U_{G'_S} = U_S$. If $n = 0$, then $G'_S \in \mathcal{S}$ and by Theorem 9, $\varepsilon(G'(S)) \leq 3$ and $U_{G'_S}$ is a k-kernel of G'_S, where $k \in \{2, 3\}$.

Suppose that $n \geq 1$. Let f be an edge ordering of G'_S, and f' the edge ordering of S induced by f. If there exists a $(U_S \cup Y_S)$-avoiding maximal f'-ascent of length at most three, then $h(f) \leq 3$. Suppose then that there does not exist a $(U_S \cup Y_S)$-avoiding f'-ascent of length at most three. By Theorem 9 there exists a U_S-avoiding maximal f'-ascent of length at most three in S, thus all maximal U_S-avoiding $(2, f')$- or $(3, f')$-ascents start or end in Y_S.

Without loss of generality we assume there exists a maximal U_S-avoiding ascent of length at most three which ends in Y_S. By Lemma 8, S has property **P1**, which implies that there exists a maximal f'-ascent $\lambda = aby_1$ or $\lambda = acby_1$ such that $y_1 \in Y_S$ and $f'(by_1) = \max_{v \in N_S(y_1)}\{f'(vy_1)\}$. Suppose that in G'_S there exists an edge y_1w such that $f(y_1w) = \max_{v \in N_{G'_S}(y_1)}\{f(vy_1)\} > f(by_1)$ and w is not a vertex of λ. Necessarily, $y_1w \notin E(S)$ which implies $w \in Y_S$. Let $w = y_2$, and x_1 and x_2 be the vertices of degree one adjacent to y_1 and y_2 respectively. Since λ is a maximal f'-ascent in S, it follows that $f(y_1x_1) < f(by_1) < f(y_1y_2)$. Therefore, either $x_1y_1y_2x_2$ (if $f(y_2x_2) > f(y_1y_2)$) or $x_2y_2y_1$ (if $f(y_2x_2) < f(y_1y_2)$) is a $U_{G'_S}$-avoiding maximal f-ascent. Hence $U_{G'_S}$ is a k-kernel of G'_S, where $k \in \{2, 3\}$.

Let $G_S \in \mathcal{G}$ be constructed from G'_S by adding any set $A = A(G_S)$ of new vertices to G'_S and arbitrary edges between vertices in $A \cup U_{G'_S}$. Then by Theorem 3, $\varepsilon(G_S) \leq 3$. □

Note that $\kappa(G_S) = 1$ for each $G_S \in \mathcal{G}_S$. We also note that for each graph G in the classes of graphs with depression three defined in [6], [10], and [13], either $\text{diam}(L(G)) = 2$ or $\kappa(G) = 1$. The graph H shown in Figure 7 is an example of a graph with $\kappa(H) > 1$, $\text{diam}(L(H)) > 2$, and $\varepsilon(H) = 3$. We provide the following argument to support the claim that $\varepsilon(H) = 3$. Suppose to the contrary that $\varepsilon(H) > 3$. Let $f : E(H) \to \{1, 2, ..., 8\}$ be an edge ordering of H such that every maximal f-ascent has length at least 4. Since e_1 and e_8 are the only edges in H which are at distance three in $L(H)$, it follows that $\{f(e_1), f(e_8)\} = \{1, 8\}$. If not, then there exists a maximal f-ascent of length at most three which begins and ends with the edges assigned 1 and 8 under f, a contradiction.

Without loss of generality we may assume that $f(e_1) = 1$ and $f(e_8) = 8$. Without loss of generality we may also assume that $f(e_5) = \max\{f(e_2), f(e_3), f(e_4), f(e_5)\}$. Then, since $h(f) > 3$ and $f(e_4) < f(e_5)$, it follows that $e_7e_2e_4e_5$ is a maximal f-ascent. However, this implies e_1e_2 is a maximal f-ascent, a contradiction.

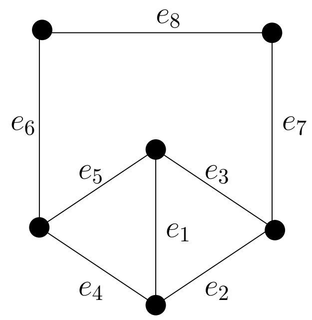

Figure 7: A graph H with $\kappa(H) > 1$, $\text{diam}(L(H)) > 2$, and $\varepsilon(H) = 3$.

5 Open Problems

1. Characterize the class of graphs with depression three.
2. Does there exist a finite number of operations of the type **O1-O7** that would yield all graphs with depression three?
3. Use a similar construction to produce large classes of graphs with depression $k \geq 4$.

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(Received 1 Jan 2013; revised 3 Nov 2013)