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MacWilliams identities for m-spotty weight enumerators of codes over rings

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Abstract

Computer memory systems using high-density RAM chips are vulnerable to m-spotty byte errors when they are exposed to high-energy particles. These errors can be effectively detected or corrected using (m-spotty) byte error-control codes. In order to study the properties of these codes and to measure their error-detection and error-correction performance, m-spotty weight enumerators are introduced and studied. In this paper, we extend notions of m-spotty weight enumerator, split m-spotty weight enumerator, r-fold joint m-spotty weight enumerator, complete m-spotty weight enumerator and byte-weight enumerator (with respect to both m-spotty Hamming and m-spotty Lee metrics) for byte error-control codes over the finite chain ring $\mathbb{F}_q + u\mathbb{F}_q + u^2\mathbb{F}_q + \cdots + u^{e-1}\mathbb{F}_q$ $(u^e = 0)$ or the ring $\mathbb{F}_q+u\mathbb{F}_q+v\mathbb{F}_q+uv\mathbb{F}_q$ $(u^2 = 0, v^2 = 0, uv = vu)$, where $\mathbb{F}_q$ is the finite field of order q. We also discuss some of their applications and establish MacWilliams identities for each of the above-mentioned enumerators.

1 Introduction

Nowadays, high-density RAM chips with wide I/O data (called a byte) are being widely used in computer memory systems, as they ensure faster communication and storage of data in computers, mobile phones, etc. However, these chips are highly susceptible to multiple random bit errors when exposed to high energy particles. In order to detect or correct these errors, Reed-Solomon codes were used initially, but these codes require a large number of parity-check bits leading to a low information rate of the code. To overcome this problem, these errors are modeled as spotty [21] and multiple spotty [19] (m-spotty) byte errors. Further, to quantify these spotty byte errors, the m-spotty Hamming and the m-spotty Lee metrics (weights) are introduced and studied extensively. Several constructions of the codes that can

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detect or correct multiple spotty byte errors have been proposed and their (m-spotty) byte error-detecting and byte error-correcting properties are also studied [2, 20, 21]. It is also shown that these codes have higher information rate as compared to the previously used Reed-Solomon codes [2, 19, 20, 21].

In order to study properties of (m-spotty) byte error-control codes and to measure their error-performance, some special type of polynomials, called m-spotty weight enumerators, have been introduced and studied with respect to both the m-spotty Hamming and m-spotty Lee metrics [6, 8, 9, 10, 11, 13, 18]. If a byte error-control code $\mathcal{C}$ contains a large number of codewords, then it is generally very hard to determine m-spotty weight enumerators of the code. However, the dual code of $\mathcal{C}$ is of comparatively smaller size, so it is easier to determine the weight enumerator of the dual code. An identity relating the weight enumerator of a code with that of its dual code is called the MacWilliams identity, and it serves as a tool to determine the weight enumerator of a code from its dual code.

Recently, the m-spotty Hamming and m-spotty Lee metrics have attracted a lot of attention. A brief survey of the results known on m-spotty weight enumerators is as follows:

Suzuki et al. [18] introduced the notion of m-spotty Hamming weight enumerator for binary byte error-control codes and derived a MacWilliams identity for the same. Ozen and Siap [6], Siap [12] and Siap and Ozen [15] further extended this work to arbitrary finite fields, to the rings $\mathbb{F}_2 + u\mathbb{F}_2$ $(u^2 = 0)$ and $\mathbb{F}_2 + u\mathbb{F}_2$ $(u^2 = u)$, respectively. We introduced split m-spotty Hamming weight enumerator and r-fold joint m-spotty Hamming weight enumerator of byte error-control codes over R, where R is either a finite field or an integer residue class ring [8, 9]. In the same work, we also derived MacWilliams identities for these two enumerators and also discussed their applications.

The above mentioned enumerators are defined relative to the Hamming metric, which is more suitable for orthogonal modulated channels. However for the transmission of non-binary signals over noisy phase modulated and amplitude modulated channels, it is observed that the Lee metric is more suitable ([1], pp. 3-4). Siap [13] introduced the notions of m-spotty Lee weight and m-spotty Lee weight enumerator for byte error-control codes over the ring of integers modulo 4 and derived a MacWilliams type identity for the same. While extending this work to byte error-control codes over R (R is either a finite field or an integer residue class ring), we also introduced the notions of split m-spotty Lee weight enumerator and r-fold joint m-spotty Lee weight enumerator for codes over R, derived MacWilliams identities for each of these enumerators and discussed their applications [10, 11].

On the other hand, Suzuki and Fujiwara [17] introduced the complete m-spotty weight enumerator for a binary byte error-control code and derived a MacWilliams identity for the same. In another work, Suzuki [16] related complete m-spotty weight enumerators of binary Type II codes with the Jacobi forms.

In another direction, Wadayama et al. [22] introduced the byte-weight enumerator for a binary byte error-control code and derived a MacWilliams identity for the same. Recently, Ozen and Siap [7] introduced the notion of m-spotty Rosenbloom-Tsfasman metric and defined the corresponding m-spotty Rosenbloom-Tsfasman weight enumerator of a binary byte error-control code. They also derived a MacWilliams identity for the m-spotty Rosenbloom-Tsfasman weight enumerator of a binary byte error-control code.

Throughout this paper, let the ring R be either the finite chain ring $R_1 = \mathbb{F}_q + u\mathbb{F}_q + u^2\mathbb{F}_q + \cdots + u^{e-1}\mathbb{F}_q$ $(u^e = 0)$ or the ring $R_2 = \mathbb{F}_q + u\mathbb{F}_q + v\mathbb{F}_q + uv\mathbb{F}_q$ $(u^2 = 0, v^2 = 0, uv = vu)$, where $\mathbb{F}_q$ is the finite field with q elements. The aim of this paper is to extend the earlier work [8, 9, 10, 11, 17] to byte error-control codes over R.

This paper is organized as follows: In Section 2, we state some preliminary results that we need to derive our main results. In Section 3, we define the m-spotty Hamming weight enumerator, split m-spotty Hamming weight enumerator, r-fold joint m-spotty Hamming weight enumerator and complete m-spotty Hamming weight enumerator for byte error-control codes over R. We also derive MacWilliams identities for each of these enumerators and discuss their applications. In Section 4, we define the m-spotty Lee weight enumerator, split m-spotty Lee weight enumerator, r-fold joint m-spotty Lee weight enumerator and complete m-spotty Lee weight enumerator for byte error-control codes over R. We also derive MacWilliams identities for each of these enumerators and discuss some of their applications. In Section 5, we derive a MacWilliams identity for the byte-weight enumerator of a byte error-control code over R. In Section 6, we mention a brief conclusion and discuss a few interesting open problems.

2 Some preliminaries

If $\mathbb{F}_q$ is the finite field of order q and having characteristic p, then $R_1 = \mathbb{F}_q + u\mathbb{F}_q + u^2\mathbb{F}_q + \cdots + u^{e-1}\mathbb{F}_q$ with $u^e = 0$, is a finite commutative ring with unity. Also every element $r \in R_1$ can be uniquely written as $r = a_0 + ua_1 + \cdots + u^{e-1}a_{e-1}$, where a_i's are in $\mathbb{F}_q$. It is easy to see that the only ideals of R_1 are $\{0\}$, R_1 and $\langle u^k \rangle = u^k R_1 = u^k\mathbb{F}_q + u^{k+1}\mathbb{F}_q + \cdots + u^{e-1}\mathbb{F}_q = \{u^k a_k + u^{k+1}a_{k+1} + \cdots + u^{e-1}a_{e-1} \mid a_k, a_{k+1}, \ldots, a_{e-1} \in \mathbb{F}_q\}$ for $1 \leq k \leq e-1$, and they satisfy

$$\{0\} \subseteq \langle u^{e-1} \rangle \subseteq \langle u^{e-2} \rangle \cdots \subseteq \langle u^2 \rangle \subseteq \langle u \rangle \subseteq R_1.$$

Therefore R_1 is a finite chain ring. If ζ_p is a complex primitive pth root of unity and $Tr_{q/p}$ is the trace function from $\mathbb{F}_q$ to $\mathbb{F}_p$, then the map $\chi_1 : R_1 \to \mathbb{C}$ defined as

$$\chi_1(r) = \zeta_p^{Tr_{q/p}(a_0 + a_1 + \cdots + a_{e-1})} \text{ for all } r = a_0 + ua_1 + \cdots + u^{e-1}a_{e-1} \in R_1, \quad (1)$$

is a non-trivial additive character on R_1. From this, we make the following observation.

Lemma 1. *For any non-zero ideal H of R_1, we have*

$$\sum_{h \in H} \chi_1(h) = 0.$$

Proof. Since χ_1 is a non-trivial additive character on R_1, by Theorem 5.4 of [3], we have $\sum_{r\in R_1} \chi_1(r) = 0$. So to prove this lemma, it is enough to prove that $\sum_{h\in H} \chi_1(h) = 0$ for all $H = \langle u^k\rangle$, where $1 \leq k \leq e-1$. Since every element $h \in \langle u^k\rangle$ can be uniquely written as $h = u^k a_k + u^{k+1}a_{k+1} + \cdots + u^{e-1}a_{e-1}$ (a_i's in $\mathbb{F}_q$), we have

$$\sum_{r\in\langle u^k\rangle} \chi_1(r) = \sum_{\substack{a_i\in\mathbb{F}_q\\ k\leq i\leq e-1}} \chi_1(u^k a_k + u^{k+1}a_{k+1} + \cdots + u^{e-1}a_{e-1}).$$

As χ_1 is an additive character on R_1, the above sum can be rewritten as

$$\sum_{r\in\langle u^k\rangle} \chi_1(r) = \prod_{i=k}^{e-1}\left(\sum_{a_i\in\mathbb{F}_q} \chi_1(u^i a_i)\right) = \prod_{i=k}^{e-1}\left(\sum_{a_i\in\mathbb{F}_q} \zeta_p^{Tr_{q/p}(a_i)}\right) = \left(\sum_{a\in\mathbb{F}_q} \zeta_p^{Tr_{q/p}(a)}\right)^{e-k}.$$

It is easy to see that the map $a \mapsto \zeta_p^{Tr_{q/p}(a)}$ is a non-trivial additive character on $\mathbb{F}_q$, which again by Theorem 5.4 of [3], gives $\sum_{a\in\mathbb{F}_q} \zeta_p^{Tr_{q/p}(a)} = 0$. Therefore the sum $\sum_{r\in\langle u^k\rangle} \chi_1(r) = 0$, which completes the proof. □

Next consider the ring $R_2 = \mathbb{F}_q + u\mathbb{F}_q + v\mathbb{F}_q + uv\mathbb{F}_q$ with $u^2 = 0, v^2 = 0$ and $uv = vu$. Note that R_2 is a finite commutative ring with unity and every element $r \in R_2$ can be uniquely written as $r = a + ub + vc + uvd$, where a, b, c, d are in $\mathbb{F}_q$. It is easy to see that the only ideals of R_2 are $\{0\}, \langle u\rangle = uR_2 = u\mathbb{F}_q + uv\mathbb{F}_q = \{ua + uvb \mid a, b \in \mathbb{F}_q\}, \langle v\rangle = vR_2 = v\mathbb{F}_q + uv\mathbb{F}_q = \{va + uvb \mid a, b \in \mathbb{F}_q\}, \langle uv\rangle = uvR_2 = uv\mathbb{F}_q = \{uva \mid a \in \mathbb{F}_q\}, \langle u + v\rangle = (u + v)R_2 = (u + v)\mathbb{F}_q + uv\mathbb{F}_q = \{(u + v)a + uvb \mid a, b \in \mathbb{F}_q\}, \langle u, v\rangle = uR_2 + vR_2 = u\mathbb{F}_q + v\mathbb{F}_q + uv\mathbb{F}_q = \{ua + vb + uvc \mid a, b, c \in \mathbb{F}_q\}$ and R_2. Further, if ζ_p is a complex primitive pth root of unity and $Tr_{q/p}$ is the trace function from $\mathbb{F}_q$ to $\mathbb{F}_p$, then the map $\chi_2 : R_2 \to \mathbb{C}$ defined as

$$\chi_2(r) = \zeta_p^{Tr_{q/p}(a+b+c+d)} \text{ for all } r = a + ub + vc + uvd \in R_2, \tag{2}$$

is a non-trivial additive character on R_2. Here also, we observe the following:

Lemma 2. *If H is any non-zero ideal of R_2, then*

$$\sum_{h\in H} \chi_2(h) = 0.$$

Proof. Its proof is similar to that of Lemma 1. □

Throughout this paper, let R be either the ring R_1 or R_2, and let b, n be fixed positive integers. Let R^{bn} denote the R-module of all bn-tuples over R. Note that every vector $v \in R^{bn}$ can be written as $v = (v_1, v_2, \ldots, v_n)$, where each $v_i = (v_{i1}, v_{i2}, \ldots, v_{ib}) \in R^b$ and is called the ith byte of v.

A byte error-control code $\mathcal{C}$ of length bn and byte length b over R is defined as an R-submodule of R^{bn}. The byte error-control codes can be used to detect or correct special type of byte errors called spotty byte errors, which are as defined below:

Definition 1. [21] For a fixed positive integer t $(1 \leq t \leq b)$, a byte error is said to be a spotty byte error (or t/b-error) if t or fewer random bit errors occur in a b-bit byte. If none of the bits in a b-bit byte are in error, then we say that no spotty byte error (or t/b-error) has occurred.

When high energy particles hit high-density RAM chips, more than t bits of a b-bit byte may be distorted by noise. This led to the notion of m-spotty (or multiple spotty) byte errors, which are as defined below:

Definition 2. [19] A byte error is said to be an m-spotty byte error (or multiple spotty byte error) if at least one spotty byte error (or t/b-error) occurs in a b-bit byte.

In order to illustrate the above definitions, we let $t = 3$ and $b = 12$. If 8 random bits in a 12-bit byte are in error, then we say that three spotty byte errors (or 3/12-errors) have occurred. If 2 random bits of a 12-bit byte are in error, then we say that one spotty byte error (or 3/12-error) has occurred.

In order to study the properties of byte error-control codes over R and to determine their (m-spotty) byte error-detecting and error-correcting capabilities relative to various channels, several m-spotty weight enumerators have been introduced and studied for byte error-control codes over various finite commutative rings. For each of these enumerators, various MacWilliams identities have been derived, which relate the m-spotty weight enumerator of a byte error-control code with that of its dual code as defined below:

If $\mathcal{C}$ is a byte error-control code of length bn and byte length b over R, then the dual code of $\mathcal{C}$, denoted by $\mathcal{C}^{\perp}$, is defined as

$$\mathcal{C}^{\perp} = \{v \in R^{bn} \ : \ \langle u, v\rangle = 0 \text{ for all } u \in \mathcal{C}\},$$

where $\langle \cdot, \cdot \rangle$ denotes the standard inner product in R^{bn}. One can easily observe that $\mathcal{C}^{\perp}$ is also a byte error-control code of length bn and byte length b over R.

The following lemma is an important tool in deriving MacWilliams identities for m-spotty weight enumerators of byte error-control codes over R.

Lemma 3. *Let $\mathcal{C}$ be a byte error-control code of length bn and byte length b over R and f be a function defined from R^{bn} into $\mathbb{C}[z]$, where $\mathbb{C}$ is the set of complex numbers. For $u \in R^{bn}$, define*

$$\tilde{f}(u) = \sum_{v \in R^{bn}} \chi(\langle u, v\rangle) f(v),$$

where $\chi = \chi_1$ if $R = R_1$ and $\chi = \chi_2$ if $R = R_2$. Then we have

$$\sum_{v \in \mathcal{C}} f(v) = \frac{1}{|\mathcal{C}^{\perp}|} \sum_{u \in \mathcal{C}^{\perp}} \tilde{f}(u),$$

where $\mathcal{C}^{\perp}$ denotes the dual code of $\mathcal{C}$. (Throughout this paper, $|A|$ denotes the cardinality of the set A.)

Proof. Its proof is similar to that of Lemma 2.8 of Siap [14]. □

3 Weight enumerators with respect to the m-spotty Hamming metric

In this section, we discuss various m-spotty weight enumerators defined with respect to the m-spotty Hamming metric, which is as defined below:

Definition 3. [21] The m-spotty Hamming distance between any two vectors u, v in R^{bn}, denoted by $d_M(u, v)$, is defined as

$$d_M(u, v) = \sum_{i=1}^{n} \left\lceil \frac{d_H(u_i, v_i)}{t} \right\rceil,$$

where $d_H(u_i, v_i)$ $(1 \leq i \leq n)$ denotes the Hamming distance of u_i and v_i. (Here $\lceil x \rceil$ denotes the ceiling of any real number x.)

When $t = 1$, the m-spotty Hamming distance is same as the Hamming distance over R. When $t = b$, the m-spotty Hamming distance coincides with the Hamming distance over R^b.

It is easy to see that d_M is a metric on R^{bn} and is called the m-spotty Hamming metric on R^{bn}.

Definition 4. [19] The m-spotty Hamming weight of a vector $v \in R^{bn}$, denoted by $w_M(v)$, is defined as

$$w_M(v) = \sum_{i=1}^{n} \left\lceil \frac{w_H(v_i)}{t} \right\rceil,$$

where $w_H(v_i)$ denotes the Hamming weight of the ith byte v_i of v. Note that $d_M(u, v) = w_M(u - v)$.

3.1 m-spotty Hamming weight enumerator

In this subsection, we define the m-spotty Hamming weight enumerator of a byte error-control code over R, derive a MacWilliams identity for the same and discuss its application.

Definition 5. [18] Let $\mathcal{C}$ be a byte error-control code of length bn and byte length b over R. The m-spotty Hamming weight enumerator of $\mathcal{C}$ is given by

$$W_{\mathcal{C}}(z) = \sum_{u \in \mathcal{C}} z^{w_M(u)}.$$

When $t = 1$, the m-spotty Hamming weight enumerator of a byte error-control code $\mathcal{C}$ coincides with the Hamming weight enumerator of $\mathcal{C}$.

For any vector $v \in R^{bn}$, let α_i $(1 \leq i \leq n)$ be the Hamming weight of the ith byte v_i of v. Then the vector $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ is called the Hamming weight distribution vector of v. Further, if $A_{(\alpha_1, \alpha_2, \ldots, \alpha_n)}$ denotes the number of codewords in $\mathcal{C}$ having the

Hamming weight distribution vector as $(\alpha_1, \alpha_2, \ldots, \alpha_n)$, then the m-spotty Hamming weight enumerator can be rewritten as

$$W_{\mathcal{C}}(z) = \sum_{\substack{(\alpha_1,\alpha_2,\ldots,\alpha_n) \\ 0\le\alpha_1,\alpha_2,\ldots,\alpha_n\le b}} A_{(\alpha_1,\alpha_2,\ldots,\alpha_n)} \prod_{i=1}^{n} z^{\lceil \alpha_i/t \rceil}.$$

In the following theorem, we derive a MacWilliams identity for the m-spotty Hamming weight enumerator of a byte error-control code over R.

Theorem 1. *Let $\mathcal{C}$ be a byte error-control code of length bn and byte length b over R and $\mathcal{C}^\perp$ be its dual code. Then the m-spotty Hamming weight enumerator of $\mathcal{C}$ is given by*

$$W_{\mathcal{C}}(z) = \frac{1}{|\mathcal{C}^\perp|} \sum_{\substack{(\alpha_1,\alpha_2,\ldots,\alpha_n) \\ 0\le\alpha_1,\alpha_2,\ldots,\alpha_n\le b}} A_{(\alpha_1,\alpha_2,\ldots,\alpha_n)} \prod_{i=1}^{n} g_{\alpha_i}^{(t)}(z),$$

where $A_{(\alpha_1,\alpha_2,\ldots,\alpha_n)}$ is the number of codewords in $\mathcal{C}^\perp$ having the Hamming weight distribution vector as $(\alpha_1, \alpha_2, \ldots, \alpha_n)$, and the polynomials $g_{\alpha_i}^{(t)}(z)$'s are defined as

$$g_{\alpha_i}^{(t)}(z) = \sum_{p_i=0}^{b} K_{p_i}(\alpha_i) z^{\lceil p_i/t \rceil} \text{ with } K_p(X) = \sum_{a=0}^{p} (-1)^a (|R|-1)^{p-a} \binom{X}{a}\binom{b-X}{p-a},$$

assuming $\binom{j}{\ell} = 0$ for $j < \ell$ and $\binom{0}{0} = 0$. (Note that $K_p(X)$ is the well-known Krawtchouk polynomial [4]*).*

Remark 1. **(i)** When $t = 1$, Theorem 1 provides the MacWilliams identity for Hamming weight enumerator of a linear code of length bn over R.

(ii) When the code $\mathcal{C}$ is of large size, it is very hard (in general) to compute the numbers $A_{(\alpha_1,\alpha_2,\ldots,\alpha_n)}$ (and hence the m-spotty Hamming weight enumerator) for the code $\mathcal{C}$. However, the dual code $\mathcal{C}^\perp$ of $\mathcal{C}$ is of relatively smaller size and so it is comparatively easier to compute the numbers $A_{(\alpha_1,\alpha_2,\ldots,\alpha_n)}$ for the dual code $\mathcal{C}^\perp$. Therefore by applying the MacWilliams identity (Theorem 1), one can obtain the m-spotty Hamming weight enumerator of $\mathcal{C}$.

Proof of Theorem 1. Let $f(v) = \prod_{i=1}^{n} z^{w_M(v_i)}$ for $v = (v_1, v_2, \ldots, v_n) \in R^{bn}$ with each $v_i \in R^b$. Then by Lemma 3, for $u = (u_1, u_2, \ldots, u_n) \in R^{bn}$, $\tilde{f}(u)$ is given by

$$\tilde{f}(u) = \sum_{v=(v_1,v_2,\ldots,v_n)\in R^{bn}} \chi(\langle u, v\rangle) \prod_{i=1}^{n} z^{w_M(v_i)} = \prod_{i=1}^{n} \left(\sum_{v_i \in R^b} \chi(\langle u_i, v_i\rangle) z^{w_M(v_i)} \right).$$

For each i $(1 \le i \le n)$, if $w_H(u_i) = \alpha_i$, then working as in Lemma 27 of Sharma et al. [8], we get

$$\sum_{v_i \in R^b} \chi(\langle u_i, v_i\rangle) z^{w_M(v_i)} = g_{\alpha_i}^{(t)}(z).$$

This gives

$$\tilde{f}(u) = \prod_{i=1}^{n} g_{\alpha_i}^{(t)}(z),$$

where $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ is the Hamming weight distribution vector of u.

Now if $A_{(\alpha_1,\alpha_2,\ldots,\alpha_n)}$ is the number of codewords in $\mathcal{C}^\perp$ having $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ as the Hamming weight distribution vector, then

$$\sum_{u\in\mathcal{C}^\perp} \tilde{f}(u) = \sum_{(\alpha_1,\alpha_2,\ldots,\alpha_n)} A_{(\alpha_1,\alpha_2,\ldots,\alpha_n)} \prod_{i=1}^{n} g_{\alpha_i}^{(t)}(z), \tag{3}$$

where the summation runs over all n-tuples $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ satisfying $0 \leq \alpha_i \leq b$.

Again applying Lemma 3 and using (3), we get

$$W_{\mathcal{C}}(z) = \sum_{v\in\mathcal{C}} f(v) = \frac{1}{|\mathcal{C}^\perp|} \sum_{u\in\mathcal{C}^\perp} \tilde{f}(u) = \frac{1}{|\mathcal{C}^\perp|} \sum_{(\alpha_1,\alpha_2,\ldots,\alpha_n)} A_{(\alpha_1,\alpha_2,\ldots,\alpha_n)} \prod_{i=1}^{n} g_{\alpha_i}^{(t)}(z),$$

which proves the identity. □

An application

Here we observe that the m-spotty Hamming weight enumerator is useful in determining the (m-spotty) error-detecting and error-correcting properties of a byte error-control code over R.

For this, let $\mathcal{C}$ be a byte error-control code of length bn and byte length b over R. Then the m-spotty Hamming distance of the code $\mathcal{C}$ is defined as

$$d_M(\mathcal{C}) = \min\{d_M(u, v) : u, v \in \mathcal{C}, u \neq v\}.$$

It is easy to see that $d_M(\mathcal{C}) = \min\{w_M(u) : u \in \mathcal{C}, u \neq 0\}$. From this, it follows that $d_M(\mathcal{C})$ equals the least positive integer d such that the coefficient of z^d in $W_{\mathcal{C}}(z)$ is non-zero. Thus one can compute the m-spotty Hamming distance of a code knowing its m-spotty Hamming weight enumerator.

In the following theorem, it is shown that the m-spotty Hamming distance $d_M(\mathcal{C})$ of a code $\mathcal{C}$ measures its (m-spotty) error-detecting and error-correcting capabilities.

Theorem 2. *Let $\mathcal{C}$ be a byte error-control code of length bn and byte length b over R. Then we have the following:*

(i) *The code $\mathcal{C}$ can detect any m-spotty byte error e satisfying $w_M(e) < d$ if and only if $d_M(\mathcal{C}) \geq d$.*

(ii) *If $d_M(\mathcal{C}) = d$, then $\mathcal{C}$ can correct all m-spotty byte errors e satisfying $w_M(e) < d/2$, and $\mathcal{C}$ cannot correct any m-spotty byte error e satisfying $w_M(e) \geq d/2$.*

Proof. Its proof is similar to that of Theorem 31 of Sharma et al. [8]. □

3.2 Split m-spotty Hamming weight enumerator

In this subsection, we define the split m-spotty Hamming weight enumerator of a byte error-control code over R, derive a MacWilliams identity for the same and discuss some of its applications.

Definition 6. [8] Let $\mathcal{C}$ be a byte error-control code of length bn and byte length b over R. Then the split m-spotty Hamming weight enumerator of the code $\mathcal{C}$ is defined as

$$\mathcal{S}_{\mathcal{C}}(z_i : i = 1, 2, \ldots, n) = \sum_{(u_1,u_2,\ldots,u_n)\in\mathcal{C}} \left(\prod_{i=1}^{n} z_i^{w_M(u_i)} \right).$$

If $A_{(\alpha_1,\alpha_2,\ldots,\alpha_n)}$ is the number of codewords in $\mathcal{C}$ having the Hamming weight distribution vector as $(\alpha_1, \alpha_2, \ldots, \alpha_n)$, then the split m-spotty Hamming weight enumerator of the code $\mathcal{C}$ can be rewritten as

$$\mathcal{S}_{\mathcal{C}}(z_i : i = 1, 2, \ldots, n) = \sum_{(\alpha_1,\alpha_2,\ldots,\alpha_n)} A_{(\alpha_1,\alpha_2,\ldots,\alpha_n)} \prod_{i=1}^{n} z_i^{\lceil \frac{\alpha_i}{t} \rceil}, \tag{4}$$

where the summation runs over all n-tuples $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ satisfying $0 \le \alpha_i \le b$ for each i.

If we take $z_1 = z_2 = \cdots = z_n = z$, then the split m-spotty Hamming weight enumerator of a byte error-control code coincides with the m-spotty Hamming weight enumerator of the code. When $t = 1$, the split m-spotty Hamming weight enumerator of a byte error-control code coincides with the split Hamming weight enumerator of the code.

In the following theorem, we derive a MacWilliams identity for the split m-spotty Hamming weight enumerator of a byte error-control code over R.

Theorem 3. [8] *Let $\mathcal{C}$ be a byte error-control code of length bn and byte length b over R and $\mathcal{C}^{\perp}$ be its dual code. Then the split m-spotty Hamming weight enumerator of $\mathcal{C}$ over R is given by*

$$\mathcal{S}_{\mathcal{C}}(z_i : i = 1, 2, \ldots, n) = \frac{1}{|\mathcal{C}^{\perp}|} \sum_{\substack{(\alpha_1,\alpha_2,\ldots,\alpha_n) \\ 0\le\alpha_1,\alpha_2,\ldots,\alpha_n\le b}} A_{(\alpha_1,\alpha_2,\ldots,\alpha_n)} \prod_{i=1}^{n} g_{\alpha_i}^{(t)}(z_i),$$

where $A_{(\alpha_1,\alpha_2,\ldots,\alpha_n)}$ is the number of codewords in $\mathcal{C}^{\perp}$ having Hamming weight distribution vector as $(\alpha_1, \alpha_2, \ldots, \alpha_n)$, and the polynomials $g_{\alpha_i}^{(t)}(z_i)$'s are given by

$$g_{\alpha_i}^{(t)}(z_i) = \sum_{p_i=0}^{b} K_{p_i}(\alpha_i) z_i^{\lceil p_i/t \rceil}, \tag{5}$$

with each $K_p(X) = \sum_{a=0}^{p} (-1)^a (|R| - 1)^{p-a} \binom{X}{a} \binom{b-X}{p-a}$. (Here also, $K_p(X)$ is the well-known Krawtchouk polynomial [4]*.)*

Remark 2. (i) When $z_1 = z_2 = \cdots = z_n = z$, Theorem 3 provides a MacWilliams identity for m-spotty Hamming weight enumerator.

(ii) When $t = 1$, Theorem 3 gives a MacWilliams identity for the split Hamming weight enumerator of a code over R.

(iii) For a large code $\mathcal{C}$, it is generally very hard to compute the numbers $A_{(\alpha_1,\alpha_2,\ldots,\alpha_n)}$, and hence the split m-spotty Hamming weight enumerator. However by applying Theorem 3, one can compute the split m-spotty Hamming weight enumerator of $\mathcal{C}$ from the numbers $A_{(\alpha_1,\alpha_2,\ldots,\alpha_n)}$ for the dual code $\mathcal{C}^\perp$ of $\mathcal{C}$, which are easier to compute, as $\mathcal{C}^\perp$ is of comparatively smaller size.

Proof of Theorem 3. For any vector $v = (v_1, v_2, \ldots, v_n) \in R^{bn}$ with each $v_i \in R^b$, let us define $f(v) = \prod_{i=1}^{n} z_i^{w_M(v_i)}$. Then by Lemma 3, for $u = (u_1, u_2, \ldots, u_n) \in R^{bn}$, $\tilde{f}(u)$ is given by

$$\tilde{f}(u) = \sum_{v=(v_1,v_2,\ldots,v_n)\in R^{bn}} \chi(\langle u, v\rangle) \prod_{i=1}^{n} z_i^{w_M(v_i)} = \prod_{i=1}^{n} \left(\sum_{v_i\in R^b} \chi(\langle u_i, v_i\rangle) z_i^{w_M(v_i)} \right).$$

Let us suppose that $w_H(u_i) = \alpha_i$ for $1 \le i \le n$. Then working as in Lemma 27 of Sharma et al. [8], we get

$$\sum_{v_i\in R^b} \chi(\langle u_i, v_i\rangle) z_i^{w_M(v_i)} = g_{\alpha_i}^{(t)}(z_i)$$

for each i.

This gives

$$\tilde{f}(u) = \prod_{i=1}^{n} g_{\alpha_i}^{(t)}(z_i),$$

where $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ is the Hamming weight distribution vector of u.

Now if $A_{(\alpha_1,\alpha_2,\ldots,\alpha_n)}$ is the number of codewords in $\mathcal{C}^\perp$ having the Hamming weight distribution vector as $(\alpha_1, \alpha_2, \ldots, \alpha_n)$, then we have

$$\sum_{u\in\mathcal{C}^\perp} \tilde{f}(u) = \sum_{(\alpha_1,\alpha_2,\ldots,\alpha_n)} A_{(\alpha_1,\alpha_2,\ldots,\alpha_n)} \prod_{i=1}^{n} g_{\alpha_i}^{(t)}(z_i), \tag{6}$$

where the summation runs over all n-tuples $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ satisfying $0 \le \alpha_i \le b$.

Again applying Lemma 3 and using (6), we get

$$\begin{aligned} \mathcal{S}_\mathcal{C}(z_i : i = 1, 2, \ldots, n) &= \sum_{v\in\mathcal{C}} f(v) = \frac{1}{|\mathcal{C}^\perp|} \sum_{u\in\mathcal{C}^\perp} \tilde{f}(u) \\ &= \frac{1}{|\mathcal{C}^\perp|} \sum_{(\alpha_1,\alpha_2,\ldots,\alpha_n)} A_{(\alpha_1,\alpha_2,\ldots,\alpha_n)} \prod_{i=1}^{n} g_{\alpha_i}^{(t)}(z_i), \end{aligned}$$

which proves the identity. □

In the following theorem, we see that equivalent byte error-control codes have the same m-spotty Hamming weight enumerator but their split m-spotty Hamming weight enumerators may be different.

Theorem 4. [8] *Let $\mathcal{C}, \mathcal{D}$ be byte error-control codes of length bn and byte length b over R having m-spotty Hamming weight enumerators as $W_{\mathcal{C}}(z)$, $W_{\mathcal{D}}(z)$ and split m-spotty Hamming weight enumerators as $\mathcal{S}_{\mathcal{C}}(z_i : i = 1, 2, \ldots, n)$, $\mathcal{S}_{\mathcal{D}}(Z_i : i = 1, 2, \ldots, n)$, respectively. Then*

(i) *the direct sum*

$$\mathcal{C} \oplus \mathcal{D} = \{(u|v) \; : \; u \in \mathcal{C}, \; v \in \mathcal{D}\}$$

has m-spotty Hamming weight enumerator as $W_{\mathcal{C}}(z)W_{\mathcal{D}}(z)$ and split m-spotty Hamming weight enumerator as $\mathcal{S}_{\mathcal{C}}(z_i : i = 1, 2, \ldots, n) \; \mathcal{S}_{\mathcal{D}}(Z_i : i = 1, 2, \ldots, n)$.

(ii) *assuming n even, the code*

$$\mathcal{C} \parallel \mathcal{D} = \{(u'|v'|u''|v'') \; : \; u = (u'|u'') \in \mathcal{C}, \; v = (v'|v'') \in \mathcal{D}\}$$

(where u and v have each been broken into two equal halves) has m-spotty Hamming weight enumerator as $W_{\mathcal{C}}(z)W_{\mathcal{D}}(z)$ and split m-spotty Hamming weight enumerator as $\mathcal{S}_{\mathcal{C}}(z_i; Z_i : i = 1, 2, \ldots, n/2) \, \mathcal{S}_{\mathcal{D}}(z_i; Z_i : i = (n/2) + 1, \ldots, n)$.

Proof. For proof, see Theorem 28 of Sharma et al. [8]. □

Some Applications

Let $\mathcal{C}$ be a byte error-control code of length bn and byte length b over R. Suppose that the codewords of $\mathcal{C}$ are transmitted through the $|R|$-ary memoryless channel $\mathfrak{C}$, defined as follows:

(i) For each i ($1 \leq i \leq n$), the bit-error probability in the ith byte is p_i.

(ii) In order to ensure reliable data transmission and storage, it is assumed that $0 < p_i < 1/2$ for $1 \leq i \leq n$.

(iii) Within a given byte, all bit-errors are equally likely.

Now let us define

$$\delta_M(\mathcal{C}) = \min \{\varpi_M(u) : u \in \mathcal{C}, u \neq 0\},$$

where $\varpi_M(u) = \sum_{i=1}^{n} \mathfrak{p}_i w_M(u_i)$ for $u = (u_1, u_2, \ldots, u_n) \in R^{bn}$ with each $u_i \in R^b$, and $\mathfrak{p}_i = \log(\frac{1-p_i}{p_i})$ for each i.

In the following theorem, we see that $\delta_M(\mathcal{C})$ measures the (m-spotty) error-detecting and error-correcting capabilities of the code $\mathcal{C}$ relative to the channel $\mathfrak{C}$.

Theorem 5. *Let $\mathcal{C}$ be a byte error-control code of length bn and byte length b over R. Then we have the following:*

(i) The code $\mathcal{C}$ can detect any m-spotty byte error e satisfying $\varpi_M(e) < \delta$ if and only if $\delta_M(\mathcal{C}) \geq \delta$.

(ii) If $\delta_M(\mathcal{C}) = \delta$, then the code $\mathcal{C}$ can correct any m-spotty byte error e satisfying $\varpi_M(e) < \delta_M(\mathcal{C})/2$ and $\mathcal{C}$ cannot correct any m-spotty byte error e satisfying $\varpi_M(e) \geq \delta_M(\mathcal{C})/2$.

Proof. For proof, see Sharma et al. [8, Theorem 31]. □

Note that if we take $z_i = z^{\mathfrak{p}_i}$ $(1 \leq i \leq n)$ in the split m-spotty Hamming weight enumerator $\mathcal{S}_{\mathcal{C}}(z_i : i = 1, 2, \ldots, n)$ of $\mathcal{C}$, then $\delta_M(\mathcal{C})$ is the least positive real number δ such that the coefficient of z^δ in $\mathcal{S}_{\mathcal{C}}(z^{\mathfrak{p}_i} : i = 1, 2, \ldots, n)$ is non-zero. Thus one can obtain the number $\delta_M(\mathcal{C})$ from the split m-spotty Hamming weight enumerator of the byte error-control code $\mathcal{C}$.

In the following theorem, we prove that the split m-spotty Hamming weight enumerator of a byte error-control code $\mathcal{C}$ also measures the probability of an undetected m-spotty byte error in any codeword of $\mathcal{C}$ assuming the channel of transmission as $\mathfrak{C}$.

Theorem 6. *Let $\mathcal{C}$ be a byte error-control code of length bn and byte length b over R, whose codewords are transmitted through the channel $\mathfrak{C}$. Then the probability of an undetected m-spotty byte error in any codeword of $\mathcal{C}$ is given by*

$$P_{UDE}(p_i : 1 \leq i \leq n) = \sum_{\mu=1}^{n\lceil \frac{b}{t} \rceil} \sum_{\substack{\mu_1+\cdots+\mu_n=\mu \\ 0 \leq \mu_i \leq \lceil \frac{b}{t} \rceil}} B_{(\mu_1,\ldots,\mu_n)} P_{\mu_1}(p_1) \ldots P_{\mu_n}(p_n),$$

where $B_{(\mu_1,\ldots,\mu_n)}$ is the number of codewords in $\mathcal{C}$ having m-spotty Hamming weights of first, second, …, nth bytes as $\mu_1, \mu_2, \ldots, \mu_n$, respectively and $P_{\mu_i}(p_i)$ $(1 \leq i \leq n)$ is given by

$$P_{\mu_i}(p_i) = \sum_{j=(\mu_i-1)t+1}^{\mu_i t} \binom{b}{j} (|R|-1)^j \Big(\frac{p_i}{|R|-1}\Big)^j (1-p_i)^{b-j}.$$

Remark 3. Note that

$$B_{(\mu_1,\ldots,\mu_n)} = \sum_{(\alpha_1,\alpha_2,\ldots,\alpha_n)} A_{(\alpha_1,\alpha_2,\ldots,\alpha_n)},$$

where the summation $\sum\limits_{(\alpha_1,\alpha_2,\ldots,\alpha_n)}$ runs over all n-tuples $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ satisfying $\lceil \frac{\alpha_i}{t} \rceil = \mu_i$ for each i. Therefore $B_{(\mu_1,\ldots,\mu_n)}$ equals the coefficient of $\prod\limits_{i=1}^{n} z_i^{\mu_i}$ in the split m-spotty Hamming weight enumerator of $\mathcal{C}$. This shows that one can compute the probability of an undetected m-spotty byte error in a codeword of $\mathcal{C}$ knowing its split m-spotty Hamming weight enumerator.

Proof of Theorem 6. Let us suppose that a codeword c is transmitted through the channel $\mathfrak{C}$ and a vector $r \in R^{bn}$ is received resulting in the error $e = r - c$. Then the error e will remain undetected if the received vector r is itself a codeword.

Without any loss of generality, we assume that the zero codeword has been sent. So to find the probability of an undetected error during this transmission, it is enough to find the probability that the received vector is a codeword.

For this, we first observe that μ number of spotty byte errors occur in a b-bit byte if the number of erroneous bits j in the byte satisfies $(\mu - 1)t + 1 \le j \le \min(\mu t, b)$. Therefore the probability that μ number of spotty byte errors occur in the ith byte is given by

$$P_\mu(p_i) = \sum_{j=(\mu-1)t+1}^{\mu t} \binom{b}{j} (|R| - 1)^j \Big(\frac{p_i}{|R| - 1}\Big)^j (1 - p_i)^{b-j},$$

where p_i is the bit-error probability in the ith byte of the codewords of $\mathcal{C}$. Further, if $B_{(\mu_1,\ldots,\mu_n)}$ is the number of codewords in $\mathcal{C}$ having the m-spotty Hamming weight of the ith byte as μ_i for each i, then the probability that the received vector is itself a codeword, is given by

$$P_{UDE}(p_i : 1 \le i \le n) = \sum_{\mu=1}^{n\lceil \frac{b}{t} \rceil} \sum_{\substack{\mu_1+\cdots+\mu_n=\mu \\ 0\le\mu_i\le\lceil \frac{b}{t}\rceil}} B_{(\mu_1,\ldots,\mu_n)} P_{\mu_1}(p_1) \ldots P_{\mu_n}(p_n),$$

which equals the probability of an undetected error in any codeword of $\mathcal{C}$. □

In the following theorem, we see that the split m-spotty Hamming weight enumerator of a byte error-control code over R measures the probability of decoding error in the channel $\mathfrak{C}$ when a bounded distance decoder is used for decoding.

Theorem 7. *Let $\mathcal{C}$ be a byte error-control code of length bn and byte length b over R having m-spotty Hamming distance as d_M. If the bounded distance decoder is used for decoding, then the probability of decoding error is given by*

$$P(p_i : 1 \le i \le n) = \sum_{\mu=d_M}^{n\lceil \frac{b}{t} \rceil} \sum_{\substack{\mu_1+\cdots+\mu_n=\mu \\ 0\le\mu_i\le\lceil \frac{b}{t}\rceil}} B_{(\mu_1,\ldots,\mu_n)} \sum_{\delta=0}^{\lfloor \frac{d_M-1}{2} \rfloor} \sum_{\substack{\delta_1+\cdots+\delta_n=\delta \\ 0\le\delta_i\le\lceil \frac{b}{t}\rceil}} R_{\delta_1,\mu_1}(p_1) \ldots R_{\delta_n,\mu_n}(p_n),$$

where $B_{(\mu_1,\ldots,\mu_n)}$ is the number of codewords in $\mathcal{C}$ having m-spotty Hamming weight of the ith byte as μ_i for each i and R_{δ_i,μ_i}'s are given by

$$R_{\delta_i,\mu_i}(p_i) = \sum_{j_i=(\mu_i-1)t+1}^{\mu_i t} \sum_{k_i=(\delta_i-1)t+1}^{\delta_i t} \sum_{a_1=0}^{k_i} \sum_{a_2=0}^{a_1} \binom{b-j_i}{k_i-a_1} \binom{j_i}{a_1} \binom{a_1}{a_2} (|R|-1)^{k_i-a_1} (|R|-2)^{a_2} \left(\frac{p_i}{|R|-1}\right)^{k_i+j_i-2a_1+a_2} (1-p_i)^{b-j_i-k_i+2a_1-a_2}.$$

Note that the numbers $B_{(\mu_1,\ldots,\mu_n)}$ of a code $\mathcal{C}$ can be computed from the split m-spotty Hamming weight enumerator of $\mathcal{C}$ as the coefficient of $\prod_{i=1}^{n} z_i^{\mu_i}$ (see Remark 3).

To prove this theorem, we need to prove the following lemma.

Lemma 4. *Let $\mathcal{C}$ be a byte error-control code of length bn and byte length b over R, whose codewords are transmitted through the channel $\mathfrak{C}$. Let $c = (c_1, c_2, \ldots, c_n)$ be a codeword of $\mathcal{C}$ such that the m-spotty Hamming weight of the ith byte c_i of c is μ_i for each i. Then the probability that a received vector is at m-spotty distance δ from the codeword c is given by*

$$\sum\nolimits_{\delta} R_{\delta_1,\mu_1}(p_1) \ldots R_{\delta_n,\mu_n}(p_n),$$

where the sum $\sum_\delta$ runs over all n-tuples $(\delta_1, \delta_2, \ldots, \delta_n)$ satisfying $0 \le \delta_i \le \lceil \frac{b}{t} \rceil$ and $\delta_1 + \cdots + \delta_n = \delta$.

Proof. Without any loss of generality, suppose that the zero codeword is sent through the channel $\mathfrak{C}$ and a vector $r \in R^{bn}$ is received. In order to calculate the probability that the received vector r is at m-spotty distance δ from the codeword c, we write $r = (r_1, r_2, \ldots, r_n) \in R^{bn}$ and suppose that $d_M(r_i, c_i) = \delta_i$ for each i, so that $\delta_1 + \delta_2 + \cdots + \delta_n = \delta$. Note that δ_i varies from 0 to $\lceil \frac{b}{t} \rceil$ for each i. Let $w_H(c_i) = j_i$ and without any loss of generality, suppose that all the non-zero entries of c_i appear at first j_i positions. Let a_1 be the number of positions at which r_i differs from the non-zero bits of c_i, and out of these a_1 positions, let a_2 be the number of positions at which both r_i and c_i have non-zero entries. If $d_H(r_i, c_i) = k_i$, then the number of positions at which r_i differs from the zero positions of c_i is $k_i - a_1$ (as shown in Figure 1).

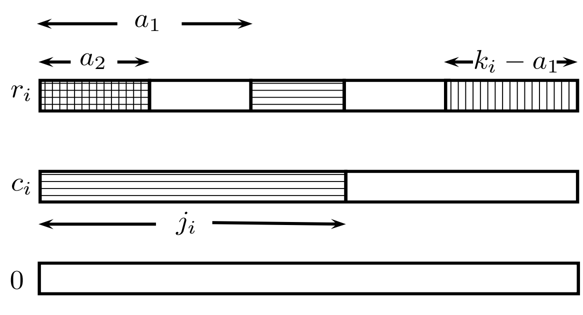

Figure 1: The ith bytes of r_i and c_i.

(The shaded region represents non-zero bits)

From Figure 1, it is clear that the $k_i - a_1$ non-zero bits of r_i can be chosen from $b - j_i$ zero bits of c_i in $(|R|-1)^{k_i - a_1}\binom{b-j_i}{k_i-a_1}$ ways and the a_1 bits of r_i can be chosen from the j_i non-zero bits of c_i in $(|R|-2)^{a_2}\binom{j_i}{a_1}\binom{a_1}{a_2}$ ways (as each of the k_2 non-zero bits of r_i can be chosen in $(|R|-2)$ ways). Since $w_M(c_i) = \mu_i$ and $d_M(c_i, r_i) = \delta_i$, we have $(\mu_i - 1)t + 1 \le j_i \le \min(\mu_i t, b)$ and $(\delta_i - 1)t + 1 \le k_i \le \min(\delta_i t, b)$. Therefore if p_i

is the bit-error probability in the ith byte of any codeword of $\mathcal{C}$, then the probability of receiving ith byte $r_i \in R^b$ (provided **0**-byte is sent) satisfying $d_M(r_i, c_i) = \delta_i$ for each i, is given by

$$R_{\delta_i,\mu_i}(p_i) = \sum_{j_i=(\mu_i-1)t+1}^{\mu_i t} \sum_{k_i=(\delta_i-1)t+1}^{\delta_i t} \sum_{a_1=0}^{k_i} \sum_{a_2=0}^{a_1} \binom{b-j_i}{k_i-a_1}\binom{j_i}{a_1}\binom{a_1}{a_2}(|R|-1)^{k_i-a_1}(|R|-2)^{a_2} \left(\frac{p_i}{|R|-1}\right)^{k_i+j_i-2a_1+a_2}(1-p_i)^{b-j_i-k_i+2a_1-a_2}.$$

Hence the probability that the received vector r is at a m-spotty Hamming distance δ from the codeword c is given by

$$\sum\nolimits_{\delta} R_{\delta_1,\mu_1}(p_1)\dots R_{\delta_n,\mu_n}(p_n),$$

where the sum $\sum_\delta$ runs over all n-tuples $(\delta_1, \delta_2, \dots, \delta_n)$ satisfying $\delta_1 + \cdots + \delta_n = \delta$ and $0 \le \delta_i \le \lceil \frac{b}{t} \rceil$ for each i. ☐

Proof of Theorem 7. Let $c' \in \mathcal{C}$ be transmitted through the channel $\mathfrak{C}$ and a vector $r \in R^{bn}$ is received. Then a decoding error occurs if $d_M(r, c^*) \le \lfloor \frac{d_M-1}{2} \rfloor$ from some other codeword $c^* \in \mathcal{C}$, and we say that the received vector r lies in the decoding sphere of c^* (here d_M is the m-spotty Hamming distance of $\mathcal{C}$).

Without any loss of generality, we can assume that the zero codeword is sent and a vector $r \in R^{bn}$ is received. Let c be a codeword such that the m-spotty Hamming weight of the ith byte of c is μ_i $(1 \le i \le n)$. Then the probability that the received vector r will lie in the decoding sphere of c is given by

$$\sum_{\delta=0}^{\lfloor \frac{d_M-1}{2} \rfloor} \sum_{\substack{\delta_1+\cdots+\delta_n=\delta \\ 0\le\delta_i\le\lceil \frac{b}{t} \rceil}} R_{\delta_1,\mu_1}(p_1)\dots R_{\delta_n,\mu_n}(p_n).$$

Further if $B_{(\mu_1,\dots,\mu_n)}$ is the number of codewords in $\mathcal{C}$ having m-spotty Hamming weight of the ith byte as μ_i for each i, then the probability of decoding error when a bounded distance decoder is used for decoding, is given by

$$P(p_i : 1 \le i \le n) = \sum_{\mu=d_M}^{n\lceil \frac{b}{t} \rceil} \sum_{\substack{\mu_1+\cdots+\mu_n=\mu \\ 0\le\mu_i\le\lceil \frac{b}{t} \rceil}} B_{(\mu_1,\dots,\mu_n)} \sum_{\delta=0}^{\lfloor \frac{d_M-1}{2} \rfloor} \sum_{\substack{\delta_1+\cdots+\delta_n=\delta \\ 0\le\delta_i\le\lceil \frac{b}{t} \rceil}} R_{\delta_1,\mu_1}(p_1)\dots R_{\delta_n,\mu_n}(p_n),$$

which proves the theorem. ☐

3.3 r-fold joint m-spotty Hamming weight enumerator

In this subsection, we define the r-fold joint m-spotty Hamming weight enumerator of r byte error-control codes over R and derive some MacWilliams identities for the same. For this, we need to define the following:

Let $\mathbb{F}_2 = \{0, 1\}$ and $\mathbb{F}_2^r$ be the vector space of all r-tuples over $\mathbb{F}_2$. For each $a \in \mathbb{F}_2^r$, let $[a]_i$ $(1 \leq i \leq r)$ denote the ith coordinate of a. Define $S_i = \{a \in \mathbb{F}_2^r : [a]_i = 1\}$ for each i.

For positive integers k and m, let $(R^m)^k = \underbrace{R^m \times R^m \times \cdots \times R^m}_{k \text{ times}}$.

Definition 7. [9] For each $a \in (\mathbb{F}_2^r)^* = \mathbb{F}_2^r \setminus \{0\}$, define a function $n_a : (R^b)^r \to \mathbb{Z}$ as

$$n_a(c_1, c_2, \ldots, c_r) = |\{k : 1 \leq k \leq b, (\widehat{c_{1k}}, \widehat{c_{2k}}, \ldots, \widehat{c_{rk}}) = a\}|,$$

where for each i, $c_i = (c_{i1}, c_{i2}, \ldots, c_{ib}) \in R^b$ and $\widehat{c_{ik}}$ $(1 \leq k \leq b)$ is given by

$$\widehat{c_{ik}} = \begin{cases} 1 & \text{if } c_{ik} \neq 0; \\ 0 & \text{if } c_{ik} = 0. \end{cases}$$

Definition 8. [9] Let $\{e_1, e_2, \ldots, e_r\}$ be the standard ordered basis of $\mathbb{F}_2^r$ over $\mathbb{F}_2$. Now for each $a \in (\mathbb{F}_2^r)^*$, define a function $N_a : (R^b)^r \to \mathbb{Z}$ as

$$N_a(c_1, c_2, \ldots, c_r) = \left\lfloor \frac{n_a(c_1, c_2, \ldots, c_r)}{t} \right\rfloor + \Omega_a(c_1, c_2, \ldots, c_r),$$

where the number $\Omega_a(c_1, c_2, \ldots, c_r)$ is given by

$$\Omega_a(c_1, c_2, \ldots, c_r) = \begin{cases} \left\lceil \dfrac{\sum\limits_{\beta \in S_i} \overline{n_\beta(c_1, c_2, \ldots, c_r)}}{t} \right\rceil & \text{if } a = e_i \quad (1 \leq i \leq r); \\ 0 & \text{otherwise.} \end{cases}$$

(Here $\bar{x}$ denotes the least non-negative residue of x modulo t.)

We extend the functions N_a $(a \in (\mathbb{F}_2^r)^*)$ defined on $(R^b)^r$ to the elements of $(R^{bn})^r$ as

$$N_a(c_1, c_2, \ldots, c_r) = \sum_{j=1}^{n} N_a(c_1^{(j)}, c_2^{(j)}, \ldots, c_r^{(j)}), \tag{7}$$

where each $c_i = (c_i^{(1)}, c_i^{(2)}, \ldots, c_i^{(n)}) \in R^{bn}$ with $c_i^{(j)} = (c_{i1}^{(j)}, c_{i2}^{(j)}, \ldots, c_{ib}^{(j)}) \in R^b$.

Now we extend the definition of r-fold joint m-spotty Hamming weight enumerator for r byte error-control codes over R as follows:

Definition 9. [9] Let $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_r$ be r byte error-control codes of length bn and having the same byte length b over R. Then the r-fold joint m-spotty Hamming weight enumerator of the codes $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_r$ is defined as

$$\mathcal{J}_{\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_r}(x_a : a \in (\mathbb{F}_2^r)^*) = \sum_{c_1 \in \mathcal{C}_1} \sum_{c_2 \in \mathcal{C}_2} \cdots \sum_{c_r \in \mathcal{C}_r} \prod_{a \in (\mathbb{F}_2^r)^*} x_a^{N_a(c_1, c_2, \ldots, c_r)},$$

where each $c_i = (c_i^{(1)}, c_i^{(2)}, \ldots, c_i^{(n)})$ with $c_i^{(j)} = (c_{i1}^{(j)}, c_{i2}^{(j)}, \ldots, c_{ib}^{(j)})$ for $1 \le j \le n$; and the numbers $N_a(c_1, c_2, \ldots, c_r)$ are as defined above.

Remark 4. (i) When $t = 1$, the r-fold joint m-spotty Hamming weight enumerator of r byte error-control codes over R is the r-fold joint Hamming weight enumerator of the codes over R.
(ii) When $r = 1$, the r-fold joint m-spotty Hamming weight enumerator of r byte error-control codes over R coincides with the m-spotty Hamming weight enumerator of a byte error-control over R.
(iii) When $r = 2$, the r-fold joint m-spotty Hamming weight enumerator of r byte error-control codes over R is same as the joint m-spotty Hamming weight enumerator of two byte error-control codes over R.

In the following theorem, we show that the r-fold joint m-spotty Hamming weight enumerator generalizes the m-spotty Hamming weight enumerator just like the joint probability density function generalizes single probability density function.

Theorem 8. [9] *Let $\mathcal{J}_{\mathcal{C}_1,\mathcal{C}_2,\ldots,\mathcal{C}_r}(x_a : a \in (\mathbb{F}_2^r)^*)$ be the r-fold joint m-spotty Hamming weight enumerator of byte error-control codes $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_r$ of length bn and byte length b over R. Then we have the following:*

(i) $\mathcal{J}_{\mathcal{C}_1,\mathcal{C}_2,\ldots,\mathcal{C}_r}(1, 1, \ldots, 1) = |\mathcal{C}_1||\mathcal{C}_2| \ldots |\mathcal{C}_r|$.

(ii) *For integers $1 \le i, j \le r$, the r-fold joint m-spotty Hamming weight enumerator of the codes $\mathcal{C}_1, \ldots, \mathcal{C}_j, \ldots, \mathcal{C}_i, \ldots, \mathcal{C}_r$ (i.e., the same sequence of codes except for $\mathcal{C}_i$ and $\mathcal{C}_j$ interchanged) is given by $\mathcal{J}_{\mathcal{C}_1,\mathcal{C}_2,\ldots,\mathcal{C}_r}(x_{\tilde{a}} : \tilde{a} \in (\mathbb{F}_2^r)^*)$, where for each $a \in (\mathbb{F}_2^r)^*$, the tuples $\tilde{a} \in (\mathbb{F}_2^r)^*$ are defined as*

$$[\tilde{a}]_k = \begin{cases} [a]_j & \text{if } k = i; \\ [a]_i & \text{if } k = j; \\ [a]_k & \text{otherwise} \end{cases}$$

for $1 \le k \le r$.

(iii) *The m-spotty Hamming weight enumerator of the code $\mathcal{C}_i$ $(1 \le i \le n)$ is given by*

$$W_{\mathcal{C}_i}(z) = \frac{1}{\prod_{j}{}_i |\mathcal{C}_j|} \mathcal{J}_{\mathcal{C}_1,\mathcal{C}_2,\ldots,\mathcal{C}_r}(x_a : a \in (\mathbb{F}_2^r)^*) \text{ with } x_a = \begin{cases} z & \text{if } a \in S_i; \\ 1 & \text{otherwise,} \end{cases}$$

where the product $\prod_{j}{}_i$ is extended over all integers j satisfying $1 \le j \le r$ and $j \ne i$.

Proof. For proof, see Sharma et al. [9, Theorem 3.6]. □

Now to derive MacWilliams identities for r-fold joint m-spotty Hamming weight enumerator of r byte error-control codes over R, we need to define the following:

For each $a \in \mathbb{F}_2^r$ and each integer i $(1 \leq i \leq r)$, define the $r+1$-tuples $\sigma_i(a)$, $\mu_i(a) \in \mathbb{F}_2^{r+1}$ as

$$[\sigma_i(a)]_j = \begin{cases} [a]_j & \text{if } 1 \leq j \leq i-1; \\ 1 & \text{if } j = i; \\ [a]_{j-1} & \text{if } i+1 \leq j \leq r+1, \end{cases} \qquad [\mu_i(a)]_j = \begin{cases} [a]_j & \text{if } 1 \leq j \leq i-1; \\ 0 & \text{if } j = i; \\ [a]_{j-1} & \text{if } i+1 \leq j \leq r+1. \end{cases}$$

Note that $\mathbb{F}_2^{r+1} = \bigcup_{a \in \mathbb{F}_2^r} \{\sigma_i(a), \mu_i(a)\}$ for each i.

Definition 10. [9] Let t $(1 \leq t \leq b)$ and q $(1 \leq q \leq r)$ be fixed integers. Let $\delta = (\delta_a : a \in \mathbb{F}_2^r)$ be a 2^r-tuple over $\{0, 1, 2, \ldots, b\}$ satisfying $\sum_{a \in \mathbb{F}_2^r} \delta_a = b$. Let $\mathcal{A}_p$ $(0 \leq p \leq b)$ be the set of all tuples $\alpha = (\alpha_g : g \in \mathbb{F}_2^{r+1})$ of non-negative integers α_g's satisfying the following:

$$\sum_{a \in \mathbb{F}_2^r} \alpha_{\sigma_{q+1}(a)} = p \quad \text{and} \quad \alpha_{\sigma_{q+1}(a)} + \alpha_{\mu_{q+1}(a)} = \delta_a \text{ for each } a \in \mathbb{F}_2^r.$$

Then the polynomial $G_\delta(x_a : a \in (\mathbb{F}_2^r)^*)$ is defined as

$$\sum_{p=0}^{b} \sum\nolimits_p (-1)^{h_p(\alpha)} (|R|-1)^{p-h_p(\alpha)} \prod_{a \in \mathbb{F}_2^r} \binom{\delta_a}{\alpha_{\sigma_{q+1}(a)}} \prod_{a \in (\mathbb{F}_2^r)^*} x_a^{\left\lfloor \frac{\alpha_{\sigma_q(a)} + \alpha_{\mu_q(a)}}{t} \right\rfloor + \theta_a^{(\alpha)}}, \tag{8}$$

where for each p $(0 \leq p \leq b)$, the summation $\sum_p$ runs over the set $\mathcal{A}_p$; and further for each tuple $\alpha \in \mathcal{A}_p$, the numbers $\theta_a^{(\alpha)}$ and $h_p(\alpha)$ are given by

$$\theta_a^{(\alpha)} = \begin{cases} \left\lceil \dfrac{\sum_{\beta \in S_i} (\overline{\alpha_{\sigma_q(\beta)} + \alpha_{\mu_q(\beta)}})}{t} \right\rceil & \text{if } a = e_i \ \ (1 \leq i \leq r); \\ 0 & \text{otherwise} \end{cases} \tag{9}$$

and

$$h_p(\alpha) = \sum_{a \in S_q} \alpha_{\sigma_q(a)}.$$

Definition 11. [9] Let $\delta_i = (\delta_a^{(i)} : a \in \mathbb{F}_2^r)$ be a 2^r-tuple over $\{0, 1, 2, \ldots, b\}$ satisfying $\sum_{a \in \mathbb{F}_2^r} \delta_a^{(i)} = b$ for $1 \leq i \leq n$. Then for $\delta = (\delta_1, \delta_2, \ldots, \delta_n)$, we define the polynomial

$$G_\delta(x_a : a \in (\mathbb{F}_2^r)^*) = \prod_{i=1}^{n} G_{\delta_i}(x_a : a \in (\mathbb{F}_2^r)^*). \tag{10}$$

Definition 12. [9] Let $c = (c_1, c_2, \ldots, c_r) \in (R^b)^r$ with each $c_i = (c_{i1}, c_{i2}, \ldots, c_{ib}) \in R^b$. Then the joint composition vector of c, denoted by $j(c)$, is defined as the tuple $\delta = (\delta_a : a \in \mathbb{F}_2^r)$, where for each $a \in \mathbb{F}_2^r$, δ_a is given by

$$\delta_a = |\{k : 1 \leq k \leq b, (\hat{c}_{1k}, \hat{c}_{2k}, \ldots, \hat{c}_{rk}) = a\}| \text{ with } \hat{c}_{ik} = \begin{cases} 1 & \text{if } c_{ik} \neq 0; \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that $\sum\limits_{a \in \mathbb{F}_2^r} \delta_a = b$.

Let $c = (c^{(1)}, c^{(2)}, \ldots, c^{(r)}) \in (R^{bn})^r$ with each $c^{(i)} = (c_{i1}, c_{i2}, \ldots, c_{in}) \in R^{bn}$. Then the joint composition vector of c is defined as

$$j(c) = \delta = (\delta_1, \delta_2, \ldots, \delta_n),$$

where $\delta_k = j(c_{1k}, c_{2k}, \ldots, c_{rk})$ for $1 \leq k \leq n$ with each $c_{ik} \in R^b$.

In the following theorem, we derive some MacWilliams identities for the r-fold joint m-spotty Hamming weight enumerator of r byte error-control codes over R.

Theorem 9. [9] *Let $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_r$ be byte error-control codes of length bn and byte length b over R. Let $P_q(\delta)$ $(1 \leq q \leq r)$ be the number of r-tuples $(c_1, c_2, \ldots, c_r)$ of codewords $c_i \in \mathcal{C}_i$ $(1 \leq i \leq r, i \neq q)$ and $c_q \in \mathcal{C}_q^{\perp}$ having the joint composition vector as δ. Then we have*

$$\mathcal{J}_{\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_r}(x_a : a \in (\mathbb{F}_2^r)^*) = \frac{1}{|\mathcal{C}_q^{\perp}|} \sum P_q(\delta) G_\delta(x_a : a \in (\mathbb{F}_2^r)^*),$$

where the summation runs over all n-tuples $\delta = (\delta_1, \delta_2, \ldots, \delta_n)$ such that each $\delta_i = (\delta_a^{(i)} : a \in \mathbb{F}_2^r)$ is a 2^r-tuple over $\{0, 1, 2, \ldots, b\}$ satisfying $\sum\limits_{a \in \mathbb{F}_2^r} \delta_a^{(i)} = b$, and the polynomials $G_\delta(x_a : a \in (\mathbb{F}_2^r)^*)$'s are as defined by (10).

Remark 5. (i) When at least one of the codes $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_r$ (say $\mathcal{C}_q$) is of large size, it is usually very hard to determine the r-fold joint m-spotty Hamming weight enumerator for them. However, by applying Theorem 9, it is comparatively easier to determine the same from the list of numbers $P_q(\delta)$'s for the codes $\mathcal{C}_1, \ldots, \mathcal{C}_{q-1}, \mathcal{C}_q^{\perp}, \mathcal{C}_{q+1}, \ldots, \mathcal{C}_r$.

(ii) When $r = 1$, Theorem 9 gives a MacWilliams identity for the m-spotty Hamming weight enumerator of a byte error-control code over R; and when $r = 2$, Theorem 9 gives a MacWilliams identity for the joint m-spotty Hamming weight enumerator of two byte error-control codes over R.

Proof of Theorem 9. The r-fold joint m-spotty Hamming weight enumerator of the codes $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_r$ is given by

$$\mathcal{J}_{\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_r}(x_a : a \in (\mathbb{F}_2^r)^*) = \sum \prod_{a \in (\mathbb{F}_2^r)^*} x_a^{N_a(c_1, c_2, \ldots, c_r)},$$

where the summation $\sum$ runs over all the codewords $c_k \in \mathcal{C}_k$ for $1 \leq k \leq r$.

For $v \in R^{bn}$, let $f(v) = \sum_q \prod_{a \in (\mathbb{F}_2^r)^*} x_a^{N_a(c_1,\ldots,c_{q-1},v,c_{q+1},\ldots,c_r)}$, where the summation $\sum_q$ runs over all codewords $c_k \in \mathcal{C}_k$ for $1 \leq k \leq r$ and $k \neq q$. Then for $u = (u_1, u_2, \ldots, u_n) \in R^{bn}$, using Lemma 3, we have

$$\begin{aligned}\tilde{f}(u) &= \sum_{v \in R^{bn}} \chi(\langle u, v\rangle) \sum_q \prod_{a \in (\mathbb{F}_2^r)^*} x_a^{N_a(c_1,\ldots,c_{q-1},v,c_{q+1},\ldots,c_r)} \\ &= \sum_q \sum_{v=(v_1,v_2,\ldots,v_n) \in R^{bn}} \chi\Big(\sum_{i=1}^n \langle u_i, v_i\rangle\Big) \prod_{a \in (\mathbb{F}_2^r)^*} x_a^{N_a(c_1,\ldots,c_{q-1},v,c_{q+1},\ldots,c_r)} \\ &= \sum_q \prod_{i=1}^n \left\{ \sum_{v_i \in R^b} \chi\big(\langle u_i, v_i\rangle\big) \prod_{a \in (\mathbb{F}_2^r)^*} x_a^{N_a(c_{1i},\ldots,c_{(q-1)i},v_i,c_{(q+1)i},\ldots,c_{ri})} \right\},\end{aligned}$$

where each $c_k = (c_{k1}, c_{k2}, \ldots, c_{kn}) \in \mathcal{C}_k$ for $1 \leq k \leq r$ and $k \neq q$.

If the joint composition vector of the r-tuple $(c_{1i}, \ldots, c_{(q-1)i}, u_i, c_{(q+1)i}, \ldots, c_{ri})$ is δ_i for each i, then working as in Lemma 4.7 of Sharma et al. [9], we get

$$\sum_{v_i \in R^b} \chi\big(\langle u_i, v_i\rangle\big) \prod_{a \in (\mathbb{F}_2^r)^*} x_a^{N_a(c_{1i},\ldots,c_{(q-1)i},v_i,c_{(q+1)i},\ldots,c_{ri})} = G_{\delta_i}(x_a : a \in (\mathbb{F}_2^r)^*).$$

This gives

$$\sum_{c_q \in \mathcal{C}_q^\perp} \tilde{f}(c_q) = \sum \prod_{i=1}^n G_{\delta_i}(x_a : a \in (\mathbb{F}_2^r)^*),$$

where the summation $\sum$ runs over all codewords $c_k = (c_{k1}, c_{k2}, \ldots, c_{kn}) \in \mathcal{C}_k$ $(1 \leq k \leq r, k \neq q)$ and $c_q = (c_{q1}, c_{q2}, \ldots, c_{qn}) \in \mathcal{C}_q^\perp$ satisfying $j(c_{1i}, c_{2i} \ldots, c_{ri}) = \delta_i$ for each i.

Further suppose that the number of r-tuples $(c_1, c_2, \ldots, c_r)$ of codewords $c_k \in \mathcal{C}_k$ $(1 \leq k \leq r, k \neq q)$ and $c_q \in \mathcal{C}_q^\perp$ having $j(c_1, c_2, \ldots, c_r) = \delta$ is $P_q(\delta)$. Then using (10), we have

$$\sum_{c_q \in \mathcal{C}_q^\perp} \tilde{f}(c_q) = \sum P_q(\delta) G_\delta(x_a : a \in (\mathbb{F}_2^r)^*), \tag{11}$$

where the summation $\sum$ runs over all n-tuples $\delta = (\delta_1, \delta_2, \ldots, \delta_n)$ such that each $\delta_i = (\delta_a^{(i)} : a \in \mathbb{F}_2^r)$ is a 2^r-tuple over $\{0, 1, 2, \ldots, b\}$ satisfying $\sum_{a \in \mathbb{F}_2^r} \delta_a^{(i)} = b$. Again applying Lemma 3 and using (11), we get

$$\begin{aligned}\mathcal{J}_{\mathcal{C}_1,\mathcal{C}_2,\ldots,\mathcal{C}_r}(x_a : a \in (\mathbb{F}_2^r)^*) &= \sum_{v \in \mathcal{C}_q} f(v) = \frac{1}{|\mathcal{C}_q^\perp|} \sum_{c_q \in \mathcal{C}_q^\perp} \tilde{f}(c_q) \\ &= \frac{1}{|\mathcal{C}_q^\perp|} \sum P_q(\delta) G_\delta(x_a : a \in (\mathbb{F}_2^r)^*),\end{aligned}$$

which proves the theorem. □

3.4 Complete m-spotty Hamming weight enumerator

In this subsection, we extend the definition of complete m-spotty Hamming weight enumerator for a byte error-control code over R and derive a MacWilliams identity for the same.

Definition 13. Let $\mathcal{C}$ be a byte error-control code of length bn and byte length b over R. Then the complete m-spotty Hamming weight enumerator of $\mathcal{C}$ is defined as

$$\mathfrak{CH}_{\mathcal{C}}(z_0, z_1, \ldots, z_b) = \sum_{u=(u_1,u_2,\ldots,u_n)\in\mathcal{C}} \prod_{i=1}^{n} z_{w_H(u_i)}.$$

Note that the m-spotty Hamming weight enumerator of a byte error-control code $\mathcal{C}$ over R can be obtained from the complete m-spotty Hamming weight enumerator of $\mathcal{C}$ by replacing z_j with $z^{\lceil j/t \rceil}$ for each j.

If $A_{(\alpha_1,\alpha_2,\ldots,\alpha_n)}$ denotes the number of codewords in $\mathcal{C}$ having the Hamming weight distribution vector as $(\alpha_1, \alpha_2, \ldots, \alpha_n)$, then the complete m-spotty Hamming weight enumerator can be rewritten as

$$\mathfrak{CH}_{\mathcal{C}}(z_0, z_1, \ldots, z_b) = \sum_{(\alpha_1,\alpha_2,\ldots,\alpha_n)} A_{(\alpha_1,\alpha_2,\ldots,\alpha_n)} \prod_{i=1}^{n} z_{\alpha_i},$$

where the summation runs over all n-tuples $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ satisfying $0 \le \alpha_i \le b$ for each i.

In the following theorem, we derive a MacWilliams identity for the complete m-spotty Hamming weight enumerator of a byte error-control code over R.

Theorem 10. *Let $\mathcal{C}$ be a byte error-control code of length bn and byte length b over R with $\mathcal{C}^\perp$ being its dual code. Then the complete m-spotty Hamming weight enumerator of $\mathcal{C}$ is given by*

$$\mathfrak{CH}_{\mathcal{C}}(z_0, z_1, \ldots, z_b) = \frac{1}{|\mathcal{C}^\perp|} \sum A_{(\alpha_1,\alpha_2,\ldots,\alpha_n)} \prod_{i=1}^{n} \left(\sum_{p_i=0}^{b} K_{p_i}(\alpha_i) z_{p_i} \right),$$

where the summation $\sum$ runs over all n-tuples $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ satisfying $0 \le \alpha_i \le b$ for each i, $A_{(\alpha_1,\alpha_2,\ldots,\alpha_n)}$ is the number of codewords in $\mathcal{C}^\perp$ having the Hamming weight distribution vector as $(\alpha_1, \alpha_2, \ldots, \alpha_n)$, and $K_p(X) = \sum_{a=0}^{p} (-1)^a (|R|-1)^{p-a} \binom{X}{a} \binom{b-X}{p-a}$ (assuming $\binom{j}{\ell} = 0$ for $j < \ell$ and $\binom{0}{0} = 0$).

Proof. Let $f(v) = \prod_{i=1}^{n} z_{w_H(v_i)}$ for $v = (v_1, v_2, \ldots, v_n) \in R^{bn}$. Then by Lemma 3, for $u = (u_1, u_2, \ldots, u_n) \in R^{bn}$, $\tilde{f}(u)$ is given by

$$\tilde{f}(u) = \sum_{v=(v_1,v_2,\ldots,v_n)\in R^{bn}} \chi(\langle u, v\rangle) \prod_{i=1}^{n} z_{w_H(v_i)} = \prod_{i=1}^{n} \left(\sum_{v_i \in R^b} \chi(\langle u_i, v_i\rangle) z_{w_H(v_i)} \right). \quad (12)$$

If $w_H(u_i) = \alpha_i$ for $1 \leq i \leq n$, then working as in Lemma 27 of Sharma et al. [8], we get

$$\sum_{v_i \in R^b} \chi(\langle u_i, v_i \rangle) z_{w_H(v_i)} = \sum_{p_i=0}^{b} K_{p_i}(\alpha_i) z_{p_i}$$

for each i. Thus

$$\tilde{f}(u) = \prod_{i=1}^{n} \left(\sum_{p_i=0}^{b} K_{p_i}(\alpha_i) z_{p_i} \right),$$

where $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ is the Hamming weight distribution vector of u.

Let $A_{(\alpha_1,\alpha_2,\ldots,\alpha_n)}$ be the number of codewords in $\mathcal{C}^\perp$ having the Hamming weight distribution vector as $(\alpha_1, \alpha_2, \ldots, \alpha_n)$. Then we have

$$\sum_{u \in \mathcal{C}^\perp} \tilde{f}(u) = \sum A_{(\alpha_1,\alpha_2,\ldots,\alpha_n)} \prod_{i=1}^{n} \left(\sum_{p_i=0}^{b} K_{p_i}(\alpha_i) z_{p_i} \right), \tag{13}$$

where the summation $\sum$ runs over all n-tuples $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ satisfying $0 \leq \alpha_i \leq b$ for each i.

Now applying Lemma 3 and using (13), we get

$$\begin{aligned} \mathfrak{CH}_{\mathcal{C}}(z_0, z_1, \ldots, z_b) &= \sum_{v \in \mathcal{C}} f(v) = \frac{1}{|\mathcal{C}^\perp|} \sum_{u \in \mathcal{C}^\perp} \tilde{f}(u) \\ &= \frac{1}{|\mathcal{C}^\perp|} \sum A_{(\alpha_1,\alpha_2,\ldots,\alpha_n)} \prod_{i=1}^{n} \left(\sum_{p_i=0}^{b} K_{p_i}(\alpha_i) z_{p_i} \right). \end{aligned}$$

This proves the theorem. □

4 Weight enumerators with respect to the m-spotty Lee metric

In this section, we discuss various weight enumerators defined with respect to the m-spotty Lee metric and derive MacWilliams identities for them. First of all, we define the Lee weight in $\mathbb{F}_q$ as follows:

Definition 14. [5] Let $e_1, e_2, \ldots, e_k$ be the standard basis of the real k-dimensional space $\mathbb{R}^k$ over $\mathbb{R}$. Now define a map $\varphi : \mathbb{Z}^k \to \mathbb{F}_q$ as $\varphi(x) = \sum_{i=1}^{k} a_i x_i \pmod{p}$ for $x = \sum_{i=1}^{k} x_i e_i$ ($x_i \in \mathbb{Z}$), where $a_1, a_2, \ldots, a_k$ are distinct elements of the finite field $\mathbb{F}_q$ chosen in such a way that the map φ is surjective. Then the Lee weight $w_L(a)$ of an element $a \in \mathbb{F}_q$ is defined as

$$w_L(a) = \min_{\substack{x=(x_i)\in\mathbb{Z}^k \\ \varphi(x)=a}} \left\{ \sum_{i=1}^{k} |x_i| \right\},$$

where $|x|$ denotes the absolute value of a real number x.
The Lee weight of a vector over $\mathbb{F}_q$ is defined as the sum of Lee weights of all its components.

Next we define Lee weights in R_1 and R_2, in terms of certain gray maps, as follows:

Definition 15. Let $\phi_1 : R_1 \to \mathbb{F}_q^e$ be a gray map defined as follows:

$$\phi_1(r) = \left(\sum_{i=0}^{e-1} a_i(r), \sum_{i=1}^{e-1} a_i(r), \ldots, a_{e-1}(r) \right)$$

for each $r = a_0(r) + ua_1(r) + \cdots + u^{e-1}a_{e-1}(r) \in R_1$ with $a_i(r)$'s in $\mathbb{F}_q$. Then the Lee weight of an element $r \in R_1$ is defined as the Lee weight of the e-tuple $\phi_1(r) \in \mathbb{F}_q^e$. The Lee weight of an n-tuple over R_1 is defined as the sum of Lee weights of its components.

Definition 16. Let $\phi_2 : R_2 \to \mathbb{F}_q^4$ be a gray map defined as

$$\phi_2(r) = (a_0(r) + a_1(r) + a_2(r) + a_3(r), a_2(r) + a_3(r), a_1(r) + a_3(r), a_3(r))$$

for $r = a_0(r) + ua_1(r) + va_2(r) + uva_3(r) \in R_2$ with each $a_i(r) \in \mathbb{F}_q$. Then we define the Lee weight of an element $r \in R_2$ as the Lee weight of the 4-tuple $\phi_2(r) \in \mathbb{F}_q^4$. Further, the Lee weight of an n-tuple over R_2 is defined as the sum of Lee weights of its components.

Let R be either the ring R_1 or the ring R_2. Then we extend the notion of m-spotty Lee distance in R^{bn} as follows:

Definition 17. [10, 13] Let u, v be two vectors in R^{bn} having u_i, v_i as their ith bytes. Then the m-spotty Lee distance between u and v, denoted by $d_{ML}(u, v)$, is defined as

$$d_{ML}(u, v) = \sum_{i=1}^{n} \left\lceil \frac{d_L(u_i, v_i)}{t} \right\rceil,$$

where $d_L(u_i, v_i)$ denotes the Lee distance between ith bytes u_i and v_i of u and v, respectively.

Note that d_{ML} is a metric on R^{bn}. When $t = 1$, the m-spotty Lee distance between any two vectors in R^{bn} is same as the Lee distance between those two vectors.

Definition 18. [10, 13] The m-spotty Lee weight of a vector $u = (u_1, u_2, \ldots, u_n) \in R^{bn}$, denoted by $w_{ML}(u)$, is defined as

$$w_{ML}(u) = \sum_{i=1}^{n} \left\lceil \frac{w_L(u_i)}{t} \right\rceil,$$

where $w_L(u_i)$ denotes the Lee weight of ith byte u_i of u.

Observe that $d_{ML}(u, v) = w_{ML}(u - v)$. When $t = 1$, the m-spotty Lee weight of a vector in R^{bn} is same as the Lee weight of the vector.

4.1 m-spotty Lee weight enumerator

In this subsection, we define the m-spotty Lee weight enumerator of a byte error-control code over R, derive a MacWilliams identity for the same and discuss its application. Let $|R| = \ell$, and let the elements of R be listed as $r_0 = 0, r_1, \ldots, r_{\ell-1}$.

Definition 19. [10, 13] Let $\mathcal{C}$ be a byte error-control code of length bn and byte length b over R. Then the m-spotty Lee weight enumerator of $\mathcal{C}$ is defined as

$$L_{\mathcal{C}}(z) = \sum_{u \in \mathcal{C}} z^{w_{ML}(u)} = \sum_{u=(u_1,\ldots,u_n)\in\mathcal{C}} \prod_{i=1}^{n} z^{\lceil w_L(u_i)/t \rceil}.$$

When $t = 1$, the m-spotty Lee weight enumerator of $\mathcal{C}$ coincides with the Lee weight enumerator of $\mathcal{C}$.

Let $u = (u_1, u_2, \ldots, u_n)$ be a vector in R^{bn} with each $u_i \in R^b$. For $0 \le k \le \ell - 1$, if j_{ik} is the number of bits in u_i that are equal to r_k, then the composition of the ith byte u_i of u is defined as the ℓ-tuple $J_i = (j_{i0}, j_{i1}, \ldots, j_{i,\ell-1})$ and the composition vector of u is defined as the tuple $J = (J_1, J_2, \ldots, J_n)$.

Now if $A(J)$ denotes the number of codewords in $\mathcal{C}$ having the composition vector as J, then the m-spotty Lee weight enumerator can be rewritten as

$$L_{\mathcal{C}}(z) = \sum_{J} A(J) \prod_{i=1}^{n} z^{\lceil \rho(J_i)/t \rceil},$$

where the summation $\sum\limits_{J}$ runs over all n-tuples $J = (J_1, J_2, \ldots, J_n)$ with each $J_i = (j_{i0}, j_{i1}, \ldots, j_{i(\ell-1)})$ an ℓ-tuple over $\{0, 1, 2, \ldots, b\}$, and $\rho(J_i) = \sum\limits_{k=0}^{\ell-1} w_L(r_k) j_{ik}$.

Definition 20. For a fixed positive integer t and an ℓ-tuple $J = (j_0, j_1, \ldots, j_{\ell-1})$ over $\{0, 1, 2, \ldots, b\}$, we define a polynomial $g_J^{(t)}(z)$ as

$$g_J^{(t)}(z) = \sum_{s_{kp}} \left(\prod_{k=0}^{\ell-1} \frac{j_k!}{\prod\limits_{p=0}^{\ell-1} s_{kp}!} \chi\left(\sum_{p=0}^{\ell-1} r_k r_p s_{kp} \right) \right) z^{\left\lceil \sum\limits_{k=0}^{\ell-1} \left(\sum\limits_{p=1}^{\ell-1} w_L(r_p) s_{kp} \right) / t \right\rceil}, \quad (14)$$

where the summation $\sum\limits_{s_{kp}}$ runs over all non-negative integers s_{kp} $(0 \le k, p \le \ell - 1)$ satisfying $\sum\limits_{p=0}^{\ell-1} s_{kp} = j_k$ for each k, and χ is the non-trivial additive character on R, given by

$$\chi = \begin{cases} \chi_1 & \text{when } R = R_1, \\ \chi_2 & \text{when } R = R_2 \end{cases}$$

with χ_1 and χ_2 as defined by (1) and (2).

In the following theorem, we derive a MacWilliams identity for the m-spotty Lee weight enumerator of a byte error-control code over R.

Theorem 11. [10] *Let $\mathcal{C}$ be a byte error-control code of length bn and byte length b over R and $\mathcal{C}^\perp$ be its dual code. Then the m-spotty Lee weight enumerator of $\mathcal{C}$ is given by*

$$L_{\mathcal{C}}(z) = \frac{1}{|\mathcal{C}^\perp|} \sum_{(J_1,J_2,\ldots,J_n)} A(J_1, J_2, \ldots, J_n) \prod_{i=1}^{n} g_{J_i}^{(t)}(z),$$

where the summation $\sum\limits_{(J_1,J_2,\ldots,J_n)}$ runs over all n-tuples $(J_1, J_2, \ldots, J_n)$ with each $J_i = (j_{i0}, j_{i1}, \ldots, j_{i,\ell-1})$ an ℓ-tuple over $\{0, 1, 2, \ldots, b\}$, $A(J_1, J_2, \ldots, J_n)$ is the number of codewords in $\mathcal{C}^\perp$ having the composition vector as $(J_1, J_2, \ldots, J_n)$ and for $1 \le i \le n$, the polynomial $g_{J_i}^{(t)}(z)$ is as defined by (14).

Remark 6. The computation of the numbers $A(J_1, J_2, \ldots, J_n)$ is, in general, very hard for a code $\mathcal{C}$ of large size. But the numbers $A(J_1, J_2, \ldots, J_n)$'s are comparatively easier to compute for the dual code $\mathcal{C}^\perp$, which is of relatively smaller size. Thus by applying the MacWilliams identity (Theorem 11), one can easily obtain the m-spotty Lee weight enumerator of $\mathcal{C}$.

Proof of Theorem 11. Let $f(v) = \prod\limits_{i=1}^{n} z^{\lceil w_L(v_i)/t \rceil}$ for $v = (v_1, v_2, \ldots, v_n) \in R^{bn}$, where each $v_i \in R^b$. Then for $u = (u_1, u_2, \ldots, u_n) \in R^{bn}$, by Lemma 3, $\tilde{f}(u)$ is given by

$$\tilde{f}(u) = \sum_{v=(v_1,v_2,\ldots,v_n)\in R^{bn}} \chi(\langle u, v \rangle) \prod_{i=1}^{n} z^{\lceil w_L(v_i)/t \rceil} = \prod_{i=1}^{n} \left(\sum_{v_i \in R^b} \chi(\langle u_i, v_i \rangle) z^{\lceil w_L(v_i)/t \rceil} \right).$$

Now if J_i $(1 \le i \le n)$ is the composition of the ith byte u_i of u, then working as in Lemma 2 of Sharma et al. [10], we get

$$\sum_{v_i \in R^b} \chi(\langle u_i, v_i \rangle) z^{\lceil w_L(v_i)/t \rceil} = g_{J_i}^{(t)}(z)$$

for each i. This gives

$$\tilde{f}(u) = \prod_{i=1}^{n} g_{J_i}^{(t)}(z).$$

If $A(J_1, J_2, \ldots, J_n)$ is the number of codewords in $\mathcal{C}^\perp$ having the composition vector as $(J_1, J_2, \ldots, J_n)$, then

$$\sum_{u \in \mathcal{C}^\perp} \tilde{f}(u) = \sum_{(J_1,J_2,\ldots,J_n)} A(J_1, J_2, \ldots, J_n) \prod_{i=1}^{n} g_{J_i}^{(t)}(z), \tag{15}$$

where the summation runs over all n-tuples $(J_1, J_2, \ldots, J_n)$ with each J_i, an ℓ-tuple over $\{0, 1, 2, \ldots, b\}$.

Again applying Lemma 3 and using (15), we get

$$\begin{aligned} L_{\mathcal{C}}(z) &= \sum_{v\in\mathcal{C}} f(v) = \frac{1}{|\mathcal{C}^{\perp}|}\sum_{u\in\mathcal{C}^{\perp}} \tilde{f}(u) \\ &= \frac{1}{|\mathcal{C}^{\perp}|}\sum_{(J_1,J_2,\ldots,J_n)} A(J_1,J_2,\ldots,J_n)\prod_{i=1}^{n} g_{J_i}^{(t)}(z), \end{aligned}$$

which proves the theorem. □

An application

For a byte error-control code $\mathcal{C}$ of length bn and byte length b over R, the m-spotty Lee distance of the code $\mathcal{C}$ is defined as $d_{ML}(\mathcal{C}) = \min\{d_{ML}(u,v) : u,v\in\mathcal{C}, u\neq v\}$. It is easy to see that $d_{ML}(\mathcal{C}) = \min\{w_{ML}(u) : u\in\mathcal{C}, u\neq 0\}$.

In the following theorem, it is proved that the m-spotty Lee distance of a code measures the m-spotty error-detecting and error-correcting capabilities of the code $\mathcal{C}$.

Theorem 12. [10] *Let $\mathcal{C}$ be a byte error-control code of length bn and byte length b over R. Then we have the following:*

(i) *The code $\mathcal{C}$ can detect any m-spotty byte error e satisfying $w_{ML}(e) < d$ if and only if $d_{ML}(\mathcal{C}) \geq d$.*

(ii) *If $d_{ML}(\mathcal{C}) = d$, then $\mathcal{C}$ can correct all m-spotty byte errors e satisfying $w_{ML}(e) < d/2$, and $\mathcal{C}$ cannot correct any m-spotty byte error e satisfying $w_{ML}(e) \geq d/2$.*

Proof. For proof, see Sharma et al. [10, Theorems 2-3]. □

Observe that the m-spotty Lee distance of $\mathcal{C}$ is the least positive integer d such that the coefficient of z^d in $L_{\mathcal{C}}(z)$ is non-zero. Thus knowing the m-spotty Lee weight enumerator $L_{\mathcal{C}}(z)$ of a code $\mathcal{C}$, one can compute its m-spotty Lee distance.

4.2 Split m-spotty Lee weight enumerator

In this subsection, we define the split m-spotty Lee weight enumerator of a byte error-control code over R, derive a MacWilliams identity for the same and discuss its application.

Definition 21. [10] Let $\mathcal{C}$ be a byte error-control code of length bn and byte length b over R. Then the split m-spotty Lee weight enumerator of $\mathcal{C}$ is defined as

$$S_{\mathcal{C}}(z_i : i = 1,2,\ldots,n) = \sum_{(c_1,c_2,\ldots,c_n)\in\mathcal{C}} \prod_{i=1}^{n} z_i^{w_{ML}(c_i)}.$$

If $A(J_1,J_2,\ldots,J_n)$ is the number of codewords in $\mathcal{C}$ having the composition vector as $(J_1,J_2,\ldots,J_n)$, then the split m-spotty Lee weight enumerator of $\mathcal{C}$ can be rewritten as

$$S_{\mathcal{C}}(z_i : i = 1,2,\ldots,n) = \sum_{(J_1,J_2,\ldots,J_n)} A(J_1,J_2,\ldots,J_n)\prod_{i=1}^{n} z_i^{\lceil \frac{\rho(J_i)}{t}\rceil},$$

where each $J_i = (j_{i0}, j_{i1}, \ldots, j_{i,\ell-1})$ is an ℓ-tuple over $\{0, 1, \ldots, b\}$, and $\rho(J_i) = \sum_{k=0}^{\ell-1} w_L(r_k) j_{ik}$ for each i.

When $z_1 = z_2 = \ldots = z_n = z$, the split m-spotty Lee weight enumerator of $\mathcal{C}$ coincides with the m-spotty Lee weight enumerator of $\mathcal{C}$.

In the following theorem, we derive a MacWilliams identity for the split m-spotty Lee weight enumerator of a byte error-control code over R.

Theorem 13. [10] *Let $\mathcal{C}$ be a byte error-control code of length bn and byte length b over R and $\mathcal{C}^\perp$ be its dual code. Then the split m-spotty Lee weight enumerator of $\mathcal{C}$ is given by*

$$\mathcal{S}_{\mathcal{C}}(z_i : i = 1, 2, \ldots, n) = \frac{1}{|\mathcal{C}^\perp|} \sum_{(J_1, J_2, \ldots, J_n)} A(J_1, J_2, \ldots, J_n) \prod_{i=1}^{n} g_{J_i}^{(t)}(z_i),$$

where the summation runs over all n-tuples $(J_1, J_2, \ldots, J_n)$ with each J_i, an ℓ-tuple over $\{0, 1, 2, \ldots, b\}$, $A(J_1, J_2, \ldots, J_n)$ is the number of codewords in $\mathcal{C}^\perp$ having the composition vector as $(J_1, J_2, \ldots, J_n)$, and the polynomials $g_{J_i}^{(t)}(z_i)$'s are as defined by (14).

Remark 7. (i) When $z_1 = z_2 = \ldots = z_n = z$, Theorem 11 follows from Theorem 13.

(ii) It is generally very hard to compute the numbers $A(J_1, J_2, \ldots, J_n)$ for a code $\mathcal{C}$ of large size, and hence its split m-spotty Lee weight enumerator. However, the dual code $\mathcal{C}^\perp$ of $\mathcal{C}$ is of relatively smaller size, so it is comparatively easier to compute the numbers $A(J_1, J_2, \ldots, J_n)$ for the dual code $\mathcal{C}^\perp$. From this, one can easily obtain the split m-spotty Lee weight enumerator of $\mathcal{C}$ by applying the MacWilliams identity (Theorem 13).

Proof of Theorem 13. We will prove this theorem by applying Lemma 3. For this, let $f(v) = \prod_{i=1}^{n} z_i^{\lceil w_L(v_i)/t \rceil}$ for $v = (v_1, v_2, \ldots, v_n) \in R^{bn}$, where each $v_i \in R^b$. Then by Lemma 3, for $u = (u_1, u_2, \ldots, u_n) \in R^{bn}$, $\tilde{f}(u)$ is given by

$$\tilde{f}(u) = \sum_{v=(v_1,v_2,\ldots,v_n)\in R^{bn}} \chi(\langle u, v\rangle) \prod_{i=1}^{n} z_i^{\lceil w_L(v_i)/t \rceil} = \prod_{i=1}^{n} \left(\sum_{v_i \in R^b} \chi(\langle u_i, v_i\rangle) z_i^{\lceil w_L(v_i)/t \rceil} \right).$$

Let the composition of the ith byte u_i of u be J_i $(1 \le i \le n)$. Then working as in Lemma 2 of Sharma et al. [10], we get

$$\sum_{v_i \in R^b} \chi(\langle u_i, v_i\rangle) z_i^{\lceil w_L(v_i)/t \rceil} = g_{J_i}^{(t)}(z_i)$$

for each $i,\ 1 \le i \le n$. This gives

$$\tilde{f}(u) = \prod_{i=1}^{n} g_{J_i}^{(t)}(z_i).$$

If $A(J_1, J_2, \ldots, J_n)$ is the number of codewords in $\mathcal{C}^\perp$ having the composition vector as $(J_1, J_2, \ldots, J_n)$, then we have

$$\sum_{u \in \mathcal{C}^\perp} \tilde{f}(u) = \sum_{(J_1, J_2, \ldots, J_n)} A(J_1, J_2, \ldots, J_n) \prod_{i=1}^{n} g_{J_i}^{(t)}(z_i), \tag{16}$$

where the summation runs over all n-tuples $(J_1, J_2, \ldots, J_n)$ with each J_i, an ℓ-tuple over $\{0, 1, 2, \ldots, b\}$.

Again applying Lemma 3 and using (16), we get

$$\begin{aligned} \mathcal{S}_{\mathcal{C}}(z_i : i = 1, 2, \ldots, n) &= \sum_{v \in \mathcal{C}} f(v) = \frac{1}{|\mathcal{C}^\perp|} \sum_{u \in \mathcal{C}^\perp} \tilde{f}(u) \\ &= \frac{1}{|\mathcal{C}^\perp|} \sum_{(J_1, J_2, \ldots, J_n)} A(J_1, J_2, \ldots, J_n) \prod_{i=1}^{n} g_{J_i}^{(t)}(z_i), \end{aligned}$$

which proves the theorem. □

In the following theorem, we show that two equivalent byte error-control codes may have the same m-spotty Lee weight enumerator but their split m-spotty Lee weight enumerators may be different.

Theorem 14. [10] *Let $\mathcal{C}, \mathcal{D}$ be byte error-control codes of length bn and byte length b over R having the m-spotty Lee weight enumerators as $L_{\mathcal{C}}(z)$, $L_{\mathcal{D}}(z)$ and split m-spotty Lee weight enumerators as $\mathcal{S}_{\mathcal{C}}(z_i : i = 1, 2, \ldots, n)$, $\mathcal{S}_{\mathcal{D}}(Z_i : i = 1, 2, \ldots, n)$, respectively. Then*

(i) *the direct sum*

$$\mathcal{C} \oplus \mathcal{D} = \{(u|v) \; : \; u \in \mathcal{C}, \; v \in \mathcal{D}\}$$

has m-spotty Lee weight enumerator as $L_{\mathcal{C}}(z)L_{\mathcal{D}}(z)$ and split m-spotty Lee weight enumerator as $\mathcal{S}_{\mathcal{C}}(z_i : i = 1, 2, \ldots, n)\ \mathcal{S}_{\mathcal{D}}(Z_i : i = 1, 2, \ldots, n)$.

(ii) *assuming n even, the code*

$$\mathcal{C} \parallel \mathcal{D} = \{(u'|v'|u''|v'') \; : \; u = (u'|u'') \in \mathcal{C}, \; v = (v'|v'') \in \mathcal{D}\}$$

(where u and v have each been broken into two equal halves) has m-spotty Lee weight enumerator as $L_{\mathcal{C}}(z)L_{\mathcal{D}}(z)$ and split m-spotty Lee weight enumerator as $\mathcal{S}_{\mathcal{C}}(z_i; Z_i : i = 1, 2, \ldots, n/2)\, \mathcal{S}_{\mathcal{D}}(z_i; Z_i : i = (n/2) + 1, \ldots, n)$.

Proof. For proof, see Theorem 4 of Sharma et al. [10]. □

An application

Let $\mathcal{C}$ be a byte error-control code of length bn and byte length b over R. Suppose that the codewords of $\mathcal{C}$ are transmitted through the channel $\mathfrak{C}$ (as defined in Section 3.2). Then we define

$$\delta_{ML}(\mathcal{C}) = \min \{\varpi_{ML}(u) : u \in \mathcal{C}, u \neq 0\},$$

where for each $u = (u_1, u_2, \ldots, u_n) \in R^{bn}$ with each $u_i \in R^b$, $\varpi_{ML}(u) = \sum_{i=1}^{n} \mathfrak{p}_i w_{ML}(u_i)$ and $\mathfrak{p}_i = \log(\frac{1-p_i}{p_i})$ for each i.

In the following theorem, we see that $\delta_{ML}(\mathcal{C})$ is a measure of (m-spotty) error-detecting and error-correcting capabilities of the code $\mathcal{C}$.

Theorem 15. *(i) The code $\mathcal{C}$ can detect any m-spotty byte error e satisfying $\varpi_{ML}(e) < \delta$ if and only if $\delta_{ML}(\mathcal{C}) \geq \delta$.*

(ii) If $\delta_{ML}(\mathcal{C}) = \delta$, then the code $\mathcal{C}$ can correct all m-spotty byte errors e satisfying $\varpi_{ML}(e) < \delta_{ML}(\mathcal{C})/2$ and $\mathcal{C}$ cannot correct any m-spotty byte error e satisfying $\varpi_{ML}(e) \geq \delta_{ML}(\mathcal{C})/2$.

Proof. For proof, see Sharma et al. [10, Theorems 6-7]. □

Note that the number $\delta_{ML}(\mathcal{C})$ can be computed from the split m-spotty Lee weight enumerator $\mathcal{S}_{\mathcal{C}}(z_i : i = 1, 2, \ldots, n)$ of $\mathcal{C}$ by taking $z_i = z^{\mathfrak{p}_i}$ for $1 \leq i \leq n$. Hence $\delta_{ML}(\mathcal{C})$ equals the least positive real number δ such that the coefficient of z^δ in $\mathcal{S}_{\mathcal{C}}(z^{\mathfrak{p}_i} : i = 1, 2, \ldots, n)$ is non-zero.

4.3 r-fold joint m-spotty Lee weight enumerator

In this subsection, we define the r-fold joint m-spotty Lee weight enumerator of r byte error-control codes over R, derive some MacWilliams identities for the same and discuss its properties. For this, we need to define the following:

For $\mathfrak{L} = \{0, 1, 2, \ldots, \ell - 1\}$, let $\mathfrak{L}^r$ be the set of all r-tuples over $\mathfrak{L}$. Then for $1 \leq i \leq r$, define

$$Q_i = \{a \in \mathfrak{L}^r \ : \ [a]_i \neq 0 \text{ and } [a]_j = 0 \text{ for all } j \neq i\}$$

and

$$T_i = \{a \in \mathfrak{L}^r \ : \ [a]_i \neq 0 \text{ and } [a]_j \neq 0 \text{ for some } j \neq i\}.$$

Definition 22. [11] For each $a \in \mathfrak{L}^r$, define a function $f_a \ : \ (R^b)^r \to \mathbb{Z}$ as

$$f_a(u_1, u_2, \ldots, u_r) = |\{p : 1 \leq p \leq b, (u_{1p}, u_{2p}, \ldots, u_{rp}) = (r_{[a]_1}, r_{[a]_2}, \ldots, r_{[a]_r})\}|,$$

where each $u_i = (u_{i1}, u_{i2}, \ldots, u_{ib}) \in R^b$.

Definition 23. [11] For $1 \leq i \leq r$, define $K_i, L_i : (R^b)^r \to \mathbb{Z}$ as

$$K_i(u) = \begin{cases} \lfloor \frac{A_i(u)}{t} \rfloor & \text{if } \overline{A_i(u)} + \overline{B_i(u)} = 0; \\ \lfloor \frac{A_i(u)}{t} \rfloor + 1 & \text{if } 0 < \overline{A_i(u)} + \overline{B_i(u)} \leq t; \\ \lfloor \frac{A_i(u)}{t} \rfloor + 2 & \text{if } t < \overline{A_i(u)} + \overline{B_i(u)} \leq 2t - 2, \end{cases} \tag{17}$$

$$L_i(u) = \Big\lfloor \frac{B_i(u)}{t} \Big\rfloor, \tag{18}$$

where $u = (u_1, u_2, \ldots, u_r)$ with each $u_i \in R^b$, and $A_i(u)$'s and $B_i(u)$'s are given by

$$A_i(u) = \sum_{a \in Q_i} w_L(r_{[a]_i}) f_a(u), \quad B_i(u) = \sum_{a \in T_i} w_L(r_{[a]_i}) f_a(u). \tag{19}$$

Now for each i, we extend the functions K_i, L_i defined on $(R^b)^r$ to the elements of $(R^{bn})^r$ as

$$K_i(u) = \sum_{j=1}^{n} K_i(u_1^{(j)}, u_2^{(j)}, \ldots, u_r^{(j)}), \qquad L_i(u) = \sum_{j=1}^{n} L_i(u_1^{(j)}, u_2^{(j)}, \ldots, u_r^{(j)}), \qquad (20)$$

where $u = (u_1, u_2, \ldots, u_r) \in (R^{bn})^r$ with each $u_i = (u_i^{(1)}, u_i^{(2)}, \ldots, u_i^{(n)}) \in R^{bn}$ and $u_i^{(j)} \in R^b$ for $1 \leq j \leq n$.

Now we extend the definition of r-fold joint m-spotty Lee weight enumerator for r byte error-control codes over R.

Definition 24. [11] Let $\mathcal{C}_1, \ \mathcal{C}_2, \ldots, \ \mathcal{C}_r$ be r byte error-control codes of length bn and byte length b over R. Then the r-fold joint m-spotty Lee weight enumerator of the codes $\mathcal{C}_1, \ \mathcal{C}_2, \ldots, \ \mathcal{C}_r$ is defined as

$$\mathcal{J}_{\mathcal{C}_1,\ldots,\mathcal{C}_r}(x_i, y_i : 1 \leq i \leq r) = \sum_{c_1 \in \mathcal{C}_1} \cdots \sum_{c_r \in \mathcal{C}_r} \prod_{i=1}^{r} x_i^{K_i(c_1,\ldots,c_r)} y_i^{L_i(c_1,\ldots,c_r)},$$

where $K_i, \ L_i$'s are as defined by (17)-(20).

Remark 8. The r-fold joint m-spotty Lee weight enumerator coincides with

(i) the m-spotty Lee weight enumerator when $r = 1$;

(ii) the joint m-spotty Lee weight enumerator when $r = 2$.

In the following theorem, we show that the r-fold joint m-spotty Lee weight enumerator generalizes m-spotty Lee weight enumerator just like the joint probability density function generalizes single probability density function.

Theorem 16. *Let $\mathcal{J}_{\mathcal{C}_1,\mathcal{C}_2,\ldots,\mathcal{C}_r}(x_i, y_i : 1 \leq i \leq r)$ be the r-fold joint m-spotty Lee weight enumerator of byte error-control codes $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_r$ over R. Then we have the following:*

(i) $\mathcal{J}_{\mathcal{C}_1,\mathcal{C}_2,\ldots,\mathcal{C}_r}(1, 1, \ldots, 1) = |\mathcal{C}_1||\mathcal{C}_2| \ldots |\mathcal{C}_r|$.

(ii) *For integers $1 \leq p < q \leq r$, the r-fold joint m-spotty Lee weight enumerator of the codes $\mathcal{C}_1, \ldots, \mathcal{C}_q, \ldots, \mathcal{C}_p, \ldots, \mathcal{C}_r$ (i.e., for the same sequence of codes except for $\mathcal{C}_p$ and $\mathcal{C}_q$ interchanged) is given by $\mathcal{J}_{\mathcal{C}_1,\mathcal{C}_2,\ldots,\mathcal{C}_r}(\tilde{x}_i, \tilde{y}_i : 1 \leq i \leq r)$, where for each x_i or y_i $(1 \leq i \leq r)$*

$$\tilde{x}_i = \begin{cases} x_q & \text{if } i = p; \\ x_p & \text{if } i = q; \\ x_i & \text{otherwise} \end{cases} \quad \text{and} \quad \tilde{y}_i = \begin{cases} y_q & \text{if } i = p; \\ y_p & \text{if } i = q; \\ y_i & \text{otherwise.} \end{cases}$$

(iii) *The m-spotty Lee weight enumerator of the code $\mathcal{C}_s$ $(1 \le s \le r)$ is given by*

$$L_{\mathcal{C}_s}(z) = \frac{1}{\prod_j^s |\mathcal{C}_j|} \mathcal{J}_{\mathcal{C}_1,\ldots,\mathcal{C}_r}(x_i, y_i : 1 \le i \le r) \text{ with } x_i = y_i = \begin{cases} z & \text{if } i = s; \\ 1 & \text{otherwise,} \end{cases}$$

where the product $\prod_j^s$ is extended over all integers j satisfying $1 \le j \le r$ and $j \ne s$.

Proof. For proof, see Sharma et al. [11, Theorem 3.6]. □

To derive MacWilliams identities, we define the following:

For integers $1 \le i \le r$, $j \in \mathfrak{L}$ and for each tuple $a \in \mathfrak{L}^r$, define the vector $\sigma_{ij}(a) \in \mathfrak{L}^{r+1}$ as

$$[\sigma_{ij}(a)]_k = \begin{cases} [a]_k & \text{if } 1 \le k \le i-1; \\ j & \text{if } k = i; \\ [a]_{k-1} & \text{if } i+1 \le k \le r+1. \end{cases}$$

Note that $\mathfrak{L}^{r+1} = \bigcup_{a \in \mathfrak{L}^r} \left\{ \bigcup_{j \in \mathfrak{L}} \sigma_{ij}(a) \right\}$ for each i.

Definition 25. [11] Let t $(1 \le t \le b)$ and q $(1 \le q \le r)$ be fixed integers. Let $\delta = (\delta_a : a \in \mathfrak{L}^r)$ be an ℓ^r-tuple over $\{0, 1, 2, \ldots, b\}$ satisfying $\sum_{a \in \mathfrak{L}^r} \delta_a = b$. For an integer p $(0 \le p \le b)$, let $\mathcal{A}_p$ be the set of all tuples $\alpha = (\alpha_w : w \in \mathfrak{L}^{r+1})$ of non-negative integers α_w's satisfying the following:

$$\sum_{a \in \mathfrak{L}^r} \sum_{j \in \mathfrak{L}^*} \alpha_{\sigma_{(q+1)j}}(a) = p \quad \text{and} \quad \sum_{j \in \mathfrak{L}} \alpha_{\sigma_{(q+1)j}}(a) = \delta_a \text{ for each } a \in \mathfrak{L}^r,$$

where $\mathfrak{L}^* = \mathfrak{L} \setminus \{0\}$. Then define the polynomial $G_\delta(x_i, y_i : 1 \le i \le r)$ as

$$\sum_{p=0}^{b} \sum\nolimits_p h_p(\alpha) \prod_{i=1}^{r} x_i^{\left\lfloor \frac{A_i(\alpha)}{t} \right\rfloor + \theta_i^{(\alpha)}} y_i^{\left\lfloor \frac{B_i(\alpha)}{t} \right\rfloor}, \tag{21}$$

where for each p $(0 \le p \le b)$, the summation $\sum_p$ runs over the set $\mathcal{A}_p$; and further for each tuple $\alpha \in \mathcal{A}_p$, the coefficient $h_p(\alpha)$ is given by

$$h_p(\alpha) = \prod_{a \in \mathfrak{L}^r} \left(\frac{\delta_a!}{\prod_{j \in \mathfrak{L}} \alpha_{\sigma_{(q+1)j}(a)}!} \chi\Big(\sum_{j \in \mathfrak{L}^*} r_{[a]_q} r_j \alpha_{\sigma_{qj}(a)} \Big) \right),$$

the integers $A_i(\alpha)$, $B_i(\alpha)$ are given by

$$A_i(\alpha) = \sum_{a \in S_i} \sum_{j \in \mathfrak{L}} w_L(r_{[a]_i}) \alpha_{\sigma_{qj}(a)}, \quad B_i(\alpha) = \sum_{a \in T_i} \sum_{j \in \mathfrak{L}} w_L(r_{[a]_i}) \alpha_{\sigma_{qj}(a)}, \tag{22}$$

and the number $\theta_i^{(\alpha)}$ is given by

$$\theta_i^{(\alpha)} = \begin{cases} 0 & \text{if } \overline{A_i(\alpha)} + \overline{B_i(\alpha)} = 0; \\ 1 & \text{if } 0 < \overline{A_i(\alpha)} + \overline{B_i(\alpha)} \leq t; \\ 2 & \text{if } t < \overline{A_i(\alpha)} + \overline{B_i(\alpha)} \leq 2t-2. \end{cases} \tag{23}$$

Definition 26. [11] For $1 \leq j \leq n$, let $\delta_j = (\delta_a^{(j)} : a \in \mathfrak{L}^r)$ be an ℓ^r-tuple over $\{0, 1, 2, \ldots, b\}$ satisfying $\sum_{a \in \mathfrak{L}^r} \delta_a^{(j)} = b$. Then for $\delta = (\delta_1, \delta_2, \ldots, \delta_n)$, we define the polynomial

$$G_\delta(x_i, y_i : 1 \leq i \leq r) = \prod_{j=1}^{n} G_{\delta_j}(x_i, y_i : 1 \leq i \leq r). \tag{24}$$

Definition 27. [11] The joint composition vector of an r-tuple $(c_1, c_2, \ldots, c_r) \in (R^b)^r$, denoted by $j(c_1, c_2, \ldots, c_r)$, is defined as the tuple $\delta = (\delta_a : a \in \mathfrak{L}^r)$, where for each $a \in \mathfrak{L}^r$, δ_a is given by

$$\delta_a = |\{k : 1 \leq k \leq b, (\hat{c}_{1k}, \hat{c}_{2k}, \ldots, \hat{c}_{rk}) = a\}| \text{ with } \hat{c}_{ik} = s \text{ if } c_{ik} = r_s.$$

It is easy to see that $\sum_{a \in \mathfrak{L}^r} \delta_a = b$.

The joint composition vector of an r-tuple $(c^{(1)}, c^{(2)}, \ldots, c^{(r)}) \in (R^{bn})^r$ is defined as

$$j(c^{(1)}, c^{(2)}, \ldots, c^{(r)}) = \delta = (\delta_1, \delta_2, \ldots, \delta_n),$$

where for each i, $c^{(i)} = (c_{i1}, c_{i2}, \ldots, c_{in}) \in R^{bn}$ with each $c_{ik} \in R^b$ and $\delta_k = j(c_{1k}, c_{2k}, \ldots, c_{rk})$ for each k.

Theorem 17. [11] *Let $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_r$ be byte error-control codes of length bn and byte length b over R. For $1 \leq q \leq r$, let $P_q(\delta)$ be the number of r-tuples $(c_1, c_2, \ldots, c_r)$ of codewords $c_i \in \mathcal{C}_i$ $(1 \leq i \leq r, i \neq q)$ and $c_q \in \mathcal{C}_q^\perp$ having the joint composition vector as δ. Then we have*

$$\mathcal{J}_{\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_r}(x_i, y_i : 1 \leq i \leq r) = \frac{1}{|\mathcal{C}_q^\perp|} \sum P_q(\delta) G_\delta(x_i, y_i : 1 \leq i \leq r),$$

where the summation runs over all n-tuples $\delta = (\delta_1, \delta_2, \ldots, \delta_n)$ such that each $\delta_j = (\delta_a^{(j)} : a \in \mathfrak{L}^r)$ is an ℓ^r-tuple over $\{0, 1, 2, \ldots, b\}$ satisfying $\sum_{a \in \mathfrak{L}^r} \delta_a^{(j)} = b$, and the polynomial $G_\delta(x_i, y_i : 1 \leq i \leq r)$ is as defined by (24).

Remark 9. When one of the codes, say $\mathcal{C}_q$, is of large size, the computation of the numbers $P_q(\delta)$'s for the codes $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_r$ becomes very tedious. However it is easier to compute these numbers for the codes $\mathcal{C}_1, \ldots, \mathcal{C}_{q-1}, \mathcal{C}_q^\perp, \mathcal{C}_{q+1}, \ldots, \mathcal{C}_r$, as the dual code $\mathcal{C}_q^\perp$ of $\mathcal{C}_q$ is of relatively smaller size. From this, one can obtain the r-fold joint m-spotty Lee weight enumerator for the codes $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_r$, using Theorem 17.

Remark 10. The MacWilliams identity for m-spotty Lee weight enumerator follows from Theorem 17 when $r = 1$, and that for joint m-spotty Lee weight enumerator follows from Theorem 17 when $r = 2$.

Proof of Theorem 17. The r-fold joint m-spotty Lee weight enumerator of the codes $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_r$ is defined as

$$\mathcal{J}_{\mathcal{C}_1,\mathcal{C}_2,\ldots,\mathcal{C}_r}(x_i, y_i : 1 \le i \le r) = \sum \prod_{i=1}^{r} x_i^{K_i(c_1,c_2,\ldots,c_r)} y_i^{L_i(c_1,c_2,\ldots,c_r)},$$

where the summation $\sum$ runs over all the codewords $c_i \in \mathcal{C}_i$ for $1 \le i \le r$.

In order to prove this result, we will apply Lemma 3. For this, let

$$f(v) = \sum\nolimits_q \prod_{i=1}^{r} x_i^{K_i(c_1,\ldots,c_{q-1},v,c_{q+1},\ldots,c_r)} y_i^{L_i(c_1,\ldots,c_{q-1},v,c_{q+1},\ldots,c_r)}$$

for $v \in R^{bn}$, where the summation $\sum_q$ runs over all codewords $c_i \in \mathcal{C}_i$ $(1 \le i \le r, i \ne q)$. Then by Lemma 3, for $u = (u_1, u_2, \ldots, u_n) \in R^{bn}$, $\tilde{f}(u)$ is given by

$$\begin{aligned}
&\sum_{v\in R^{bn}} \chi(\langle u, v\rangle) \sum\nolimits_q \prod_{i=1}^{r} x_i^{K_i(c_1,\ldots,c_{q-1},v,c_{q+1},\ldots,c_r)} y_i^{L_i(c_1,\ldots,c_{q-1},v,c_{q+1},\ldots,c_r)} \\
&= \sum\nolimits_q \prod_{j=1}^{n} \Bigg\{ \sum_{v_j\in R^b} \chi\big(\langle u_j, v_j\rangle\big) \\
&\qquad \prod_{i=1}^{r} x_i^{K_i(c_{1j},\ldots,c_{(q-1)j},v_j,c_{(q+1)j},\ldots,c_{rj})} y_i^{L_i(c_{1j},\ldots,c_{(q-1)j},v_j,c_{(q+1)j},\ldots,c_{rj})} \Bigg\},
\end{aligned}$$

where $c_i = (c_{i1}, c_{i2}, \ldots, c_{in}) \in \mathcal{C}_i$ for $1 \le i \le r$ with $i \ne q$.

If the joint composition vector of the r-tuple $(c_{1j}, \ldots, c_{(q-1)j}, u_j, c_{(q+1)j}, \ldots, c_{rj})$ is δ_j for each j, then working as in Lemma 4.7 of Sharma et al. [11], we obtain

$$\begin{aligned}
&\sum_{v_j\in R^b} \chi\big(\langle u_j, v_j\rangle\big) \prod_{i=1}^{r} x_i^{K_i(c_{1j},\ldots,c_{(q-1)j},v_j,c_{(q+1)j},\ldots,c_{rj})} y_i^{L_i(c_{1j},\ldots,c_{(q-1)j},v_j,c_{(q+1)j},\ldots,c_{rj})} \\
&\quad = G_{\delta_j}(x_i, y_i : 1 \le i \le r) \text{ for each } j.
\end{aligned}$$

This gives

$$\sum_{c_q\in\mathcal{C}_q^\perp} \tilde{f}(c_q) = \sum \prod_{j=1}^{n} G_{\delta_j}(x_i, y_i : 1 \le i \le r),$$

where the summation $\sum$ runs over all codewords $c_i = (c_{i1}, c_{i2}, \ldots, c_{in}) \in \mathcal{C}_i$ for $1 \le i \le r$ $(i \ne q)$ and $c_q = (c_{q1}, c_{q2}, \ldots, c_{qn}) \in \mathcal{C}_q^\perp$ satisfying $j(c_{1j}, c_{2j}, \ldots, c_{rj}) = \delta_j$ for each j.

Now if $P_q(\delta)$ is the number of r-tuples $(c_1, c_2, \ldots, c_r)$ of codewords $c_i \in \mathcal{C}_i$ $(1 \leq i \leq r, i \neq q)$ and $c_q \in \mathcal{C}_q^{\perp}$ such that $j(c_1, c_2, \ldots, c_r) = \delta$, then using (24), we get

$$\sum_{c_q \in \mathcal{C}_q^{\perp}} \tilde{f}(c_q) = \sum P_q(\delta) G_\delta(x_i, y_i : 1 \leq i \leq r), \tag{25}$$

where the summation $\sum$ runs over all n-tuples $\delta = (\delta_1, \delta_2, \ldots, \delta_n)$ such that each $\delta_j = (\delta_a^{(j)} : a \in \mathfrak{L}^r)$ is an ℓ^r-tuple over $\{0, 1, 2, \ldots, b\}$ satisfying $\sum\limits_{a \in \mathfrak{L}^r} \delta_a^{(j)} = b$. Again applying Lemma 3 and using (25), we get

$$\begin{aligned} \mathcal{J}_{\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_r}(x_i, y_i : 1 \leq i \leq r) &= \sum_{v \in \mathcal{C}_q} f(v) = \frac{1}{|\mathcal{C}_q^{\perp}|} \sum_{c_q \in \mathcal{C}_q^{\perp}} \tilde{f}(c_q) \\ &= \frac{1}{|\mathcal{C}_q^{\perp}|} \sum P_q(\delta) G_\delta(x_i, y_i : 1 \leq i \leq r), \end{aligned}$$

which proves the theorem. □

4.4 Complete m-spotty Lee weight enumerator

In this subsection, we define the complete m-spotty Lee weight enumerator of a byte error-control code over R and derive a MacWilliams identity for the same.

Definition 28. Let $\mathcal{C}$ be a byte error-control code of length bn and byte length b over R. Then the complete m-spotty Lee weight enumerator of $\mathcal{C}$ is defined as

$$\mathfrak{CL}_{\mathcal{C}}(z_0, z_1, \ldots, z_M) = \sum_{u=(u_1, \ldots, u_n) \in \mathcal{C}} \prod_{i=1}^{n} z_{w_L(u_i)},$$

where $M = \max\limits_{r \in R} \{w_L(r)\}$.

If $A(J)$ is the number of codewords in $\mathcal{C}$ having the composition vector as J, then the complete m-spotty Lee weight enumerator can be rewritten as

$$\mathfrak{CL}_{\mathcal{C}}(z_0, z_1, \ldots, z_M) = \sum_{J} A(J) \prod_{i=1}^{n} z_{\rho(J_i)},$$

where $J = (J_1, J_2, \ldots, J_n)$ with each $J_i = (j_{i0}, j_{i1}, \ldots, j_{i,\ell-1})$ an ℓ-tuple over $\{0, 1, 2, \ldots, b\}$ and $\rho(J_i) = \sum\limits_{k=0}^{\ell-1} w_L(r_k) j_{ik}$.

Definition 29. For a fixed positive integer t and an ℓ-tuple $J = (j_0, j_1, \ldots, j_{\ell-1})$ over $\{0, 1, 2, \ldots, b\}$, we define a polynomial $h_J^{(t)}(z_0, z_1, \ldots, z_M)$ as

$$h_J^{(t)}(z_0, z_1, \ldots, z_M) = \sum_{s} \left(\prod_{k=0}^{\ell-1} \frac{j_k!}{\prod\limits_{p=0}^{\ell-1} s_{kp}!} \chi\left(\sum_{p=0}^{\ell-1} r_k r_p s_{kp} \right) \right) z_{\omega_s}, \tag{26}$$

where the summation $\sum\limits_{s}$ runs over all non-negative integers s_{kp} $(0 \leq k, p \leq \ell - 1)$ satisfying $\sum\limits_{p=0}^{\ell-1} s_{kp} = j_k$ for each k, $\omega_s = \sum\limits_{k=0}^{\ell-1}\Big(\sum\limits_{p=1}^{\ell-1} w_L(r_p)s_{kp}\Big)$ and χ is the non-trivial additive character on R that equals χ_1 when $R = R_1$ and equals χ_2 when $R = R_2$ (χ_1 and χ_2 are as defined by (1) and (2)).

In the following theorem, we derive a MacWilliams identity for the complete m-spotty Lee weight enumerator of a byte error-control code over R.

Theorem 18. *Let $\mathcal{C}$ be a byte error-control code of length bn and byte length b over R and $\mathcal{C}^\perp$ be its dual code. Then the complete m-spotty Lee weight enumerator of $\mathcal{C}$ is given by*

$$\mathfrak{CL}_{\mathcal{C}}(z_0, z_1, \ldots, z_M) = \frac{1}{|\mathcal{C}^\perp|} \sum_{(J_1,J_2,\ldots,J_n)} A(J_1, J_2, \ldots, J_n) \prod_{i=1}^{n} h_{J_i}^{(t)}(z_0, z_1, \ldots, z_M),$$

where the summation runs over all n-tuples $(J_1, J_2, \ldots, J_n)$ with each $J_i = (j_{i0}, j_{i1}, \ldots, j_{i,\ell-1})$ an ℓ-tuple over $\{0, 1, 2, \ldots, b\}$, $A(J_1, J_2, \ldots, J_n)$ is the number of codewords in $\mathcal{C}^\perp$ having the composition vector as $(J_1, J_2, \ldots, J_n)$ and for $1 \leq i \leq n$, the polynomial $h_{J_i}^{(t)}(z_0, z_1, \ldots, z_M)$ is as defined by (26).

Proof. Let $f(v) = \prod\limits_{i=1}^{n} z_{w_L(v_i)}$ for $v = (v_1, v_2, \ldots, v_n) \in R^{bn}$, where each $v_i \in R^b$. Then by Lemma 3, for $u = (u_1, u_2, \ldots, u_n) \in R^{bn}$, $\tilde{f}(u)$ is given by

$$\tilde{f}(u) = \sum_{v=(v_1,v_2,\ldots,v_n)\in R^{bn}} \chi(\langle u, v\rangle) \prod_{i=1}^{n} z_{w_L(v_i)} = \prod_{i=1}^{n} \left(\sum_{v_i \in R^b} \chi(\langle u_i, v_i\rangle) z_{w_L(v_i)} \right).$$

Now if J_i $(1 \leq i \leq n)$ is the composition of the ith byte u_i of u, then working as in Lemma 2 of Sharma et al. [10], we get

$$\sum_{v_i \in R^b} \chi(\langle u_i, v_i\rangle) z_{w_L(v_i)} = h_{J_i}^{(t)}(z_0, z_1, \ldots, z_M)$$

for each i. This gives

$$\tilde{f}(u) = \prod_{i=1}^{n} h_{J_i}^{(t)}(z_0, z_1, \ldots, z_M).$$

If $A(J_1, J_2, \ldots, J_n)$ is the number of codewords in $\mathcal{C}^\perp$ having the composition vector as $(J_1, J_2, \ldots, J_n)$, then

$$\sum_{u\in\mathcal{C}^\perp} \tilde{f}(u) = \sum_{(J_1,J_2,\ldots,J_n)} A(J_1, J_2, \ldots, J_n) \prod_{i=1}^{n} h_{J_i}^{(t)}(z_0, z_1, \ldots, z_M), \tag{27}$$

where the summation runs over all n-tuples $(J_1, J_2, \ldots, J_n)$ with each J_i, an ℓ-tuple over $\{0, 1, 2, \ldots, b\}$.

Again applying Lemma 3 and using (27), we get

$$\begin{aligned}\mathfrak{CL}_{\mathcal{C}}(z_0, z_1, \ldots, z_M) &= \sum_{v\in\mathcal{C}} f(v) = \frac{1}{|\mathcal{C}^{\perp}|}\sum_{u\in\mathcal{C}^{\perp}} \tilde{f}(u)\\ &= \frac{1}{|\mathcal{C}^{\perp}|}\sum_{(J_1,J_2,\ldots,J_n)} A(J_1, J_2, \ldots, J_n)\prod_{i=1}^{n} h_{J_i}^{(t)}(z_0, z_1, \ldots, z_M),\end{aligned}$$

which proves the theorem. □

5 Byte-weight enumerator

In this section, we extend the definition of byte-weight enumerator for a byte error-control code over R and derive a MacWilliams identity for the same.

Definition 30. [22] Let $\mathcal{C}$ be a byte error-control code of length bn and byte length b over R. Then the byte-weight enumerator of $\mathcal{C}$ is defined as

$$\mathcal{BW}_{\mathcal{C}}(z_k : k \in R^b) = \sum_{u=(u_1,u_2,\ldots,u_n)\in\mathcal{C}} \prod_{i=1}^{n} z_{u_i}.$$

It is easy to see that the complete m-spotty Hamming weight enumerator of a byte error-control code $\mathcal{C}$ can be obtained from the byte-weight enumerator of $\mathcal{C}$ by replacing z_{u_i} with $z_{w_H(u_i)}$ for each i and the complete m-spotty Lee weight enumerator of $\mathcal{C}$ can be obtained from the byte-weight enumerator of $\mathcal{C}$ by replacing z_{u_i} with $z_{w_L(u_i)}$ for each i.

In the following theorem, we derive a MacWilliams identity for the byte-weight enumerator of a byte error-control code over R.

Theorem 19. *Let $\mathcal{C}$ be a byte error-control code of length bn and byte length b over R and $\mathcal{C}^{\perp}$ be its dual code. Then the byte-weight enumerator of $\mathcal{C}$ is given by*

$$\mathcal{BW}_{\mathcal{C}}(z_k : k \in R^b) = \frac{1}{|\mathcal{C}^{\perp}|}\mathcal{BW}_{\mathcal{C}^{\perp}}\left(\sum_{v\in R^b} \chi(\langle u, v\rangle) z_v : u \in R^b\right),$$

where χ is the non-trivial additive character on R, given by

$$\chi = \begin{cases} \chi_1 & \text{when } R = R_1,\\ \chi_2 & \text{when } R = R_2.\end{cases}$$

Proof. Let $f(v) = \prod_{i=1}^{n} z_{v_i}$, where $v = (v_1, v_2, \ldots, v_n) \in R^{bn}$. Then by applying Lemma 3, for $u = (u_1, u_2, \ldots, u_n) \in R^{bn}$, $\tilde{f}(u)$ is given by

$$\tilde{f}(u) = \sum_{v=(v_1,v_2,\ldots,v_n)\in R^{bn}} \chi(\langle u, v\rangle)\prod_{i=1}^{n} z_{v_i} = \prod_{i=1}^{n}\left(\sum_{v_i\in R^b} \chi(\langle u_i, v_i\rangle) z_{v_i}\right).$$

This gives

$$\sum_{u \in \mathcal{C}^\perp} \tilde{f}(u) = \sum_{u \in \mathcal{C}^\perp} \prod_{i=1}^{n} \left(\sum_{v_i \in R^b} \chi(\langle u_i, v_i \rangle) z_{v_i} \right). \tag{28}$$

Again applying Lemma 3 and using (28), we get

$$\begin{aligned} \mathcal{BW}_{\mathcal{C}}(z_k : k \in R^b) &= \frac{1}{|\mathcal{C}^\perp|} \sum_{u \in \mathcal{C}^\perp} \tilde{f}(u) = \frac{1}{|\mathcal{C}^\perp|} \sum_{u \in \mathcal{C}^\perp} \prod_{i=1}^{n} \left(\sum_{v_i \in R^b} \chi(\langle u_i, v_i \rangle) z_{v_i} \right) \\ &= \frac{1}{|\mathcal{C}^\perp|} \mathcal{BW}_{\mathcal{C}^\perp} \left(\sum_{v \in R^b} \chi(\langle u, v \rangle) z_v : u \in R^b \right), \end{aligned}$$

which proves the theorem. □

6 Conclusion

Let R be either the finite chain ring $R_1 = \mathbb{F}_q + u\mathbb{F}_q + u^2\mathbb{F}_q + \cdots + u^{e-1}\mathbb{F}_q$ $(u^e = 0)$ or the ring $R_2 = \mathbb{F}_q + u\mathbb{F}_q + v\mathbb{F}_q + uv\mathbb{F}_q$ $(u^2 = 0, v^2 = 0, uv = vu)$. In this paper, we defined the m-spotty weight enumerator, split m-spotty weight enumerator, r-fold joint m-spotty weight enumerator, complete m-spotty weight enumerator and byte-weight enumerator for byte error-control codes over R with respect to both m-spotty Hamming and m-spotty Lee metrics. We also derived MacWilliams identities for these enumerators and discussed some of their applications. Further, it would be interesting to find some more applications of these enumerators and to study their invariance properties.

Note that the results on m-spotty enumerators with respect to m-spotty Hamming metric can be extended to any finite Frobenius ring, while the results on m-spotty weight enumerators with respect to m-spotty Lee metric can be extended to all those finite Frobenius rings for which a suitable notion of Lee weight can be defined.

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