AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **57** (2013), Pages 139–143

A Fan-type degree condition for k-linked graphs

Ruijuan Li*

Institute of Mathematics and Applied Mathematics
Shanxi University, 030006 Taiyuan
P.R. China
ruijuanli@sxu.edu.cn

Michael Ferrara†

Department of Mathematical and Statistical Sciences
University of Colorado Denver
Denver, CO 80217
U.S.A.
michael.ferrara@ucdenver.edu

Xinhong Zhang

Department of Applied Mathematics
Taiyuan University of Science and Technology
030024 Taiyuan
P.R. China

Shengjia Li‡

Institute of Mathematics and Applied Mathematics
Shanxi University, 030006 Taiyuan
P.R. China

Abstract

A graph G is *k-linked* if for any $2k$ vertices $s_1 \ldots, s_k, t_1, \ldots, t_k$ in G, there exist disjoint paths P_i such that P_i is an $s_i - t_i$ path for $1 \leq i \leq k$. Motivated by work of G. Fan, let $\sigma_2^*(G)$ denote the minimum degree sum of vertices at distance two in G. In this note, we prove that a graph G

* Research supported by the Mathematics Tianyuan Foundation of China under no. 11026162, SRF for ROCS, SEM and Technology Foundation for Selected Overseas Chinese Scholar, Department of Human Resources and Social Security of Shanxi Province.

† Research supported in part by Simons Foundation Grant #206692.

‡ Research partially supported by NNSFC under no. 61174082 and Shanxi Scholarship Council of China under no. 2011-006.

of order $n \geq 232k$ with $\sigma_2^*(G) \geq n+2k-3$ is k-linked. For n sufficiently large, this implies a result of Kawarabayashi et al. that gives Ore-type degree conditions for k-linkedness.

1 Introduction

All graphs in this paper are simple and finite. We let $N(v)$ denote the neighborhood of a vertex v, and let $d(v)$ denote the degree of v. If X is a set of vertices in a graph G, we will often simply write X for the induced subgraph $G[X]$ if the context is clear. Further, we let $N_X(v) = N(v) \cap X$ and $d_X(v) = |N_X(v)|$, and we also let $N(X)$ denote $\bigcup_{v \in X} N(v)$. The distance between vertices u and v in a graph is denoted $\mathrm{dist}(u, v)$.

A graph G is *k-linked* if for any $2k$ vertices $s_1 \ldots, s_k, t_1, \ldots, t_k$ in G, there exist disjoint paths P_i such that P_i is an $s_i - t_i$ path for $1 \leq i \leq k$. In [3], Kawarabayashi, Kostochka and Yu gave Ore-type degree-sum conditions that ensure a graph G of order at least $2k$ is k-linked. Let $\sigma_2(G)$ denote the minimum degree sum of a pair of nonadjacent vertices in G.

Theorem 1. *Let G be a graph on $n \geq 2k$ vertices. If*

$$\sigma_2(G) \geq \begin{cases} n+2k-3, & n \geq 4k-1 \\ \frac{2(n+5k)}{3} - 3, & 3k \leq n \leq 4k-2 \\ 2n-3, & 2k \leq n \leq 3k-1 \end{cases}$$

then G is k-linked. These bounds are best possible.

In this note, inspired by a result of G. Fan [2] for hamiltonian graphs, we are interested in studying degree conditions for k-linkedness restricted to pairs of vertices at distance two in G. Specifically, let

$$\sigma_2^*(G) = \min\{d(u) + d(v) \mid \mathrm{dist}(u, v) = 2\}$$

if G is not complete and $\sigma_2^*(G) = \infty$ if G is complete. Our main result is as follows.

Theorem 2. *Let $k \geq 1$ and let G be a graph of order $n \geq 232k$. If $\sigma_2^*(G) \geq n+2k-3$, then G is k-linked.*

We note here that Theorem 2 implies Theorem 1 for sufficiently large n. The converse, however, does not hold. Indeed, construct the graph G from $K_{m_1} \cup K_{m_2} \cup K_{m_3} \cup K_{m_4}$ with $\sum m_i = n$, where K_t denotes the complete graph of order t, by adding all edges between K_{m_i} and $K_{m_{i+1}}$ for $i = 1, 2, 3$. If $m_2 = 2k-2$, then $\sigma_2^*(G) = n+2k-4$, but G is not k-linked since if we choose $s_1, t_1, \ldots, s_{k-1}$ and t_{k-1} to be the vertices in K_{m_2}, then there is no path from K_{m_1} to K_{m_3} avoiding these vertices. Furthermore, if $m_i \geq 2k-1$ for $1 \leq i \leq 4$, then $\sigma_2^*(G) \geq n+2k-3$, so Theorem 2 implies that G is k-linked. However, $\sigma_2(G) = n-2$ in this case, so Theorem 1 does not allow us to draw the same conclusion.

2 Proof of Theorem 2

Prior to proving Theorem 2, we require several known results and lemmas.

Theorem 3 (Mader 1972 [4])**.** *If G is a graph of order n with at least $2kn$ edges, then G contains a k-connected subgraph.*

One of the central questions in the area of graph linkedness is to determine the minimum $f(k)$ such that every $f(k)$-connected graph is k-linked. The following result implies that $f(k) \leq 10k$, which currently represents the best progress towards determining $f(k)$ in general.

Theorem 4 (Thomas and Wollan 2005 [6])**.** *If G is a $2k$-connected graph with at least $5kn$ edges, then G is k-linked.*

The next two lemmas appear in several places throughout the literature on k-linked graphs.

Lemma 1 (cf. Chen, Gould and Pfender [1])**.** *If G is a $2k$-connected graph that contains a k-linked subgraph H, then G is k-linked.*

Lemma 2 (cf. Manoussakis [5])**.** *Let G be a graph and v be a vertex in G with $d(v) \geq 2k-1$. If $G - v$ is k-linked, then G is k-linked.*

Our final lemma is a straightforward analogue to the σ_2-threshold for k-connectedness.

Lemma 3. *If $p \geq 1$ and G is a graph of order n with $\sigma_2^*(G) \geq n + p - 2$, then G is p-connected and in particular has minimum degree at least p.*

Proof. Suppose that $\sigma_2^*(G) \geq n + p - 2$, but that G is not p-connected. Choose a minimum cutset S of G, so that $|S| \leq p - 1$, and note that every vertex in S has a neighbor in each component of $G - S$. Thus, there exist components X and Y of $G - S$ containing vertices x and y, respectively, such that $\text{dist}(x, y) = 2$. However, we then have that

$$d(x) + d(y) \leq (|X| - 1) + (|Y| - 1) + 2|S| \leq n + p - 3,$$

a contradiction. □

We are now ready to prove Theorem 2.

Proof: Assume that G is as given, but is not k-linked. Let S be a minimum cutset of G, so that by Theorem 4 and Lemma 3 we have that $2k-1 \leq |S| < 10k$. As $n \geq 232k$ and $\sigma_2^*(G) \geq n+2k-3$, we also have that $G-S$ has exactly two components; call them A and B, and assume without loss of generality that $|A| \leq |B|$.

First assume that $|S| = 2k - 1$, and for some $s \in S$, let a and b be vertices in $N_A(s)$ and $N_B(s)$, respectively. We have that

$$n + 2k - 3 \leq d(a) + d(b) \leq (|A| - 1) + (|B| - 1) + 2|S| = n + 2k - 3. \qquad (1)$$

Consequently, $N(a) = (A \cup S) - \{a\}$ and $N(b) = (B \cup S) - \{b\}$ for every such choice of a and b. Let $X_A = N_A(S)$ and $X_B = N_B(S)$. As S is a minimum cutset, each $s \in S$ must have neighbors in both A and B, so X_A and X_B are both necessarily complete. Furthermore, for $x_a \in X_A$ and $y \in (A \cup S) - X_A$ (respectively $x_b \in X_B$ and $y' \in (B \cup S) - X_B$), $x_a y$ (resp. $x_b y'$) is an edge in G.

We now wish to apply Lemma 2 to G. If $X_A = A$, then each vertex in A is adjacent to each vertex in S and, if $X_A \neq A$, then $|X_A| \geq 2k-1$, lest G is not $(2k-1)$-connected. In either event we may iteratively delete all vertices in $A - X_A$, followed by all vertices in A. Now, as $|B| \geq \frac{n-2k+1}{2}$ and $n \geq 232k$, we have that $|X_B| \geq 2k-1$ (as again, otherwise G is not $(2k-1)$-connected). Thus, every vertex in $S \cup B$ is adjacent to every vertex in X_B, so we may iteratively delete vertices in $(B - X_B) \cup S$ until we obtain a complete graph of order $2k$ (comprised of X_B and any vertex in S). As each deleted vertex had degree at least $2k-1$ at the time of its deletion and K_{2k} is k-linked, G is k-linked by Lemma 2.

Thus, we may assume that $2k \leq |S| < 10k$, so that in particular G is $2k$-connected. We consider two cases.

Case 1: A is not complete.

Observe that $\sigma_2^*(A) \geq (n + 2k - 3) - 2|S| \geq n - 18k - 1$. As $n \geq 232k$ and $|A| \leq \frac{n-2k+1}{2}$, we have that $\sigma_2^*(A) \geq |A| + 40k$, so that $\delta(A) \geq 40k$. Theorem 3 therefore implies that A, and hence G, contains a $10k$-connected subgraph so that G is k-linked by Lemma 1.

Case 2: A is complete.

In this case, Lemma 1 implies that $|A| < 2k$, so that for any vertex $a \in A$, $d(a) < |S| + 2k < 12k$. Let $X_b = N_B(S)$ and note that the minimality of S implies that for every $x_b \in X_b$ there is some vertex $s \in S$ and $a \in A$ such that asx_b is an induced P_3. Consequently, as $d(a) < 12k$, it follows that $d(x_b) \geq n - 10k - 2$.

We next claim that if $|X_b| > 11k$, then $B \cup S$ is k-linked. If so,

$$|E(B \cup S)| \geq \frac{1}{2}|X_b|(n - 10k - 2) > \frac{11}{2}k(n - 10k) > 5kn,$$

since $n > 32k + 4$. As S is a minimum cutset of G and G is $2k$-connected, it follows that $B \cup S$ is also $2k$-connected. This implies that $B \cup S$ is k-linked by Theorem 4 and consequently G is k-linked by Lemma 1. Thus, we will assume going forward that $|X_b| \leq 11k$, which implies that every vertex in S has degree at most $|A| + |S| - 1 + |X_b| < 23k$.

As $n \geq 232k$ and $|B| > \frac{n-10k}{2}$ we know that $B - X_b$ is nonempty, so let $X_b' = N_{B-X_b}(X_b)$. Every vertex in X_b' is distance two from some vertex in S, so for any vertex $x_b' \in X_b'$ we therefore have that $d(x_b') \geq n - 21k - 3$. As above, we claim that if $|X_b'| > 11k$, then G is k-linked. Indeed, if so then

$$|E(B \cup S)| \geq \frac{1}{2}|X_b'|(n - 21k - 3) > \frac{11}{2}k(n - 21k) \geq 5kn,$$

since $n \geq 232k$. We will therefore assume that $|X_b'| < 11k$, so that each vertex x_b in X_b has degree at most $|S| + |X_b| + |X_b'| < 32k$.

To complete the proof, choose any vertices $a \in A$ and $x_b \in X_b$ that are at distance two in G. As $d(a) < 12k$ and $d(x_b) < 32k$ we have that $n+2k-3 \leq d(a)+d(x_b) < 44k$, a contradiction to the assumption that $n \geq 232k$. $\square$

References

[1] G. Chen, R. Gould and F. Pfender, New conditions for k-ordered hamiltonian graphs, *Ars Combin.* **70** (2004), 245–255.

[2] G. Fan, New sufficient conditions for cycles in graphs, *J. Combin. Theory, Ser. B* **37** (1984), 221–227.

[3] K. Kawarabayashi, A. Kostochka and G. Yu, On sufficient degree conditions for a graph to be k-linked, *Combin. Probab. Comput.* **15** (2006), 685–694.

[4] W. Mader, Existenz n-fach zusammenhängender Teilgraphen in Graphen genügend grosser Kantendichte, *Abh. Math. Sem. Univ. Hamburg* **37** (1972), 86–97.

[5] Y. Manoussakis, k-linked and k-cyclic digraphs, *J. Combin. Theory, Ser. B* **48** (1990), 216–226.

[6] R. Thomas and P. Wollan, An improved linear edge bound for graph linkages, *European J. Combin.* **26** (2005), 309–324.

(Received 8 June 2012; revised 13 May 2013)