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Some bounds for the signed edge domination number of a graph

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Abstract

The closed neighbourhood $N_G[e]$ of an edge e in a graph G is the set consisting of e and of all edges having a common end vertex with e. Let f be a function on the edges of G into the set $\{-1, 1\}$. If $\sum_{e\in N_G[x]} f(e) \geq 1$ for every $x \in E(G)$, then f is called a signed edge domination function of G. The minimum value of $\sum_{x\in E(G)} f(x)$, taken over every signed edge domination function f of G, is called signed edge domination number of G and denoted by $\gamma'_s(G)$. It has been proved that $\gamma'_s(G) \geq n - m$

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for every graph G of order n and size m. In this paper we prove that $\gamma'_s(G) \geq \frac{2\alpha'(G)-m}{3}$ for every simple graph G, where $\alpha'(G)$ is the size of a maximum matching of G. We also prove that for a simple graph G of order n whose each vertex has an odd degree, $\gamma'_s(G) \leq n - \frac{2\alpha'(G)}{3}$.

1 Introduction

All graphs considered in this article are finite, undirected, and simple. We use [2] for any terms that are not denoted here. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. Here, $\delta(G)$ and $\alpha'(G)$ denote the minimum degree of G and the size of a maximum matching of G, respectively. The open neighborhood $N_G(e)$ of an edge $e \in E(G)$ is the set of all edges incident with e. The closed neighborhood of e is defined by $N_G[e] = N_G(e) \cup \{e\}$. By an odd graph, we mean a graph all of whose degrees are odd. For a function $f : E(G) \longrightarrow \{-1, 1\}$ and a subset S of $E(G)$, we define $f(S) = \sum_{e \in S} f(e)$. For each vertex $v \in V(G)$, we also define $f(v) = \sum_{e \in E(v)} f(e)$, where $E(v)$ is the set of all edges incident with v. A function $f : E(G) \longrightarrow \{-1, 1\}$ is called a *signed edge dominating function* (SEDF) of G, if $f(N_G[e]) \geq 1$ for each edge $e \in E(G)$. The minimum value of $f(E(G))$, taken over all signed edge dominating functions f of G, is called the *signed edge dominating number* of G. The signed edge domination number was introduced by Xu in [4] and denoted by $\gamma'_s(G)$. In 2005 Xu [3] conjectured that:

Conjecture. *For all graph of order n, $\gamma'_s(G) \leq n - 1$.*

In [3] the following was also proved.

Theorem 1. *Let G be a graph of order n and size m with $\delta(G) \geq 1$. Then $\gamma'_s(G) \geq n - m$.*

Also in [1], the following was shown.

Theorem 2. *Let G be an odd graph of order $n \geq 2$. Then $\gamma'_s(G) \leq n - 1$.*

In this paper we first show that for any graph G of size m, $\gamma'_s(G) \geq \frac{2\alpha'(G)-m}{3}$. Also, we show that for any odd graph G of order n, $\gamma'_s(G) \leq n - \frac{2\alpha'(G)}{3}$.

2 A lower bound for signed edge domination number

In this section we establish a lower bound for the signed edge domination number of a graph. This lower bound depends on the size of a maximum matching.

Lemma 1. *Let G be a graph and let f be an SEDF of G such that $\gamma'_s(G) = f(E(G))$. If m is the size of G, then for any $S \subseteq E(G)$ we have*

$$\sum_{x \in S} f(x) \geq \frac{\gamma'_s(G) - m}{2}.$$

Proof. Suppose that f is an SEDF of G and $f^+ = \{e \in E(G) \mid f(e) = 1\}$, $f^- = \{e \in E(G) \mid f(e) = -1\}$. Clearly, $-|f^-| \leq \sum_{x \in S} f(x)$, and

$$|f^+| + |f^-| = m, |f^+| - |f^-| = \gamma'_s(G), \text{ so we have}$$

$$|f^-| = \frac{m - \gamma'_s(G)}{2}.$$

This implies that $\sum_{x \in S} f(x) \geq \frac{\gamma'_s(G) - m}{2}$. □

Theorem 3. *Let G be a graph of size m. Then $\gamma'_s(G) \geq \frac{2\alpha'(G) - m}{3}$.*

Proof. Let M be a maximum matching of G and $S = L \cup M$, where L is the set of all edges of G with exactly one end vertex in $V(M)$. If f is an SEDF of G such that $\gamma'_s(G) = f(E(G))$, then for each edge $e \in E(G)$, $\displaystyle\sum_{x \in N_G[e]} f(x) \geq 1$. Therefore, we conclude that $\displaystyle\sum_{e \in M} \sum_{x \in N_G[e]} f(x) \geq \alpha'(G)$. Recall that every edge in $E(G)$ has either only one end vertex or both end vertices in $V(M)$. Then

$$\alpha'(G) \leq \sum_{e \in M} \sum_{x \in N_G[e]} f(x) = \sum_{x \in E(G)} 2f(x) - \sum_{x \in S} f(x) = 2\gamma'_s(G) - \sum_{x \in S} f(x).$$

By Lemma 1, $2\gamma'_s(G) \geq \alpha'(G) + \frac{\gamma'_s(G) - m}{2}$, which implies that $\gamma'_s(G) \geq \frac{2\alpha'(G) - m}{3}$. □

3 An upper bound for $\gamma'_s(G)$ in odd graphs

In this section we prove that for any odd graph G of order n, $\gamma'_s(G) \leq n - \frac{2\alpha'(G)}{3}$. Before proving the main result, first we need several lemmas.

Lemma 2. *Let G be a graph. Suppose that $f(u) + f(v) \geq 2$, for an edge $e = (u, v)$. Then*

$$\textstyle\sum_{x \in N_G[e]} f(x) \geq 1.$$

Proof. We know that $f(u) + f(v) = (\sum_{x \in N_G[e]} f(x)) + f(e)$. Hence we find that

$$\sum_{x\in N_G[e]} f(x) \geq 2 - f(e).$$

Since $f(e) \leq 1$, we have $\sum_{x\in N_G[e]} f(x) \geq 2 - f(e) \geq 1$. □

Lemma 3. [2, Theorem 2.4.3] *Let G be a connected graph having exactly $2k$ vertices of odd degree, with $k \geq 1$. Then $E(G)$ can be partitioned into k trails.*

Lemma 4. *Let G be a graph having exactly $2k$ vertices of odd degree, $k \geq 1$, and with each component having at least two vertices of odd degree. Then one can assign -1 and $+1$ to the edges of G in such a way that for every vertex v of odd degree, $f(v) \geq 1$, and for every vertex v of even degree, $f(v) \geq 0$. Moreover, $\sum_{e\in E(G)} f(e) \leq 2k$.*

Proof. By Lemma 3, one can partition each component of G into $\frac{r_i}{2}$ different trails, where r_i is the number of vertices of odd degree in the ith component. Therefore, $E(G)$ can be partitioned into k different trails $P_1, \ldots, P_k$. For each trail P_i we assign $+1$ and -1 to the edges of P_i alternatively starting with $+1$. If some trail ends with -1, then we change the value of the last edge to $+1$. For each trail P_i, we find that $\sum_{e\in E(P_i)} f(e) \leq 2$. Therefore, $\sum_{e\in E(G)} f(e) = \sum_{i=1}^{k} \sum_{e\in E(P_i)} f(e) \leq 2k$. Since each trail starts from a vertex of odd degree and ends in a vertex of odd degree, $f(v) \geq 1$ for every vertex v of odd degree, and $f(v) \geq 0$ for every vertex v of even degree. □

Lemma 5. *Let G be a graph of order n which is partitioned into k connected components $C_1, \ldots, C_k$ and let T be a set of $\frac{n}{2}$ $\{u_i, v_i\}$ pairs such that the following conditions hold:*

- *u_i and v_i are not adjacent.*
- *Each vertex of G is exactly in one pair of T.*
- *For each vertex $x \in V(G)$, the degree of x is even.*

We say u and v are related, if $\{u, v\} \in T$, and two components are called related, if there is a pair $\{u, v\} \in T$ such that u belongs to one of the two components and v belongs to the other one. If we choose some components of G such that there are no two related unchosen components having an odd number of edges, then there are at most $\frac{n}{6}$ unchosen components with an odd number of edges.

Proof. Let p be the number of unchosen components with an odd number of edges and let $C_{u_1}, \ldots, C_{u_p}$ be all of these components. Moreover, let $Q(C_{u_i})$ be the set of all vertices contained in C_{u_i} or related to a vertex in C_{u_i}. Since there is no vertex v in two pairs of T and there are no two unchosen related components with an odd number of edges, for every different i and j, the following condition holds:

$$Q(C_{u_i}) \cap Q(C_{u_j}) = \emptyset.$$

Since each C_{u_i} has an odd number of edges and the degree of each vertex of G is even, none of the C_{u_i} contains $1, 2$ or 4 vertices. Therefore, each C_{u_i} has one of following forms:

1. C_{u_i} has 3 vertices. In this case all pairs of the vertices of C_{u_i} are adjacent. Thus, each vertex in C_{u_i} is related to a vertex in another component. Hence $|Q(C_{u_i})| = 6$.
2. C_{u_i} has 5 vertices. In this case at least one vertex in C_{u_i} is related to a vertex in another component. Hence $|Q(C_{u_i})| \geq 6$.
3. C_{u_i} has at least 6 vertices. In this case $|Q(C_{u_i})| \geq 6$.

Since each $Q(C_{u_i})$ has at least six vertices and no two different $Q(C_{u_i})$ and $Q(C_{u_j})$ have a common vertex, $p \leq \frac{n}{6}$. □

In what follows, a chosen component is a component all of whose edges have been labeled -2 or $+1$. A component with no edge labeled is called unchosen.

Theorem 4. *Let G be an odd graph of order n. Then $\gamma'_s(G) \leq n - \frac{2\alpha'(G)}{3}$.*

Proof. According to Lemma 2, if we construct a function f in such a way that $f(v) \geq 1$ for every $v \in V(G)$, then f would be an SEDF of G. First, we temporarily assign 0 to every edge of G, and then we construct a function f in five steps to satisfy the mentioned constraint. The *overload* of a component S is $f(E(S))$ after assigning -1 and $+1$ to $E(S)$. The *total overload* of a step is $f_a(E(G)) - f_b(E(G))$, where $f_a(E(G))$ and $f_b(E(G))$ are $f(E(G))$ after and before assigning -1 and $+1$ to the edges of $E(G)$ in that step.

1. Let M be a maximum matching of G. We assign $+1$ to all edges of M and remove them from G. Now we partition G into two categories.

   (a) The set of components containing at least one vertex of odd degree.

   (b) The set of components not containing a vertex of odd degree.

   The total overload of this step is $\alpha'(G)$.

2. Since every connected component in Category (a) has at least one vertex of odd degree and there are exactly $n - 2\alpha'(G)$ vertices of odd degree in Category (a), according to Lemma 4, one can assign -1 and $+1$ to the edges of each connected component of Category (a) such that the total overload does not exceed $n - 2\alpha'(G)$.

3. The remaining unassigned edges are all edges in Category (b) which belong to the components with all vertices having even degree. Let a component be even if it contains an even number of edges and odd otherwise. In this step, for each even component in Category (b), we find its Eulerian tour and assign $+1$ and -1 to its edges alternately. The total overload of this step is equal to zero.

4. In this step, we mark some of the components by the following instruction. For every edge $e = (u, v)$ in M such that u and v belong to different unchosen odd components in Category (b), we add the edge e to G, and choose the components of u and v. Afterward, we find an Eulerian trail within the jointed component and assign $+1$ and -1 to its edges alternately. Consider that, by this assignment, the value of e will change from $+1$ to -1. Therefore, we use the overload that has already been counted in step 1, and the assignment of the joint component does not require further overload. Thus the overload of each component is zero, and consequently the total overload of this step is 0. Notice that although the value of (u, v) changes to -1 during the assignment, the values of $f(u)$ and $f(v)$ remain unchanged and they are at least $+1$.

5. According to Lemma 5, the number of remaining unchosen components is less than or equal to $\frac{\alpha'(G)}{3}$. Let H be the subgraph of G whose vertex set and edge set are $V(M)$ and all edges of components in Category (b), respectively. Let T be the set of pairs obtained from two end vertices of all edges of M. Obviously, T in H satisfies three conditions given in Lemma 5. By the method used in Step 4, no two related unchosen components in H have an odd number of edges (this is the last assumption of Lemma 5). Therefore, by Lemma 5, the number of unchosen components with an odd number of edges in H is at most $\frac{|V(M)|}{6} = \frac{\alpha'(G)}{3}$. Thus the number of remaining unchosen components in G is at most $\frac{\alpha'(G)}{3}$. In these components, we find an Eulerian tour and assign $+1$ and -1 to its edges alternately, starting with $+1$. The overload for each component is $+1$. Therefore the total overload of this step is not more than $\frac{\alpha'(G)}{3}$.

Let F be the graph which is made by the removal of unsigned edges in G after step 2. We have $f(v) \geq 1$ for every $v \in V(F)$. Let H be the induced spanning subgraph of G on the edge set $E(G)\backslash E(F)$. In the steps 3, 4 and 5 we assigned -1 and $+1$ to every $e \in E(H)$ in such a way that $f(v) \geq 0$ for every $v \in V(H)$. Since $E(G) = E(F) \cup E(H)$, the assignment method satisfies $f(v) \geq 1$ for every $v \in V(G)$. Moreover, we have:

$$\sum_{e \in E(G)} f(e) \leq \alpha'(G) + n - 2\alpha'(G) + \frac{\alpha'(G)}{3}$$

$$\sum_{e \in E(G)} f(e) \leq n - \frac{2\alpha'(G)}{3}.$$

□

References

[1] H. Karami, A. Khodkar and S.M. Sheikholeslami, An improved upper bound for signed edge domination numbers in graphs, *Utilitas Math.* 78 (2009), 121–128 .

[2] D.B. West, *Introduction to Graph Theory*, Prentice-Hall, 2000.

[3] B. Xu, Note on edge domination numbers of graphs, *Discrete Math.* 294 (2005), 311–316.

[4] B. Xu, On signed domination numbers of graphs, *Discrete Math.* 239 (2001), 179–189.

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