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Group distance magic labeling of some cycle-related graphs

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Abstract

Let $G = (V, E)$ be a graph and Γ an abelian group, both of order n. A group distance magic labeling of G is a bijection $\ell\colon V \to \Gamma$ for which there exists $\mu \in \Gamma$ such that $\sum_{x\in N(v)} \ell(x) = \mu$ for all $v \in V$, where $N(v)$ is the neighborhood of v. Froncek [*Australas. J. Combin.* 55 (2013), 167–174] showed that the cartesian product $C_m \Box C_n$, $m, n \geq 3$ is a $\mathbb{Z}_{mn}$-distance magic graph if and only if mn is even. In this paper we show some Γ-distance magic labelings for $C_m \Box C_n$ where $\Gamma \not\cong \mathbb{Z}_{mn}$. Moreover we will deal with group distance labeling of the pth power of a cycle C_n.

1 Introduction

All graphs considered in this paper are simple finite graphs. Consider a simple graph G whose order we denote by $|G| = n$. Write $V(G)$ for the vertex set and $E(G)$ for the edge set of a graph G. The *neighborhood* $N(x)$ of a vertex x is the set of vertices adjacent to x, and the degree $d(x)$ of x is $|N(x)|$, the size of the neighborhood of x. By C_n we denote a cycle on n vertices.

A *distance magic labeling* (also called a *sigma labeling*) of a graph $G = (V, E)$ of order n is a bijection $\ell\colon V(G) \to \{1, 2, \ldots, n\}$ with the property that there is a positive integer k (called the *magic constant*) such that $\sum_{y\in N_G(x)} \ell(y) = k$ for every $x \in V(G)$. If a graph G admits a distance magic labeling, then we say that G is a *distance magic graph.*

Miller et al. showed in [5]:

Theorem 1 ([5]) *The cycle C_n of length n is a distance magic graph if and only if $n = 4$.*

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The *cartesian product* $G\Box H$ of graphs G and H is a graph such that the vertex set of $G\Box H$ is the cartesian product $V(G)\times V(H)$; and any two vertices (u,u') and (v,v') are adjacent in $G\Box H$ if and only if either $u=v$ and u' is adjacent with v' in H, or $u'=v'$ and u is adjacent with v in G.

Rao et al. proved the following result for cartesian product of cycles in [6].

Theorem 2 ([6]) *The cartesian product* $C_n\Box C_m$, $n,m\geq 3$ *is a distance magic graph if and only if* $n=m\equiv 2 \pmod 4$.

The pth power of a graph G is a graph G^p with the same set of vertices as G and an edge between two vertices if and only if there is a path of length at most p between them. In this paper we will consider the pth power of a cycle C_n. Notice that C_n^p is a $2p$-regular graph and moreover it is a circulant graph. A *circulant graph* is a graph on n vertices that admits a cyclic automorphism of order n.

By $\gcd(a,b)$ we denote the greatest common divisor of a and b. It was shown in [1] that a graph C_n^2 is not distance magic graph unless $n=6$. For the case p being odd the following theorem was proved:

Theorem 3 ([1]) *If p is odd, then C_n^p is a distance magic graph if and only if* $2p(p+1)\equiv 0 \pmod n$, $n\geq 2p+2$ *and* $\frac{n}{\gcd(n,p+1)}\equiv 0 \pmod 2$.

The notion of group distance magic labeling of graphs was introduced by Froncek in [4]. A *group distance magic labeling* or a Γ-distance magic labeling of a graph $G(V,E)$ with $|V|=n$ is an injection from V to an abelian group Γ of order n such that the sum of the labels of all neighbors of every vertex $x\in V$, called the *weight* of x and denoted by $w(x)$, is equal to the same element $\mu\in\Gamma$, called the *magic constant*.

One can notice that if a graph G of order n admits a distance magic labeling, then it also admits a $\mathbb{Z}_n$-distance magic labeling and the inverse is not necessary true.

There have been considered some families of graphs that are Γ-distance magic (see [2, 3, 4]). The following was proved in [4]:

Theorem 4 ([4]) *The cartesian product* $C_m\Box C_n$, $m,n\geq 3$ *is a* $\mathbb{Z}_{mn}$*-distance magic graph if and only if* mn *is even.*

Froncek [4] also showed that the graph $C_{2^n}\Box C_{2^n}$ has a $(\mathbb{Z}_2)^{2n}$-distance magic labeling for $n\geq 2$ and $\mu=(0,0,\dots,0)$ and posted the following problem:

Problem 5 ([4]) *For a given graph* $C_m\Box C_k$, *determine all abelian groups* Γ *such that the graph* $C_m\Box C_k$ *admits a* Γ*-distance magic labeling.*

In the next section we will partially solve the above problem by showing some other groups Γ besides $\mathbb{Z}_{mn}$ for which $C_m\Box C_k$ admits a Γ-distance magic labeling;

whereas in the third section we will consider group distance magic labelings for a graph C_n^p.

Recall that any group element $g \in \Gamma$ of order 2 (i.e. $g \neq 0$ such that $2g = 0$) is called an *involution*, and that a non-trivial finite group has elements of order 2 if and only if the order of the group is even. Moreover every cyclic group of even order has exactly one involution. The fundamental theorem of finite abelian groups states that the finite abelian group Γ can be expressed as the direct sum of cyclic subgroups of prime-power order. This product is unique up to the order of the direct product. If t be the number of these cyclic components whose order is a power of 2, then Γ has $2t-1$ involutions. A subgroup generated by an element g in a group Γ we denote as $\langle g \rangle$. If H is a subgroup of an abelian group Γ, and $g \in \Gamma$, then $H + g = \{h + g : h \in H\}$ is a *coset* of H in Γ.

2 Group labeling for $C_n \square C_m$

Let $V(C_n \times C_m) = \{x_{i,j} : 0 \leq i \leq n-1, 0 \leq i \leq m-1\}$, where $N(x_{i,j}) = \{x_{i,j-1}, x_{i,j+1}, x_{i+1,j}, x_{i-1,j}\}$ and operation on the first suffix is taken modulo n and the second suffix modulo m respectively. By a diagonal D^j of $C_n \square C_m$ we mean a sequence of vertices $(x_{0,j}, x_{1,j+1}, \ldots, x_{n-1,j+n-1}, x_{0,j+n}, x_{1,j+n+1}, \ldots, x_{n-1,j-1})$ of length l. It is easy to observe that $l = \operatorname{lcm}(n,m)$, the least common multiple of n and m. As in [4] we denote the diagonal by $D^j = (d_0^j, d_1^j, \ldots, d_{l-1}^j)$, $j = 0, 1, 2, \ldots, d-1$, where $d = \gcd(m,n)$.

We will start this section by showing that if m, n are odd then there does not exist an abelian group Γ of order mn such that $C_n \square C_m$ has Γ-distance magic labeling.

Lemma 6 *If n, m are odd, then $C_n \square C_m$ is not a Γ-distance magic graph for any abelian group Γ of order nm.*

Proof. Suppose that there exists a group Γ that $C_n \square C_m$ is Γ-distance magic. Let $\ell : V(C_n \square C_m) \to \Gamma$ be a Γ-distance magic labeling and μ be the magic constant for $C_n \square C_m$. It follows $w(x_{0,0}) = w(x_{1,1}) = \ell(x_{0,m-1}) + \ell(x_{n-1,0}) + \ell(x_{1,0}) + \ell(x_{0,1}) = \ell(x_{1,0}) + \ell(x_{0,1}) + \ell(x_{2,1}) + \ell(x_{1,2})$. Hence $\ell(x_{0,m-1}) + \ell(x_{n-1,0}) = \ell(x_{2,1}) + \ell(x_{1,2})$. Comparing weights of vertices $x_{2,2}$ and $x_{3,3}$ we obtain $\ell(x_{2,1}) + \ell(x_{1,2}) = \ell(x_{4,3}) + \ell(x_{3,4})$. Consequently, we obtain

$$\ell(x_{0,m-1}) + \ell(x_{n-1,0}) = \ell(x_{2\alpha,2\alpha-1}) + \ell(x_{2\alpha-1,2\alpha})$$

for any natural number α. Let $d = \gcd(n,m)$. Since n, m are odd and taking $\alpha' = \frac{1}{2}\left(\frac{nm}{d} + 1\right)$, we obtain $2\alpha' \equiv 1 \pmod{n}$ and $2\alpha' \equiv 1 \pmod{m}$. Hence $\ell(x_{0,m-1}) + \ell(x_{n-1,0}) = \ell(x_{1,0}) + \ell(x_{0,1})$ and furthermore $2(\ell(x_{0,m-1}) + \ell(x_{n-1,0})) = \mu$.

Analogously, if we start with vertices $x_{n-2,0}$ and $x_{n-3,1}$, we obtain

$$\ell(x_{n-2,m-1}) + \ell(x_{n-1,0}) = \ell(x_{n-4,1}) + \ell(x_{n-3,2}) = \ldots = \ell(x_{n-3,0}) + \ell(x_{n-2,1}).$$

Thus $\mu = \ell(x_{n-2,m-2}) + \ell(x_{n-1,0}) + \ell(x_{n-3,0}) + \ell(x_{n-2,1})$ and $2(\ell(x_{n-2,m-1}) + \ell(x_{n-1,0})) = \mu$. Therefore $2(\ell(x_{0,m-1}) - \ell(x_{n-2,m-1})) = 0$. Since the order of the group Γ is odd, it implies that there does not exist an element of order 2 in Γ, so $\ell(x_{0,m-1}) - \ell(x_{n-2,m-1}) = 0$, a contradiction. ■

In the proof of the theorem below we are using constructions similar to those by Froncek (see [4]).

Theorem 7 *Let $l = \mathrm{lcm}(n, m)$, if n or m is even then $C_n \Box C_m$ has a $\mathbb{Z}_\alpha \times \mathcal{A}$-magic labeling for any $\alpha \equiv 0 \pmod{l}$ and any abelian group $\mathcal{A}$ of order mn/α.*

Proof. Without losing generality we can assume that n is even. Notice that $l = 2k$ for some k and $\alpha = 2kh$ for some h. Notice also that $d = \gcd(n, m)$ is the number of diagonals of $C_n \Box C_m$.

If $d = 1$, then observe that there exists only one abelian group of order nm, namely $\mathbb{Z}_{mn}$ and we are done by Theorem 4. So we can assume that $d \geq 2$.

Let $r = nm/\alpha$ and $\Gamma \cong \mathbb{Z}_\alpha \times \mathcal{A}$, thus if $g \in \Gamma$, then we can write that $g = (j, a_i)$ for $j \in \mathbb{Z}_\alpha$ and $a_i \in \mathcal{A}$ for $i = 0, 1, \ldots, r-1$. We can assume that $a_0 = 0 \in \mathcal{A}$. Let $\ell(x) = (l_1(x), l_2(x))$.

Let $H = \langle 2h \rangle$ be the subgroup of $\mathbb{Z}_\alpha$ of order k.

Label the vertices of D^0 as follows:

$$\ell(d^0_{2i}) = (2ih, a_0),\ \ell(d^0_{2i+1}) = (-2ih - 1, -a_0)$$

for $i = 0, 1, \ldots, k-1$.

The vertices in $D^1, D^2, D^3, \ldots, D^{h-1}$ will be labeled as

$$\begin{aligned} \ell(d^j_g) &= (l_0(d^{j-1}_g) + 1, a_0) \ \text{ if } g \equiv 1 \pmod{2}, \\ \ell(d^j_g) &= (l_1(d^{j-1}_g) - 1, -a_0) \ \text{ if } g \equiv -1 \pmod{2}. \end{aligned}$$

The vertices in $D^h, D^{h+1}, \ldots, D^{d-1}$ will be labeled as

$$\begin{aligned} \ell(d^j_g) &= (l_1(d^{j-h}_g), a_{\lfloor j/h \rfloor}) \ \text{ if } g \equiv 1 \pmod{2}, \\ \ell(d^j_g) &= (l_1(d^{j-h}_g), -a_{\lfloor j/h \rfloor}) \ \text{ if } g \equiv 0 \pmod{2}. \end{aligned}$$

Obviously ℓ is a bijection and moreover:

$\ell(d^j_{2i}) + \ell(d^j_{2i+1}) = (-1, 0)$ and $\ell(d^j_{2i+1}) + \ell(d^j_{2i+2}) = (2h-1, 0)$ for any i.

If $d > 2$, then the vertex $x_{i',j'} = d^j_i$ has in $C_n \Box C_m$ neighbors d^{j-1}_i, d^{j-1}_{i+1} and d^{j+1}_{i-1}, d^{j-1}_i. Therefore $w(d^j_i) = 2h - 2$ and the labeling is Γ-distance magic as desired.

If $d = 2$, then the vertex $x_{i',j'} = d^j_i$ has in $C_n \Box C_m$ neighbors d^{j+1}_a, d^{j+1}_{a+1} and d^{j+1}_b, d^{j+1}_{b+1}, for $j + 1 \pmod{2} \leq 1$, $0 \leq a < b \leq l - 1$. Since $d^{j+1}_a = x_{i',j'-1}$ and $d^{j+1}_b = x_{i',j'+1}$, thus $b + 1 = a + qn$ for some $1 \leq q < l/n$. Because n is even a and $b + 1$ have the same parity and thus $w(d^j_i) = 2h - 2$. ■

Notice that by the above theorem for an example, a graph $C_{12} \Box C_{12}$ admits a Γ-distance magic labeling to any group Γ isomorphic to $\mathbb{Z}_{12} \times \mathbb{Z}_{12}$, $\mathbb{Z}_{12} \times \mathbb{Z}_2 \times \mathbb{Z}_6$, $\mathbb{Z}_{24} \times \mathbb{Z}_6$, $\mathbb{Z}_{48} \times \mathbb{Z}_3$, $\mathbb{Z}_{36} \times \mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_{36} \times \mathbb{Z}_4$ or $\mathbb{Z}_{144}$.

3 Group labelings for C_n^p

In this section we will show some group distance magic labelings for the circulant graph C_n^p. Let $C_n = x_0x_1\dots x_{n-1}$; then

$$N(x_i) = \{x_{i-p}, x_{i-p+1}, \dots, x_{i-1}, x_{i+1}, x_{i+2}, \dots, x_{i+p}\},$$

where $i+j$ is taken modulo n, is the neighborhood of a vertex $x_i \in V(C_n^p)$.

Observation 8 *If $n < 2p+2$, then C_n^p is not a Γ-distance magic graph for any abelian group Γ of order n.*

Proof. If $n < 2p+2$, then $C_n^p \cong K_n$, so there does not exist a group Γ so that C_n^p is a Γ-distance magic graph. ■

Observation 9 *If C_n^p is a Γ-distance magic graph for a group Γ, then n is even.*

Proof. Let $\ell : V(C_n^p) \to \Gamma$ be a Γ-distance magic labeling and μ be the magic constant for C_n^p. One can check that:

$$\ell(x_i)+\ell(x_{i+1})+\ldots+\ell(x_{i+p-1}) = \ell(x_{i+2\gamma(p+1)})+\ell(x_{i+1+2\gamma(p+1)})+\ldots+\ell(x_{i+p-1+2\gamma(p+1)})$$

for any natural γ. Suppose that n is odd; then $\frac{n}{\gcd(n,p+1)} \equiv 1 \pmod 2$. Thus $\gcd(n,p+1) = \gcd(n,2p+2)$ and $\langle 2(p+1)\rangle = \langle p+1\rangle$. Hence, $p+1 = c2(p+1)$ for some $c \geq 1$. Set $\gamma = c$, $i = 0,1$ and obtain respectively:

$$\begin{aligned} \ell(x_0)+\ell(x_1)+\cdots+\ell(x_{p-1}) &= \ell(x_{p+1})+\ell(x_{p+2})+\cdots+\ell(x_{2p}), \\ \ell(x_1)+\ell(x_2)+\cdots+\ell(x_p) &= \ell(x_{p+2})+\ell(x_{p+3})+\cdots+\ell(x_{2p+1}). \end{aligned}$$

Since $N(x_i) = \{x_{i-p}, x_{i-p+1}, \dots, x_{i-1}, x_{i+1}, x_{i+2}, \dots, x_{i+p}\}$ and $C_n(1,p)$ is Γ-distance magic, we obtain:

$$\begin{aligned} 2(\ell(x_0)+\ell(x_1)+\cdots+\ell(x_{p-1})) &= \mu, \\ 2(\ell(x_1)+\ell(x_2)+\cdots+\ell(x_p)) &= \mu. \end{aligned}$$

Therefore $2(\ell(x_0)-\ell(x_p)) = 0$. Recall that n being odd implies that there does not exists an element $g \neq 0$, $g \in \Gamma$ such that $2g = 0$. Thus $\ell(x_0) = \ell(x_p)$ and we obtain a contradiction, because $n > 2p+1$. ■

Theorem 10 *Let $\gcd(n,p+1) = d$. If p is even, $n > 2p+1$ and $n = 2kd$, then C_n^p has a $\mathbb{Z}_\alpha \times \mathcal{A}$-magic labeling for any $\alpha \equiv 0 \pmod{2k}$ and any abelian group $\mathcal{A}$ of order n/α.*

Proof. Let $n/\alpha = r$. Since $\Gamma \cong \mathbb{Z}_\alpha \times \mathcal{A}$, thus if $g \in \Gamma$, then we can write that $g = (j, a_i)$ for $j \in \mathbb{Z}_\alpha$ and $a_i \in \mathcal{A}$ for $i = 0, 1, \ldots, r-1$. We can assume that $a_0 = 0 \in \mathcal{A}$. Let $\ell(x) = (l_1(x), l_2(x))$.

Let $X = \langle p+1 \rangle$ be the subgroup of $\mathbb{Z}_n$ of order $2k$. Let us denote for $j = 1, 2, \ldots, d-1$ by X_j the set of all vertices whose subscripts belong to coset $X + j$. Notice that $\alpha = 2kh$ for some h. Let $H = \langle 2h \rangle$ be the subgroup of $\mathbb{Z}_\alpha$ of order k.

Label the vertices of X_0 as follows:

$$\ell(x_{2i(p+1)}) = (2ih, a_0), \quad \ell(x_{(2i+1)(p+1)}) = (-2ih-1, -a_0), \quad i = 0, 1, \ldots, k-1.$$

If a subscript m belongs to the coset $X + j$, then denote it by m_j. Notice that a vertex x_{m_j} belongs to X_j if the vertex x_{m_j-p-2} belongs to X_{j-1}. The vertices in $X_1, X_2, X_3, \ldots, X_{d-1}$ will be labeled recursively as follows.

$$l_1(x_{m_j}) = \begin{cases} l_1(x_{m_j-p-2}) + 1 & \text{if} \quad l_1(x_{m_j-j(p+2)}) \equiv 0 \pmod{2h}, \\ l_1(x_{m_j-p-2}) - 1 & \text{if} \quad l_1(x_{m_j-j(p+2)}) \not\equiv 0 \pmod{2h}, \end{cases}$$

$$l_2(x_{m_j}) = \begin{cases} a_{\lfloor j/k \rfloor} & \text{if} \quad l_1(x_{m_j}) \equiv 0 \pmod{2}, \\ -a_{\lfloor j/k \rfloor} & \text{if} \quad l_1(x_{m_j}) \equiv 1 \pmod{2}. \end{cases}$$

Obviously ℓ is a bijection and moreover: $\ell(x_{2i}) + \ell(x_{2i+p+1}) = (-1, 0)$ and $\ell(x_{2i+1}) + \ell(x_{2i+p+2}) = (2h-1, 0)$ for any i.

Recall that $N(x_i) = \{x_{i-p}, x_{i-p+1}, \ldots, x_{i-1}, x_{i+1}, x_{i+2}, \ldots, x_{i+p}\}$. Since p is even, it implies that $w(x_i) = \sum_{j=1}^{p} (\ell(x_{i-j}) + \ell(x_{i-j+p+1}) = p(2h-2, 0)$ for any i. ■

Below, we show an example of a $\mathbb{Z}_{12} \times \mathbb{Z}_3$-distance magic labeling for C_{36}^8.

$$\begin{array}{llll}
\ell(x_0) = (0,0), & \ell(x_9) = (11,0), & \ell(x_{18}) = (6,0), & \ell(x_{27}) = (5,0), \\
\ell(x_{10}) = (1,0), & \ell(x_{19}) = (10,0), & \ell(x_{28}) = (7,0), & \ell(x_1) = (4,0), \\
\ell(x_{20}) = (2,0), & \ell(x_{29}) = (9,0), & \ell(x_2) = (8,0), & \ell(x_{11}) = (3,0), \\
\ell(x_{30}) = (3,2), & \ell(x_3) = (8,1), & \ell(x_{12}) = (9,2), & \ell(x_{21}) = (2,1), \\
\ell(x_4) = (4,1), & \ell(x_{13}) = (7,2), & \ell(x_{22}) = (10,1), & \ell(x_{31}) = (1,2), \\
\ell(x_{14}) = (5,2), & \ell(x_{23}) = (6,1), & \ell(x_{32}) = (11,2), & \ell(x_5) = (0,1), \\
\ell(x_{24}) = (6,2), & \ell(x_{33}) = (5,1), & \ell(x_6) = (12,2), & \ell(x_{15}) = (11,1), \\
\ell(x_{34}) = (7,1), & \ell(x_7) = (4,2), & \ell(x_{16}) = (13,1), & \ell(x_{25}) = (10,2), \\
\ell(x_8) = (8,2), & \ell(x_{17}) = (3,1), & \ell(x_{26}) = (14,2), & \ell(x_{35}) = (9,1).
\end{array}$$

We will use a similar method as in the proof of Theorem 3 ([1]) to show the following lemma.

Lemma 11 *If p is odd and $2p(p+1) \not\equiv 0 \pmod{n}$, then C_n^p is not a Γ-distance magic graph for any abelian group Γ of order n.*

Proof. Suppose that there exists a group Γ and C_n^p is Γ-distance magic and $2p(p+1) \not\equiv 0 \pmod{n}$. Let $\ell : V(C_n^p) \to \Gamma$ be a Γ-distance magic labeling and μ be the magic constant for C_n^p. It follows that $w(x_i) - w(x_{i+1}) = \ell(x_{i-p}) + \ell(x_{i+1}) - (\ell(x_i) + \ell(x_{i+1+p})) = 0$ for $i \in \{0, 1, \ldots, n-1\}$. Hence for any natural γ:

$$\begin{aligned}
&\ell(x_0) + \ell(x_{p+1}) = \ell(x_p) + \ell(x_{2p+1}) = \cdots = \ell(x_{\gamma p}) + \ell(x_{(\gamma+1)p+1}) = \mu_1,\\
&\ell(x_1) + \ell(x_{p+2}) = \ell(x_{p+1}) + \ell(x_{2p+2}) = \cdots = \ell(x_{\gamma p+1}) + \ell(x_{(\gamma+1)p+2}) = \mu_2,\\
&\quad\vdots\\
&\ell(x_{p-1}) + \ell(x_{2p}) = \ell(x_{2p-1}) + \ell(x_{3p}) = \cdots = \ell(x_{(\gamma+1)p-1}) + \ell(x_{(\gamma+2)p}) = \mu_p,
\end{aligned}$$

and $\mu_1 + \mu_2 + \cdots + \mu_p = \mu$.

Let $\ell(x_0) = k_0$; then

$$\ell(x_{j(p+1)}) = \sum_{i=0}^{j} (-1)^{j-i} \mu_i$$

for $j = 1, 2, \ldots, p$. If p is odd then $\ell(x_{p(p+1)}) = \mu_p - \mu_{p-1} + \mu_{p-2} - \cdots + \mu_1 - \mu_0$. It follows that

$$\begin{aligned}
\ell(x_{(p+1)(p+1)}) &= -\mu_p + \mu_{p-1} - k_{p-2} + \cdots + \mu_2 + k_0\\
\ell(x_{(p+2)(p+1)}) &= \mu_p - \mu_{p-1} + \mu_{p-2} - \cdots + \mu_3 - k_0\\
&\ \ \vdots\\
\ell(x_{2p(p+1)}) &= \mu_0.
\end{aligned}$$

It follows that $\ell(x_0) = \ell(x_{2p(p+1)}) = \mu_0$, a contradiction. ■

Recall that if p is odd, $2p(p+1) \equiv 0 \pmod{n}$, $n \geq 2p+2$ and $\frac{n}{\gcd(n,p+1)} \equiv 0 \pmod{2}$, then C_n^p is a $\mathbb{Z}_n$-distance magic graph by Theorem 3. Below, we generalize this result for some other groups Γ.

Theorem 12 *Let* $\gcd(n, p+1) = d$. *If* p *is odd,* $n = 2kd$, $p \equiv 0 \pmod{k}$ *and* $n > 2p+1$, *then* C_n^p *has a* $\mathbb{Z}_\alpha \times \mathcal{A}$*-magic labeling for any* $\alpha \equiv 0 \pmod{2k}$ *and any abelian group* $\mathcal{A}$ *of order* n/α.

Proof. Notice that $\alpha = 2kh$ for some natural h. Let $n/\alpha = r$. Since $\Gamma \cong \mathbb{Z}_\alpha \times \mathcal{A}$, if $g \in \Gamma$, then we can write We can assume that $a_0 = 0 \in \mathcal{A}$. Let $\ell(x) = (l_1(x), l_2(x))$.

Let $X = \langle p+1 \rangle$ be the subgroup of $\mathbb{Z}_n$ of order $2k$. Let us denote, for $j = 1, 2, \ldots, d-1$, by X_j the set of all vertices whose subscripts belong to the coset $X + j$.

Label the vertices of X_0 as follows:

If $k = 1$, then $\ell(x_0) = (0, a_0)$, $\ell(x_{p+1}) = (-1, -a_0)$.
If $k = 3$, then $\ell(x_0) = (0, a_0)$, $\ell(x_{p+1}) = (-2, -a_0)$, $\ell(x_{2(p+1)}) = (2, a_0)$, $\ell(x_{3(p+1)}) = (-3, -a_0)$, $\ell(x_{4(p+1)}) = (1, a_0)$, $\ell(x_{5(p+1)}) = (-1, -a_0)$.
For $k \geq 5$ let:

$\ell(x_0) = (0, a_0)$, $\ell(x_{2(p+1)}) = (2, a_0)$, $\ell(x_{4(p+1)}) = (4, a_0)$,..., $\ell(x_{2i(p+1)}) = (2i, a_0)$,..., $\ell(x_{(k-3)(p+1)}) = (k-3, a_0)$, $\ell(x_{(k-1)(p+1)}) = (k-1, a_0)$, $\ell(x_{(k+1)(p+1)}) = (k-2, a_0)$,

$\ell(x_{(k+3)(p+1)}) = (k-4, a_0)$, $\ell(x_{(k+5)(p+1)}) = (k-6, a_0)$,…,$\ell(x_{(2k-4)(p+1)}) = 3$, $\ell(x_{(2k-2)(p+1)}) = 1$, and

$\ell(x_{p+1}) = (-2, -a_0)$, $\ell(x_{3(p+1)}) = (-4, -a_0)$, $\ell(x_{5(p+1)}) = (-6, -a_0)$, …, $\ell(x_{(2i+1)(p+1)}) = (-2i-2, -a_0)$,…, $\ell(x_{(k-4)(p+1)}) = (-k+3, -a_0)$, $\ell(x_{(k-2)(p+1)}) = (-k+1, -a_0)$ $\ell(x_{k(p+1)}) = (-k, -a_0)$, $\ell(x_{(k+2)(p+1)}) = (-k+2, -a_0)$ …, $\ell(x_{(2k-3)(p+1)}) = (-3, -a_0)$, $\ell(x_{(2k-1)(p+1)}) = (-1, -a_0)$.

If a subscript m belongs to a coset $X + j$, then denote it by m_j. Notice that a vertex x_{m_j} belongs to X_j if the vertex x_{m_j+p} belongs to X_{j-1}. The vertices in $X_1, X_2, X_3 \ldots, X_{h-1}$ will be labeled recursively as follows.

$$\ell(x_{m_j}) = \begin{cases} (l_1(x_{m_j+p}) + k, a_0) & \text{if} \quad m_j + jp \equiv 0 \pmod{2p+2} \\ (l_1(x_{m_j+p}) - k, -a_0) & \text{if} \quad m_j + jp \not\equiv 0 \pmod{2p+2}. \end{cases}$$

Notice that a vertex x_{m_j} belongs to X_j if the vertex x_{m_j+hp} belongs to X_{j-h}. And the vertices in $X_h, X_{h+1}, \ldots, X_{d-1}$ will be labeled recursively as follows.

$$\ell(x_{m_j}) = \begin{cases} (l_1(x_{m_j+hp}), a_{\lfloor j/h \rfloor}) & \text{if} \quad l_1(x_{m_j+hp}) < \frac{\alpha}{2} \\ (l_1(x_{m_j+hp}), -a_{\lfloor j/h \rfloor}) & \text{if} \quad l_1(x_{m_j+hp}) > \frac{\alpha}{2}. \end{cases}$$

Obviously l is a bijection, and observe that if $k = 1$, then $\ell(x_i) + \ell(x_{i+p+1}) = (-1, 0)$ for any i, whereas for $k > 1$ since $p \equiv 0 \pmod{k}$:

$$\begin{aligned}
&\ell(x_{ik+0}) + \ell(x_{p+ik+1}) = (-2, 0),\\
&\ell(x_{ik+1}) + \ell(x_{p+ik+2}) = (0, 0),\\
&\ell(x_{ik+2}) + \ell(x_{p+ik+3}) = (-2, 0),\\
&\ell(x_{ik+3}) + \ell(x_{p+ik+4}) = (0, 0),\\
&\vdots\\
&\ell(x_{(i+1)k-3}) + \ell(x_{p+(i+1)k-2}) = (-2, 0),\\
&\ell(x_{(i+1)k-2}) + \ell(x_{p+(i+1)k-1}) = (0, 0),\\
&\ell(x_{(i+1)k-1}) + \ell(x_{p+(i+1)k}) = (-1, 0),
\end{aligned}$$

for $i = 0, 1, \ldots, p/k - 1$.

Furthermore, because $N(x_i) = \{x_{i-p}, x_{i-p+1}, \ldots, x_{i-1}, x_{i+1}, x_{i+2}, \ldots, x_{i+p}\}$ and p/k is odd, we obtain $w(x_i) = (-p, 0)$ for any i. ■

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