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Proof of the Caccetta-Häggkvist conjecture for in-tournaments with respect to the minimum out-degree, and pancyclicity

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Abstract

Tewes and Volkmann [*Australas. J. Combin.* 18 (1998), 293–301] proved that for an integer $k \geq 3$, every in-tournament of minimum in-degree $d \geq 1$ and of order at most kd contains a directed cycle of length at most k. In other words, they proved that the Caccetta-Häggkvist conjecture, with respect to the minimum in-degree, is true for in-tournaments. In the same paper, they proved also that every strong in-tournament of minimum in-degree d and of order at most $3d$ is pancyclic. In our paper, we prove that for an integer $k \geq 3$, every in-tournament of minimum out-degree $d \geq 1$ and of order at most kd contains a directed cycle of length at most k, which means that the Caccetta-Häggkvist conjecture, with respect to the minimum out-degree, is true for in-tournaments. We prove also that every strong in-tournament of minimum out-degree d and of order at most $3d$ is pancyclic.

1 Introduction and definitions

The definitions which follow are those of [1].

We consider oriented graphs, that is, digraphs D without loops and without parallel arcs, such that for any distinct vertices x and y, at most one of the ordered pairs (x, y) and (y, x) is an arc of D. Also $V(D)$ is the *vertex set* of D and the *order* of D is the cardinality of $V(D)$. Moreover, $\mathcal{A}(D)$ is the set of the arcs of D.

We say that a vertex y is an *out-neighbor* of a vertex x (*in-neighbor* of x) if (x, y) (respectively, (y, x)) is an arc of D. $N_D^+(x)$ is the set of the out-neighbors of x and $N_D^-(x)$ is the set of the in-neighbors of x. The cardinality of $N_D^+(x)$ is the *out-degree* $d_D^+(x)$ of x and the cardinality of $N_D^-(x)$ is the *in-degree* $d_D^-(x)$ of x. When no confusion is possible, we omit the subscript D. For a set S of vertices of D, we denote by $D[S]$ the subgraph of D induced by S, that is, the oriented graph whose vertex set is S and whose arcs are the arcs of G with both vertices in S. A vertex y is *adjacent* with a vertex x of D, if y is either an out-neighbor of x or an in-neighbor of x. An *independent set* of D is a subset S of D such that its vertices are mutually non-adjacent.

For a vertex x of D and for a subset S of $V(D)$, $N_S^+(x)$ is the set of the out-neighbors of x which are in S, and $d_S^+(x)$ is the cardinality of $N_S^+(x)$. Similarly, $N_S^-(x)$ is the set of the in-neighbors of x which are in S, and $d_S^-(x)$ is the cardinality of $N_S^-(x)$.

For two disjoint subsets A and B of $V(D)$, an *arc from A to B* is an arc (x, y) with $x \in A$ and $y \in B$. We say that A has out-neighbors in B, if there exists a vertex of A having an out-neighbor in B. We say that there are no arcs between A and B, if there are no arcs from A to B and no arcs from B to A.

A *directed walk* of length p of D is a list $P = (x_0, \ldots, x_p)$ of vertices such that $(x_{i-1}, x_i) \in \mathcal{A}(D)$ for $1 \leq i \leq p$. When the vertices of P are distinct, P is a *directed path* (from x_0 to x_p). A *directed cycle* of length $p \geq 2$ is a list $(x_0, \ldots, x_{p-1}, x_0)$ of vertices with $x_0, \ldots, x_{p-1}$ distinct, $(x_{i-1}, x_i) \in \mathcal{A}(D)$ for $1 \leq i \leq p-1$ and $(x_{p-1}, x_0) \in \mathcal{A}(D)$. From now on, we omit the adjective "directed". A p-cycle of D is a cycle of length p.
A *triangle* is a 3-cycle of D. A *Hamiltonian path* of D is a path using all the vertices of P. A *Hamiltonian cycle* of D is a cycle using all the vertices of D. The oriented graph D is said to be *Hamiltonian* if it contains a Hamiltonian cycle.

The *girth* $g(D)$ of D is the minimum length of the cycles of D. An oriented graph D of order n is *pancyclic* if for every integer i with $3 \leq i \leq n$, D contains an i-cycle. The oriented graph D is said to be *strongly connected* (or *strong* for short), if for every pair of distinct vertices x and y of D, there exists a path from x to y. It is known and easy to prove that if D is strongly connected, and S is a proper subset of $V(D)$, there exists a vertex of $V(D) \setminus S$ with al least one out-neighbor in S and a vertex of $V(D) \setminus S$ with at least one in-neighbor in S

A *tournament* is an oriented graph T such that for any two distinct vertices x and y, exactly one of the ordered pairs (x, y) and (y, x) is an arc of D. By Redei's theorem (see [7]), every tournament admits a Hamiltonian path. By Camion's Theorem (see [4]), a tournament T is strong if and only if T is Hamiltonian. By a Theorem of Moon (see [6]) a strong tournament is pancyclic (even vertex pancyclic).

An *in-tournament* is an oriented graph D such that for every vertex x of D the subgraph $D[N^-(x)]$ induced by the in-neighbors of x is a tournament. This class of oriented graphs was first studied by Bang-Jensen, who proved in 1993 with Huang and Prisner (See [2]) that an in-tournament is strong if and only if it is Hamiltonian (a generalization of Camion's Theorem). Later, the class of in-tournaments was studied in [5], [8], [9], [10] and [11].

Caccetta and Häggkvist (see [3]) conjectured in 1978 that the girth of any digraph of order n with minimum out-degree at least d is at most $\lceil n/d \rceil$. For our purpose, we give an equivalent version of the Caccetta-Häggkvist conjecture (relatively to the minimum out-degree):

Conjecture 1.1 (C-H conjecture) *For given integers $d \geq 1$ and $k \geq 3$, every oriented graph of minimum out-degree d and of order at most kd, contains a cycle of length at most k.*

Of course, the conjecture is the same, if we replace "minimum out-degree" by "minimum in-degree".

In 1998, Tewes and Volkmann proved (see [10]) that for an integer $k \geq 3$, every in-tournament of minimum in-degree $d \geq 1$ and of order at most kd contains a directed cycle of length at most k. In other words, they proved that the Caccetta-Häggkvist conjecture with respect to the minimum out-degree is true for in-tournaments. In the same paper, they proved also that every strong in-tournament of minimum in-degre d and of order at most $3d$ is pancyclic. As for us, we prove:

Theorem 1.2 *For given integers $d \geq 1$ and $k \geq 3$, every in-tournament of minimum out-degree d and of order at most kd, contains a cycle of length at most k.*

This means that the Caccetta-Häggkvist conjecture with respect to the minimum out-degree is true for in-tournaments. Our second result is:

Theorem 1.3 *Every strong in-tournament of minimum out-degree d and of order at most $3d$ is pancyclic.*

It is easy to see that $\vec{C}^{d}_{kd+1}$ (the d-th power of a directed cycle of length $kd+1$) is an in-tournament of order $kd+1$ and minimum out-degree d which has no cycle of order at most k. This example, proposed by one of the referee, shows that Theorems 1.2 and 1.3 are best possible for in-tournaments.

2 Proof of Theorem 1.2

Suppose, for the sake of a contradiction, that the theorem is not true. Then there exist in-tournaments with minimum in-degree d, of order at most kd and of girth at least $k+1$. Among these in-tournaments, we choose an in-tournament D of minimum order n. We now give several intermediate results on D.

Claim 2.1 *Let S be a proper subset of $V(D)$, and let u be a vertex of S with minimum out-degree in $D[S]$. Then $N^+(u) \setminus N^+_S(u)$ is a non-empty set disjoint with S.*

Proof. Clearly $N^+(u) \setminus N^+_S(u)$ is disjoint with S. Suppose that $N^+(u) \setminus N^+_S(u)$ is the empty set. This means that $d^+_S(u) = d^+(u) \geq d$, and then $d^+_S(v) \geq d$ for every vertex v of S, which, by minimality of n, is not possible. So $N^+(u) \setminus N^+_S(u)$ is a non-empty set. □

Claim 2.2 *For any two non-adjacent vertices u and v of D, we have $N^+(u) \cap N^+(v) = \emptyset$*

Proof. Suppose the opposite, and then let w be a common out-neighbor of u and v. Then u and v are in-neighbors of w, and since D is a in-tournament, v and u are adjacent, which is false. So the result is proved. □

Claim 2.3 *Let u be a vertex of D, and let $P = (u_0, \ldots, u_m)$ be a path of D of length $m \geq 1$, with $u_0 \neq u$, $u_i \notin N^-(u)$ for $0 \leq i \leq m$, and $u_0 \notin N^+(u)$. Then for every i with $0 \leq i \leq m$, u_i is not an out-neighbor of u.*

Proof. Suppose the opposite, and then let j be the smallest of the integers i with $u_i \in N^+(u)$. Then $j \geq 1$. But then, since u_{j-1} and u are distinct in-neighbors of u_j, and D is an in-tournament, the vertices u and u_{j-1} are adjacent. But, by hypothesis $u_{j-1} \notin N^-(u)$, so $u_{j-1} \in N^+(u)$, contradicting the minimality of j. So, the result is proved. □

Claim 2.4 *Let $P = (u_1, \ldots, u_m)$ be a path of D of order m with $3 \leq m < g(D)$, such that for every i with $1 \leq i \leq m-2$, u_{i+2} is not an out-neighbor of u_i. Then P is an induced path of D.*

Proof. Since $m < g(D)$, there are no arcs (u_j, u_i) with $j > i \geq 1$. Suppose that there exists an arc (u_i, u_j) of D with $i + 2 \leq j \leq m$. Since (u_i, u_{i+2}) is not an arc of D, $(u_{i+2}, \ldots, u_j)$ is a path of length at least 1 with $u_{i+2} \neq u_i$, $u_p \notin N^-(u_i)$ for $i + 2 \leq p \leq j$, and $u_{i+2} \notin N^+(u_i)$. Then, by Claim 2.3, u_j is not an out-neighbor of u_i, which is contradictory. So P is an induced path. □

Now we are able to prove Theorem 1.2.

We consider a vertex x of D with $d^-(x) \geq d$. Recursively, we define the sets A_i, $0 \leq i \leq k-2$, the vertices x_i, $0 \leq i \leq k-2$, the sets B_i, $0 \leq i \leq k-2$ and the integers b_i, $0 \leq i \leq k-2$, in the following way:

- $A_0 = N^+(x)$ and x_0 is a vertex of A_0 of minimum out-degree in $D[A_0]$. B_0 is the set of the out-neighbors of x_0, which are in A_0 (in other words $B_0 = N^+_{A_0}(x_0)$), and $b_0 = |B_0|$.

- For $1 \leq i \leq k-2$, $A_i = N^+(x_{i-1}) \setminus B_{i-1}$. By Claim 2.1, A_i is a non-empty set, and then we can pick a vertex x_i of A_i with minimum out-degree in $D[A_i]$. Then B_i is the set of the out-neighbors of x_i, which are in A_i (in other words $B_i = N^+_{A_i}(x_i)$), and $b_i = |B_i|$.

Each of the sets A_i, $0 \leq i \leq k-2$ is disjoint with $N^-(x)$, and there are no arcs from a set A_i, $1 \leq i \leq k-3$ to $N^-(x)$ (for otherwise we would have a cycle of length at most k, which is not possible). We put $x_{-1} = x$. By construction, $P = (x_{-1}, x_0, \ldots, x_{k-3})$ is a walk of length $k-2$, and since the girth of D is at least $k+1$, the vertices of P are distinct. So P is a path and it is easy to see that by construction, x_{i+2} is not an out-neighbor of x_i for $-1 \leq i \leq k-5$ (with $x_{-1} = x$). Consequently, by Lemma 2.4, P is an induced path of D.

We claim that the sets A_i, $0 \leq i \leq k-2$, are mutually disjoint.
Suppose to the contrary, there exist two sets A_i and A_j with $j > i \geq 0$ such that $A_i \cap A_j \neq \emptyset$. By definition of A_{i+1}, we have $A_i \cap A_{i+1} = \emptyset$ for $0 \leq i \leq k-3$, and so

$j \geq i+2$. Let u be a vertex of $A_i \cap A_j$. The vertices x_{i-1} and x_{j-1} are in-neighbors of u. Since D is an in-tournament, x_{i-1} and x_{j-1} are adjacent, which contradicts the fact that $P = (x_{-1}, x_0, \ldots, x_{k-3})$ is an induced path.

We claim also that a vertex $u \in A_i$, $1 \leq i \leq k-3$, has no out-neighbors in the sets A_j with $0 \leq j \leq i-1$. Indeed, suppose the opposite. Then u has an out-neighbor v in a set A_j with $0 \leq j \leq i-1$. The vertices x_{j-1} and u are in-neighbors of v, and consequently they are adjacent. The vertex x_{j-1} cannot be an out-neighbor of u (for otherwise $C = (x_{j-1}, \ldots x_{i-1}, u, x_{j-1})$ would be a cycle of D of length at most $k-1$). Consequently, u is an out-neighbor of x_{j-1}. Then u is either in A_{j-1} or in A_j, but as u is in A_i, this is not possible (A_i is disjoint with A_{j-1} and A_j).

We put $A = \{x\} \cup A_0 \ldots \cup A_{k-2}$. We have $|A| = 1 + d^+(x) + d^+(x_0) - b_0 + \cdots + d^+(x_{k-3}) - b_{k-3}$. Since A is disjoint with $N^-(x)$, we get

$$n \geq 1 + d + d^+(x) + d^+(x_0) + \cdots d^+(x_{k-3}) - b_0 - \cdots - b_{k-3}. \qquad (1)$$

Suppose that all the b_i, $0 \leq i \leq k-3$ are null. From (1), we get then $n \geq kd+1$, which is impossible. Consequently, there exist integers i, $0 \leq i \leq k-3$ with $bi > 0$. Let $b_{i_1}, \ldots, b_{i_r}$ with $0 \leq i_1 < \cdots < i_r \leq k-3$, be the list of the integers $b_0, \ldots, b_{k-3}$ which are positive. We get then:

$$n \geq 1 + d + \sum_{-1 \leq q \leq k-3} d^+(x_q) - b_{i_1} - \cdots - b_{i_r}. \qquad (2)$$

For j with $1 \leq j \leq r$, $D[A_{i_j}]$ is of minimum out-degree $b_{i_j} > 0$. A vertex u of B_{i_j} has at most $b_{i_j} - 1$ out-neighbors in B_{i_j}. It follows that u has at least one out-neighbor y_{i_j} in A_{i_j} but not in B_{i_j}. The vertices y_{i_j} and x_{i_j} are non-adjacent (because $y_{i_j} \notin B_{i_j}$ and D does not contain triangles).

We have seen that y_{i_j} has no out-neighbors in A_p with $p < i_j$. We claim that y_{i_j} has no out-neighbors in a set A_p with $p > i_j$. Indeed, suppose the opposite, and then let $y \in A_p$ be an out-neighbor of y_{i_j}. Then $P' = (x_{i_j}, \ldots, x_{p-1}, y)$ is a directed path of D. None of the vertices $x_{i_j}, \ldots, x_{p-1}, y$ is in $N^-(y_{i_j})$, and $x_{i_j} \notin N^+(y_{i_j})$. By claim 2.3, y is not an out-neighbor of y_{i_j}, which is contradictory. So, y_{i_j} has no out-neighbors in a set A_p with $p > i_j$. By Claim 2.2, we deduce that y_{i_j} has no out-neighbors in B_{i_j}. It is easy to see that y_{i_j} has no out neighbors in $N^-(x)$). It follows that y_{i_j} has at most $d^+(x_{i_j-1}) - b_{i_j}$ out-neighbors in $A \cup N^-(x) \cup \{x\}$, and then y_{i_j} has at least $d - d^+(x_{i_j-1}) + b_{i_j}$ out-neighbors not in $A \cup N^-(x) \cup \{x\}$. Since two vertices y_{i_j} and y_{i_p} are non-adjacent, they cannot have a common out-neighbor. We get then:

$$n \geq 1 + d + \sum_{-1 \leq q \leq k-3} d^+(x_q) - b_{i_1} - \cdots - b_{i_r} + d - d^+(x_{i_1-1}) + b_{i_1} + \cdots + d - d^+(x_{i_r-1}) + b_{i_r},$$

hence $n \geq kd+1$ which is not possible. So Theorem 1.1 is proved.

3 Proof of Theorem 1.3

We give first two intermediate results, which are certainly known:

Claim 3.1 *Let T be a tournament without 4 cycles. Then there exists a vertex of T with out-degree at most 1.*

Proof. Suppose the opposite. Let $(x_1, \dots, x_m)$ be a Hamiltonian path of T. Since x_m has at least two out-neighbors, there exists an out neighbor x_i of x_m with $i \leq m-3$. Then $C' = (x_i, \dots x_m, x_i)$ is a cycle of T, and consequently $T[V(C')]$ is a strong tournament of order at least 4. Since such a tournament is pancyclic, we get a 4-cycle, which is a contradiction. Consequently, the result is proved. $\square$

The second result is:

Claim 3.2 *Let D be an in-tournament of minimum out-degree d of order $n \leq 3d$. Suppose that A and B are two non-empty disjoint subsets of $V(D)$ such that there are no arcs between A and B. Then $D[A]$ and $D[B]$ are tournaments.*

Proof. Suppose that $D[A]$ is not a tournament. Then, there exist two non-adjacent vertices x and y of A. Let z be a vertex of B. Then $\{x, y, z\}$ is an independent set, and the sets $N^+(x), N^+(y)$ and $N^+(z)$ are disjoint (because D is an in-tournament) and disjoint with x, y and z. It follows that $n \geq 3d + 3$ which is contradictory. So $D[A]$ is a tournament, and similarly $D[B]$ is a tournament. $\square$

Now we are able to prove Theorem 1.3. Suppose that there exists an in-tournament D fulfilling the required conditions, which is not pancyclic. Then, there exists an integer p, $3 \leq p \leq n-1$, such that D contains at least one cycle of length p, but does not contain cycles of length $p+1$. Then, let $C = (x_1, \dots, x_p, x_1)$ be a p-cycle of D. The indexation of the vertices of C is taken modulo p (this means $x_{i+p} = x_i$). We put $S_0 = V(C)$. Let S_1 be the set of the vertices not in $V(C)$ with at least one out-neighbor in $V(C)$. Since $V(C) \neq V(D)$ and D is strong, S_1 is a non-empty set. We claim that for a given vertex x of S_1, every vertex of C is an out-neighbor of x. Indeed, suppose the opposite. Then, there exist two consecutive vertices x_i and x_{i+1} of C such that x_{i+1} is an out-neighbor of x, and x_i is not an out-neighbor of x. The vertices x_i and x are in-neighbors of x_{i+1}, and since D is an in-tournament, x_i and x are adjacent, which implies that x_i is an in-neighbor of x. But then $C_1 = (x_i, x, x_{i+1}, x_{i+p-1}, x_i)$ is a cycle of length $p+1$, contradicting our assumption.

Let S_2 be the set of the vertices of $V(D) \setminus V(C)$ with at least one in-neighbor in $V(C)$. Since $V(C) \neq V(D)$ and D is strong, S_2 is a non-empty set, and since every vertex of $V(C)$ is an out-neighbor of every vertex of S_1, S_1 and S_2 are disjoint. Further, there is no arc from S_2 to S_1. Indeed, suppose the opposite, and then let (x, y) be an arc of D with $x \in S_2$ and $y \in S_1$. The vertex x has an in-neighbor x_i in $V(C)$, and then $C_1 = (x, y, x_{i+2}, \dots, x_{i+p}, x)$ is a cycle of D of length $p+1$, contradicting our assumption.

We claim that $V(D) \setminus (S_0 \cup S_1 \cup S_2) \neq \emptyset$. Indeed, by the preceding arguments, S_2 has no out-neighbors in $S_0 \cup S_1$. Then $D = D[S_0 \cup S_1 \cup S_2]$ is not strong, and since D is strong, it follows $V(D) \setminus (S_0 \cup S_1 \cup S_2) \neq \emptyset$.

Observe that $V(D) \setminus (S_0 \cup S_1 \cup S_2)$ is the set of the vertices of $V(D) \setminus V(C)$ with no adjacent vertices in $V(C)$, and then by Claim 3.2, the induced in-tournaments

$D[V(C)]$ and $D[V(D)\setminus(S_0\cup S_1\cup S_2)]$ are tournaments. We claim that $V(D)\setminus(S_0\cup S_1\cup S_2)$ has no out-neighbors in S_2. Indeed, suppose the opposite, and then let (x,y) be an arc of D with $x\in V(D)\setminus(S_0\cup S_1\cup S_2)$ and $y\in S_2$. The vertex y has an in-neighbor x_i in $V(C)$. The vertices x_i and x are in-neighbors of y, and since D is an in-tournament, they are adjacent, which is impossible.

Since D is strong, there exists a vertex of S_2 with out-neighbors not in S_2, and these out-neighbors are not in $V(C)\cup S_1$. So, the set S_3 of the vertices of $V(D)\setminus S_2$ with at least one in-neighbor in S_2, is non-empty and disjoint with each of the sets S_0, S_1 and S_2. We claim that a vertex x of S_3 has no out-neighbors in S_1. Indeed, suppose the opposite, and then let (x,y) be an arc of D with $y\in S_1$. The vertex x has an in-neighbor u in S_2, and u has an in-neighbor x_i of S_0. Then (y,x_i,u,x,y) is a 4-cycle, and consequently $p\neq 3$, hence $p\geq 4$. But then $C'=(y,x_{i+3},\dots,x_{i+p},u,x,y)$ is a cycle of length $p+1$, which is impossible. So, the assertion is true, and this means that S_3 has no out-neighbors in S_1.

Since D is strong, there exists a vertex x of S_3 with at least one out-neighbor not in S_3, and by the preceding arguments, this out-neighbor is not in $S_0\cup S_1\cup S_2\cup S_3$. This means that the set S_4 of the vertices of $V(D)\setminus S_3$ with at least one in-neighbor in S_3, is non-empty and disjoint with S_0,S_1,S_2 and S_3. We claim also that every vertex of S_2 is non-adjacent with every vertex of S_4. Indeed, let x be a vertex of S_4, and let y be a vertex of S_2. Since x is not in S_3, y is not an in-neighbor of x. Suppose that y is an out-neighbor of x. Then y has an in-neighbor x_i in S_0. The vertices x and x_i are in-neighbors of y, and then, since D is an in-tournament, x and x_i are adjacent, which is false. So x and y are non adjacent, which proves our assertion. Now, we consider two cases:

Case 1: $p=3$

The vertex x_1 of S_0 has at least $d-1>0$ out-neighbors which are all in S_2, and therefore $|S_2|\geq d-1$. Since the tournament $D[S_2]$ has no 4-cycles, there exists a vertex u of S_2 with at most one out-neighbor in S_2 (by Claim 3.1). Then u has at least $d-1$ out-neighbors which are all in S_3, and consequently $|S_3|\geq d-1$. Since the tournament $D[S_3]$ has no 4-cycles, there exists a vertex v of S_3 with at most one out-neighbor in S_3 (by Claim 3.1). Then v has at least $d-1$ out-neighbors which are all in S_4, and consequently $|S_4|\geq d-1$. Since $|S_1|\geq 1$, we get $n\geq 3(d-1)+3+1$, which yields $n\geq 3d+1$, which is impossible.

Case 2: $p\geq 4$

We claim that a vertex x of S_4 has no out-neighbors in S_1. Indeed, suppose the opposite, and then let (x,y) be an arc of D with $y\in S_1$. The vertex x has an in-neighbor u in S_3, u has an in-neighbor v in S_2, and v has an in-neighbor x_i in S_0. Then y,x_i,v,u,x,y is a 5-cycle, and consequently $p\neq 4$, hence $p\geq 5$. But then $C'=(y,x_{i+4},\dots,x_{i+p},v,u,x,y)$ is a cycle of length $p+1$, which is impossible. So, the assertion is true, and this means that S_4 has no out-neighbors in S_1. We claim also that a vertex x of S_4 has no out-neighbors in S_3. Indeed suppose the opposite, and then let y be an out-neighbor of x in S_3. The vertex y has an in-neighbor z in S_2, and then z and x are in-neighbors of y, and since D is an in-tournament, z and

x are adjacent, which is contradictory. So, the assertion is true, and this means that S_4 has no out-neighbors in S_3.

Since D is strong, there exists a vertex x of S_4 with at least one out-neighbor not in S_4, and by which precedes this out-neighbor is not in $S_0 \cup S_1 \cup S_2 \cup S_3 \cup S_4$. This means that the set S_5 of the vertices of $V(D) \setminus S_4$ with at least one in-neighbor in S_4, is non-empty and disjoint with S_0, S_1, S_2, S_3 and S_4. By the preceding arguments, it is easy to prove successively that a vertex x of S_5 has out-neighbors neither in S_0, neither in S_2, neither in S_3, nor in S_4. It follows that all the out-neighbors of x are in $V(D) \setminus (S_0 \cup S_2 \cup S_3 \cup S_4)$. This implies $|V(D) \setminus (S_0 \cup S_2 \cup S_3 \cup S_4)| \geq d+1$. A vertex of S_3 has all its out-neighbors in $S_3 \cup S_4$. It follows $|S_3 \cup S_4| \geq d+1$. A vertex of S_0 has all its out-neighbors in $S_0 \cup S_2$. It follows $|S_0 \cup S_2| \geq d+1$. It follows $n \geq 3d+3$, which is impossible. Consequently, Theorem 1.3 is proved.

References

[1] J. Bang-Jensen and G. Gutin, *Digraphs*, Springer, 2002.

[2] J. Bang-Jensen, J. Huang and E. Prisner, In-tournament digraphs, *J. Combin. Theory Ser. B* 59 (1993), 267–287.

[3] L. Caccetta and R. Häggkvist, On minimal digraphs with given girth, *Congressus Numerantium* 21 (1978), 181–187.

[4] P. Camion, Chemins et circuits hamiltoniens des graphes complets, *C.R. Acad. Sci. Paris* 249 (1959), 2151–2152.

[5] D. Meierling, On the number of nonseparating vertices in strongly connected in-tournaments, *Australas. J. Combin.* 44 (2009), 123–152.

[6] J. W. Moon, On subtournaments of a tournament, *Canad. Math. Bull.* 9 (1966), 297–301.

[7] L. Rédei, Ein kombinatorischer Satz, *Acta Litt. Sci. Szeged* 7 (1934), 39–43.

[8] M. Tewes, Pancyclic in-tournaments, *Discrete Math.* 233 (1–3) (2001), 193–204.

[9] M. Tewes, Pancyclic ordering of in-tournaments, *Discrete Appl. Math.* 120 (1–3) (2002), 239–249.

[10] M. Tewes and L. Volkmann, On the cycle structure of in-tournaments, *Australas. J. Combin.* 18 (1998), 293–301.

[11] M. Tewes and L. Volkmann, Vertex pancyclic in-tournaments, *J. Graph Theory* 36 (2) (2001), 84–104.

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