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Super (a, d)-H-antimagic total labelings for shackles of a connected graph H

N. Inayah* R. Simanjuntak A.N.M. Salman

Combinatorial Mathematics Research Group
Faculty of Mathematics and Natural Sciences
Institut Teknologi Bandung
Jalan Ganesa 10 Bandung 40132
Indonesia
inayahmath@gmail.com rino@math.itb.ac.id msalman@math.itb.ac.id

K.I.A. Syuhada

Statistics Research Group
Faculty of Mathematics and Natural Sciences
Institut Teknologi Bandung
Jalan Ganesa 10 Bandung 40132
Indonesia
khreshna@math.itb.ac.id

Abstract

A simple graph $G = (V(G), E(G))$ admits an *H-covering*, if every edge in $E(G)$ belongs to at least one subgraph of G isomorphic to a given graph H. An (a, d)-*H-antimagic total labeling* of G admitting an H-covering is a bijective function $\xi : V(G) \cup E(G) \rightarrow \{1, 2, \ldots, |V(G)| + |E(G)|\}$ such that for all subgraphs $H^{'}$ isomorphic to H, the H-weights $w(H') = \sum_{v \epsilon V(H')} \xi(v) + \sum_{e \epsilon E(H')} \xi(e)$ constitute an arithmetic progression $a, a + d, a + 2d, \ldots, a + (k-1)d$ where a and d are positive integers and k is the number of subgraphs of G isomorphic to H. Such a labeling is called *super* if the smallest possible labels appear on the vertices.

This paper is devoted to studying super (a, d)-*H-antimagic total labelings* for some shackles of a connected graph H. A *shackle* of $G_1, G_2, \ldots,$ G_k, denoted by shack$(G_1, G_2, \ldots, G_k)$, is a graph constructed from non-trivial connected and ordered graphs $G_1, G_2, \ldots, G_k$ such that for every $1 \leq i, j \leq k$ with $|i - j| \geq 2$, G_i and G_j have no common vertex, and

* Permanent address: Program Studi Matematika, FST, Universitas Islam Negeri (UIN) Syarif Hidayatullah Jakarta, Jl. Ir. H. Djuanda 95, Ciputat 15412, Indonesia.

for every $1 \leq i \leq k-1$, G_i and G_{i+1} share exactly one common vertex, called a *linkage vertex*, where the $k-1$ linkage vertices are all distinct. In the case when all G_i's are isomorphic to a connected graph H, we call the resulting graph a shackle of H, denoted by shack(H,k). A technique of partitioning a multiset is introduced and an upper bound for d is obtained.

1 Introduction

We consider finite and simple graphs. Let the vertex and edge sets of a graph G be denoted by $V(G)$ and $E(G)$, respectively. An edge-covering of G is a family of different subgraphs $H_1, H_2, \ldots, H_k$ such that each edge of $E(G)$ belongs to at least one of the subgraphs H_j, $1 \leq j \leq k$. Then it is said that G admits an $(H_1, H_2, \ldots, H_k)$-(edge)covering. If every H_j is isomorphic to a given graph H, then G admits an H-covering. Suppose that G admits an H-covering. Gutiérrez and Lladó [4] defined an H-magic labeling which is a generalization of Kotzig and Rosa's edge-magic total labeling [6]. The graph G is called H-magic if there exists a total labeling $\xi : V(G) \cup E(G) \rightarrow \{1, 2, \ldots, |V(G)| + |E(G)|\}$ such that, for each subgraph H' isomorphic to H, $\sum_{v \epsilon V(H')} \xi(v) + \sum_{e \epsilon E(H')} \xi(e)$ is constant. In this case, we say that G is *H-magic*. When $\xi(V(G))$ $=\{1, 2, \ldots, |V(G)|\}$, we say that G is *H-supermagic*.

Lladó and Moragas [7] proved that wheels, prisms, books, and windmills are *cycle-magic*. Maryati et al. [8] and Salman et al. [12] proved that some families of trees are *path-supermagic*. Recently, Ngurah et al. [11] proved that chains, wheels, triangles, ladders, and grids are cycle-supermagic labelings.

For any connected graph H, Maryati et al. [9] have shown that for $k \geq 2$ and H an arbitrary graph, if shack(H,k) contains exactly k subgraphs isomorphic to H, then shack(H,k) is *H-supermagic*. A *shackle* of $G_1, G_2, \ldots, G_k$, denoted by shack$(G_1, G_2, \ldots, G_k)$, is any graph constructed from non-trivial connected and ordered graphs $G_1, G_2, \ldots, G_k$ such that for every $1 \leq i, j \leq k$ with $|i-j| \geq 2$, G_i and G_j have no common vertex, and for every $1 \leq i \leq k-1$, G_i and G_{i+1} share exactly one common vertex, called a *linkage vertex*, where the $k-1$ linkage vertices are all distinct. In the case when all G_i's are isomorphic to a connected graph H, we call any resulting graph a shackle of H, denoted by shack(H,k).

Meanwhile, Simanjuntak et al. [13] introduced an *(a,d)-edge-antimagic total labeling* of G which is defined as a bijective function $f : V(G) \cup E(G) \rightarrow \{1, 2, \ldots, |V(G)| + |E(G)|\}$ so that the set of edge-weights $\{f(u) + f(uv) + f(v) \mid uv \in E(G)\}$ is equal to the set $\{a, a+d, a+2d, \ldots, a+(|E(G)|-1)d\}$ for some positive integers a and d. An (a,d)-edge-antimagic total labeling f is called *super* if the vertex labels are the smallest possible labels. Several results related to edge-antimagic total labelings are provided; see for example [1], [2], [3], and [10].

Combining the two previous labelings, Inayah et al. [5] introduced the (a,d)-H-antimagic total labeling. An (a,d)-H-antimagic total labeling of a graph G admitting an H-covering is a bijective function $\xi : V(G) \cup E(G) \to \{1, 2, \ldots, |V(G)| + |E(G)|\}$ such that for all subgraphs H' isomorphic to H, the H-weights $w(H') = \sum_{v \epsilon V(H')} \xi(v) + \sum_{e \epsilon E(H')} \xi(e)$ constitute an arithmetic progression $a, a+d, a+2d, \ldots, a+(k-1)d$ where a and d are positive integers and k is the number of subgraphs of G isomorphic to H. In this case we say that G is (a,d)-H-antimagic. Such a labeling is called *super* if the smallest possible labels appear on the vertices and G is super (a,d)-H-antimagic.

In this paper, we present super (a,d)-H-antimagic total labelings of certain connected graphs obtained from k isomorphic copies of a graph H, namely, a shackle. In Section 2 we introduce a technique of partitioning a multiset. We will use the following notation. Let X, Y be two multisets. Then the multiset $X \biguplus Y = \{\alpha : \alpha \in X; \alpha \in Y\}$. Notice that if α appears r times in X and s times in Y, $r, s \in \{0, 1, 2, \ldots\}$, then α appears $r+s$ times in $X \biguplus Y$.

In Section 3 we show that the shackle is a super (a,d)-H-antimagic total labeling for some integers d.

2 (k, δ)-anti balanced

Let $k \in \mathbb{N}$ and let X be a multiset containing positive integers. Then X is said to be (k,δ)-*anti balanced* if there exist k subsets of X, say $X_1, X_2, X_3, \ldots, X_k$, such that for every $i \in [1,k]$, $|X_i| = \frac{|X|}{k}$, $\biguplus_{i=1}^{k} X_i = X$, and for $i \in [1, k-1]$, $\sum X_{i+1} - \sum X_i = \delta$ is satisfied. Next, we give several lemmas about (k,δ)-anti balanced multisets.

Lemma 1 *Let k be an integer such that $k \geq 2$. If*

$$X = \begin{cases} [1, k+1] \ \biguplus \ [2, k], & \text{if } k \text{ is odd}, \\ [1, \frac{k}{2}] \ \biguplus \ [\frac{k}{2}+2, k+1] \ \biguplus \ [2, k+1], & \text{if } k \text{ is even}, \end{cases}$$

then X is $(k,1)$-anti balanced.

Proof. For $i \in [1,k]$, define $X_i = \{\lceil \frac{i+1}{2} \rceil, \lfloor \frac{k}{2} \rfloor + \lceil \frac{i}{2} \rceil + 1\}$. It is easy to verify that for each $i \in [1,k]$, $|X_i| = 2$, $X_i \subset X$, and $\biguplus_{i=1}^{k} X_i = X$. Since $\sum X_i = i + 2 + \lfloor \frac{k}{2} \rfloor$ for every $i \in [1,k]$, and $\sum X_{i+1} - \sum X_i = 1$ for every $i \in [1, k-1]$, X is $(k,1)$-anti balanced. □

Lemma 2 *Let k be an integer such that $k \geq 2$. If $X = [1,k] \biguplus [2, k+1]$, then X is $(k,2)$-anti balanced.*

Proof. For $i \in [1,k]$, define $X_i = \{i, i+1\}$. It is easy to verify that for each $i \in [1,k]$, $|X_i| = 2$, $X_i \subset X$, and $\biguplus_{i=1}^{k} X_i = X$. Since $\sum X_i = 1 + 2i$ for every $i \in [1,k]$, and $\sum X_{i+1} - \sum X_i = 2$ for every $i \in [1, k-1]$, X is $(k,2)$-anti balanced. □

Lemma 3 *Let n, k be integers with $n, k \geq 2$. If $X = \{1 + (j_1 - 1)(n-1),\ j_1 = 1, 2, \ldots, k\} \biguplus \{n + (j_2 - 1)(n-1),\ j_2 = 1, 2, \ldots, k\}$, then X is $(k, 2n-2)$-anti balanced.*

Proof. For $i \in [1,k]$, define $X_i = \{2-n+(n-1)i, 1+(n-1)i\}$. It is easy to verify that for each $i \in [1,k]$, $|X_i| = 2$, $X_i \subset X$, and $\biguplus_{i=1}^{k} X_i = X$. Since $\sum X_i = 3-n+2(n-1)i$ for every $i \in [1,k]$, and $\sum X_{i+1} - \sum X_i = 2(n-1)$ for every $i \in [1,k]$, X is $(k, 2n-2)$-anti balanced. □

Lemma 4 *Let n,k be integers and $n,k \geq 2$. If $X = [k+2, (n-1)k+1]$, then X is $(k, n-2)$-anti balanced.*

Proof. For $i \in [1,k]$, define $X_i = \{kj+i+1 \mid 1 \leq j \leq n-2\}$. It is easy to verify that for each $i \in [1,k]$, $|X_i| = n-2$, $X_i \subset X$, and $\biguplus_{i=1}^{k} X_i = X$. Since $\sum X_i = (n-2) + \frac{1}{2}(n-1)(n-2)k + (n-2)i$ for every $i \in [1,k]$, and $\sum X_{i+1} - \sum X_i = n-2$ for every $i \in [1,k-1]$, X is $(k, n-2)$-anti balanced. □

Lemma 5 *Let n,k be integers and $n,k \geq 2$. If $X = \{(e+1)+(j-1)(n-1),\ e = 1,2,3,\ldots,n-2,\ j = 1,2,3,\ldots,k\}$, then X is $(k,(n^2-3n+2))$-anti balanced.*

Proof. For $i \in [1,k]$, define $X_i = \{(n-1)i+2-n+j \mid 1 \leq j \leq n-2\}$. It is easy to verify that for each $i \in [1,k]$, $|X_i| = n-2$, $X_i \subset X$, and $\biguplus_{i=1}^{k} X_i = X$. Since $\sum X_i = \frac{1}{2}n(5-n) - 3 + (n^2-3n+2)i$ for every $i \in [1,k]$, and $\sum X_{i+1} - \sum X_i = (n^2-3n+2)$ for every $i \in [1,k]$, X is $(k,(n^2-3n+2))$-anti balanced. □

3 Super (a,d)-H-antimagic total labelings for some shackles

Let H be a connected graph with $|V(H)| = n$ and $|E(H)| = m$. Then $G \cong \text{shack}(H,k)$ has $|V(G)| = (n-1)k+1$ and $|E(G)| = mk$. We derive an upper-bound of the difference d for G to be super (a,d)-H-antimagic total.

Lemma 6 *If G is super (a,d)-H-antimagic then*

$$d \ \leq\ m^2 + n^2 - n.$$

Proof. Let $|V(G)| = v_G$, $|E(G)| = e_G$, $|V(H)| = v_H$, and $|E(H)| = e_H$. Since G is super (a,d)-H-antimagic, the maximum possible H-weight is $v_G + (v_G-1) + (v_G-2) + \cdots + (v_G - v_H + 1) + (v_G + e_G) + (v_G + e_G - 1) + \cdots + (v_G + e_G - e_H + 1)$, or

$$a + (k-1)d \leq n[(n-1)k+1] - \frac{n(n-1)}{2} + m[(m+n-1)k+1] - \frac{m(m-1)}{2}. \tag{3.1}$$

On the other hand, the least possible H-weight is $1 + 2 + \cdots + v_H + (v_G+1) + (v_G+2) + \cdots + (v_G + e_H)$ or

$$a \geq \frac{n(n+1)}{2} + m[(n-1)k+1] + \frac{m(m+1)}{2}. \tag{3.2}$$

From (3.1) and (3.2), we obtain

$$\begin{aligned}(k-1)d &\leq n[(n-1)k+1] - \frac{n(n-1)}{2} + m[(m+n-1)k+1] - \frac{m(m-1)}{2} \\ &\quad - \frac{n(n+1)}{2} - m[(n-1)k+1] - \frac{m(m+1)}{2}, \\ d &\leq m^2 + n^2 - n.\end{aligned}$$

□

In the next theorem, we consider super (a, d)-H-antimagic total labelings for shackles of a connected graph H.

Theorem 1 *Let H be a non-trivial connected graph and let k be an integer, $k \geq 2$. If* shack(H, k) *contains exactly k subgraphs isomorphic to H, then* shack(H, k) *is super (a, d)-H-antimagic for $1 \leq d \leq m + n$.*

Proof. In this proof we have four separate cases. In the first three cases, we define three sets $W = [1, k+1]$, $Y = [k+2, (n-1)k+1]$, and $Z = [(n-1)k+2, (m+n-1)k+1]$. Let $L = [2, k]$ and $X = W \uplus L$. By Lemma 2, X is $(k, 2)$-anti balanced. Let X_i be an anti balanced subset of X for every $i \in [1, k]$, as defined in the proof of Lemma 2 and let Y_i be an anti balanced subset of Y, for every $i \in [1, k]$, as defined in the proof of Lemma 4.

Case 1. $m + n$ is odd and $d = 1$

For $p = \frac{m-n+1}{2}$, define

$$\begin{aligned}P_i &= \{(n + j_1 - 2)k + i + 1,\ j_1 = 1, 2, 3, \cdots, m - p + 1\} \bigcup \\ &\quad \{(n + m - p + j_2)k + 2 - i,\ j_2 = 1, 2, 3, \cdots, p - 1\} \text{ for } i \in [1, k].\end{aligned}$$

It is easy to verify that for each $i \in [1, k]$,

$$\biguplus_{i=1}^{k} P_i = Z \ , \ P_i \cap P_j = \emptyset \text{ for } i \neq j, \text{ and}$$

$$\sum P_i = (mn + p - 1)k + m + p - 1 + \frac{1}{2}(m^2 - 3m)k + (m - 2p + 2)i.$$

Hence, $\sum P_1, \sum P_2, \sum P_3, \ldots, \sum P_k$ form an arithmetic progression with common difference $(m - 2p + 2)$.

Let $u_1, u_2, u_3, \ldots, u_{k-1}$ be the linkage vertices of shack(H, k). Now, we define a total labeling f_1 of shack(H, k) as follows.

- For every $i \in [1, k-1]$, label the linkage vertex u_i with $k + 1 - i \in X_i$.

- Label any vertex other than the linkage vertices in H_1 and H_k with $k+1$ and 1, respectively.

- For every $i \in [1,k]$, label the remaining $n-2$ vertices of H_i with the elements of Y_{k+1-i}.

- For every $i \in [1,k]$, use the elements of P_i to label edges of H_i.

Under the labeling f_1, we find

$$\begin{aligned} f_1(H_i) &= f_1(V(H_i)) + f_1(E(H_i)) \\ &= \sum X_{k+1-i} + \sum Y_{k+1-i} + \sum P_i \\ &= (mn+p)k + \frac{1}{2}(m^2+n^2-3m-n)k + m + p + 2n - 2 + \\ &\quad (m-n-2p+2)i \end{aligned}$$

Since for $p = \frac{m-n+1}{2}$, the following is satisfied

$$f_1(H_i) = (mn-m-n)k + \frac{1}{2}(m^2+n^2+1)k + \frac{3}{2}(m+n-1) + i,$$

shack(H,k) admits an $(a,1)$-H-super antimagic total labeling with

$$a = (mn-m-n)k + \frac{1}{2}(m^2+n^2+1)k + \frac{3}{2}(m+n-1) + 1.$$

Case 2. $m+n$ is odd and $3 \le d \le m+n$ is odd.
For $p = \frac{m-n+1}{2}$, define
$Q_i = \{(n+j_1-2)k+i+1,\ j_1 = 1,2,3,\dots,m-p-\frac{d-1}{2}\}$
$\quad \cup\{(m+n-p-\frac{d-1}{2}+j_2)k+2-i,\ j_2 = 0,1,2,3,\dots,p+\frac{d-1}{2}-1\},\ i \in [1,k].$
It is easy to verify that for each $i \in [1,k]$,

$$\biguplus_{i=1}^{k} Q_i = Z\ ,\ Q_i \cap Q_j = \emptyset \text{ for } i \neq j,$$

and

$$\begin{aligned} \sum Q_i &= (mn+p+\frac{1}{2}m(m-3)+\frac{1}{2}(d-1))k + m + p + \frac{1}{2}(d-1) + \\ &\quad (m-2p-d+1)i. \end{aligned}$$

Hence, $\sum Q_1, \sum Q_2, \sum Q_3, \dots, \sum Q_k$ form an arithmetic progression with common difference $(m-2p-d+1)$.

Let $u_1, u_2, u_3, \dots, u_{k-1}$ be the linkage vertices of shack(H,k). Now, define a total labeling f_2 of shack(H,k) as follows.

- For every $i \in [1, k-1]$, label the linkage vertex u_i with $i+1 \in X_i$.

- Label any vertex other than the linkage vertices in H_1 and H_k with 1 and $k+1$, respectively.

- For every $i \in [1, k]$, label the remaining $n-2$ vertices of H_i with the elements of Y_i.

- For every $i \in [1, k]$, use the elements of Q_{k+1-i} to label edges of H_i.

Under the labeling f_2, we find

$$\begin{aligned} f_2(H_i) &= f_2(V(H_i)) + f_2(E(H_i)) \\ &= \sum X_i + \sum Y_i + \sum Q_{(k+1-i)} \\ &= (mn - p + \frac{1}{2}(m^2 - m - d + 1) + \frac{1}{2}(n-2)(n-1))k + 2m + n - p \\ &\quad -1 - \frac{1}{2}(d-1) + (2p + d + n - m - 1)i. \end{aligned}$$

Since for $p = \frac{m-n+1}{2}$, the following is satisfied

$$\begin{aligned} f_2(H_i) &= (mn - m - n + 1 + \frac{1}{2}(m^2 + n^2 - d))k + \\ &\quad \frac{3}{2}(m + n - 1) - \frac{1}{2}(d-1) + di, \end{aligned}$$

shack(H, k) admits an (a, d)-H-super antimagic total labeling with $a = (mn - m - n + 1 + \frac{1}{2}(m^2 + n^2 - d))k + \frac{3}{2}(m + n - 1) - \frac{1}{2}(d-1) + d$.

Next, we will show that $d \le m + n$. Since one of the edge labels in H_k is $(m + n - p - \frac{d-1}{2})k + 1$ and the smallest edge label in H_1 is $nk + 1$,

$$\begin{aligned} (m + n - p - \frac{d-1}{2})k + 1 &\ge nk + 1 \\ d &\le m + n. \end{aligned}$$

Case 3. $m + n$ is even and $d \le m + n$ is even.

For $r = \frac{m-n}{2}$, define

$$\begin{aligned} R_i = \{&(n + j_1 - 2)k + i + 1,\ j_1 = 1, 2, 3, \ldots, m - r - \tfrac{d}{2}\} \\ &\cup \{(n + m - r - \tfrac{d}{2} + j_2)k + 2 - i,\ j_2 = 0, 1, 2, 3, \ldots, r + \tfrac{d}{2} - 1\},\ i \in [1, k]. \end{aligned}$$

It is easy to verify that for each $i \in [1, k]$, we have

$$\biguplus_{i=1}^{k} R_i = Z,\ R_i \cap R_j = \emptyset \text{ for } i \ne j, \text{ and}$$

$$\sum R_i = (mn + r + \frac{1}{2}(m^2 - 3m + d))k + m + r + \frac{1}{2}d + (m - 2r - d)i.$$

Hence $\sum R_1, \sum R_2, \sum R_3, \ldots, \sum R_k$ form an arithmetic progression with common difference $(m - 2r - d)$.

Define a total labeling f_3 as follow.

- Label the vertices of shack(H, k) as in Case 2.

- For every $i \in [1, k]$, use the elements of R_{k+1-i} to label edges of H_i.

Under the labeling f_3, we find

$$\begin{aligned} f_3(H_i) &= f_3(V(H_i)) + f_3(E(H_i)) \\ &= \sum X_i + \sum Y_i + \sum R_{(k+1-i)} \\ &= (mn - r + \frac{1}{2}(m^2 - m - d) + \frac{1}{2}(n-2)(n-1))k + 2m + n - r - \\ &\quad \frac{1}{2}d - 1 + (2r + n - m + d)i. \end{aligned}$$

Since for $r = \frac{m-n}{2}$, we have

$$f_3(H_i) = (mn - m - n + 1 + \frac{1}{2}(m^2 + n^2 - d))k + \frac{1}{2}(3m + 3n - d - 2) + di,$$

shack(H, k) admits and (a, d)-H-super antimagic total labeling with $a = (mn - m - n + 1 + \frac{1}{2}(m^2 + n^2 - d))k + \frac{1}{2}(3m + 3n - d - 2) + d$.

Next, we will show that $d \leq m + n$. Since one of the edge labels in H_k is $(m + n - r - \frac{d}{2})k + 1$ and the smallest edge label in H_1 is $nk + 1$,

$$\begin{aligned} (m + n - r - \frac{s}{2})k + 1 &\geq nk + 1 \\ d &\leq m + n. \end{aligned}$$

Case 4. $m + n$ is even and $1 \leq d \leq m + n - 1$ is odd, or $m + n$ is odd and $1 \leq d \leq m + n - 1$ is even.

In this case, we define three sets $W = [1, k + 1]$, $Y = [k + 2, (n - 1)k + 1]$, and $Z = [(n - 1)k + 2, (m + n - 1)k + 1]$. Let $L = [2, k]$ and $X = W \uplus L$. By Lemma 1, we find that X is $(k, 1)$-anti balanced. For every $i \in [1, k]$, let X_i be an anti balanced subset of X as defined in the proof of Lemma 1 and let Y_i be an anti balanced subset of Y as defined in the proof of Lemma 4.

For $s = \frac{m+n-d-1}{2}$, define

$$\begin{aligned} S_i = &\{(n + j_1 - 2)k + i + 1,\ j_1 = 1, 2, 3, \cdots, m - s\} \\ &\cup \{(m + n - j_2)k + 2 - i,\ j_2 = 1, 2, 3, \cdots, s\},\ i \in [1, k]. \end{aligned}$$

It is easy to verify that for each $i \in [1, k]$,

$$\biguplus_{i=1}^{k} S_i = Z \ , \ S_i \cap S_j = \emptyset \text{ for } i \neq j, \text{ and}$$

$$\sum S_i = (mn + s + \frac{1}{2}m(m-3))k + m + s + (m-2s)i.$$

Hence $\sum S_1, \sum S_2, \sum S_3, \ldots, \sum S_k$ form an arithmetic progression with common difference $(m - 2s)$.

Now, we define a total labeling f_4 of shack(H, k) as follows.

- For every $i \in [1, k-1]$, label the linkage vertex u_i with $\lfloor \frac{k}{2} \rfloor + \lceil \frac{i}{2} \rceil + 1$ (for odd i) and $\lceil \frac{i+1}{2} \rceil$ (for even i). Note that the labels are in X_i.

- Label any vertex other than the linkage vertices in H_1 with 1.

- Label any vertex other than the linkage vertices in H_k with $k+1$ (for i is odd) and $\frac{k}{2} + 1$ (for i is even).

- For every $i \in [1, k]$, label the remaining $n - 2$ vertices of H_i with the elements of Y_i.

- For every $i \in [1, k]$, use the elements of S_i to label edges of H_i.

Under the labeling f_4, we find

$$\begin{aligned} f_4(H_i) &= f_4(V(H_i)) + f_4(E(H_i)) \\ &= \sum X_i + \sum Y_i + \sum S_i \\ &= \lfloor \frac{k}{2} \rfloor + m + n + (mn + s + \frac{1}{2}m(m-3))k + \frac{1}{2}(n-2)(n-1)k + s + di. \end{aligned}$$

Since for $s = \frac{m+n-d-1}{2}$, the following is satisfied

$$\begin{aligned} f_4(H_i) = {} & \lfloor \frac{k}{2} \rfloor + \frac{3}{2}(m+n) - \frac{1}{2}(d+1) + \\ & (mn + \frac{1}{2}(m+n) - \frac{1}{2}(d+1) + \frac{1}{2}m(m-3))k + \\ & \frac{1}{2}(n-2)(n-1)k + di, \end{aligned}$$

shack(H, k) admits an (a, d)-H-super antimagic total labeling with $a = \lfloor \frac{k}{2} \rfloor + \frac{3}{2}(m + n) - \frac{1}{2}(d+1) + (mn + \frac{1}{2}(m+n) - \frac{1}{2}(d+1) + \frac{1}{2}m(m-3))k + \frac{1}{2}(n-2)(n-1)k + d$. $\square$

In the next two theorems, we define three sets

$$\begin{aligned} W &= \{1, 1 + (n-1), 1 + (n-1)2, 1 + (n-1)3, \ldots, n + (k-1)(n-1)\}, \\ Y &= \{(e+1) + (j-1)(n-1), e = 1, 2, 3, \ldots, n-2, j = 1, 2, 3, \ldots, k\}, \text{ and} \\ Z &= [(n-1)k + 2, (m+n-1)k + 1]. \end{aligned}$$

Let $L = \{n, n+(n-1), n+2(n-1), \ldots, 1+(k-1)(n-1)\}$ and $X = W \uplus L$. By Lemma 3, we find that X is $(k, 2(n-1))$-anti balanced. For every $i \in [1,k]$, let X_i be an anti balanced subset of X as defined in the proof of Lemma 3. By Lemma 5, we obtain the fact that Y is $(k, (n^2-3n+2))$-anti balanced. For every $i \in [1,k]$, let Y_i be an anti balanced subset of Y as defined in the proof of Lemma 5.

Theorem 2 *Let H be a non-trivial connected graph, and let k, p and m be positive integers satisfying $k \geq 2$ and $p \leq m$. If* $\text{shack}(H,k)$ *contains exactly k subgraphs isomorphic to H, then* $\text{shack}(H,k)$ *is super (a,d)-H-antimagic, where $d = n^2 - n + m - 2p$.*

Proof. For $i \in [1,k]$, define

$$\begin{aligned} Z_i = &\{(n+j_1-2)k+i+1, j_1 = 1,2,3,\cdots, m-p\} \\ &\cup \{(n+j_2-1)k+2-i, j_2 = m-p+1, \cdots, m\},\ i \in [1,k]. \end{aligned}$$

It is easy to verify that for each $i \in [1,k]$, we have

$$\biguplus_{i=1}^{k} Z_i = Z\ ,\ Z_i \cap Z_j = \emptyset \text{ for } i \neq j,$$

and

$$\sum Z_i = (mn+p)k + \frac{1}{2}(m^2-3m)k + m + p + (m-2p)i.$$

Hence, $\sum Z_1, \sum Z_2, \sum Z_3, \ldots, \sum Z_k$ form an arithmetic progression with common difference $(m-2p)$.

Let $u_1, u_2, u_3, \ldots, u_{k-1}$ be the linkage vertices of $\text{shack}(H,k)$. Now, define a total labeling f_5 of $\text{shack}(H,k)$ as follows.

- For every $i \in [1,k-1]$, label the linkage vertex u_i with $2i+1 \in X_i$.

- Label any vertex other than linkage vertices in H_1 and H_k with 1 and $(n-1)k+1$, respectively.

- For every $i \in [1,k]$, label the remaining $n-2$ vertices of H_i with the elements of Y_i.

- For every $i \in [1,k]$, use the elements of Z_i to label edges of H_i.

Under the labeling f_5, we find

$$\begin{aligned} f_5(H_i) &= f_5(V(H_i)) + f_5(E(H_i)) \\ &= \sum X_i + \sum Y_i + \sum Z_i \\ &= (mn+p)k + \frac{1}{2}m(m-3)k + \frac{1}{2}(3n-n^2) + m + p + (n^2-n+m-2p)i. \end{aligned}$$

Hence $\text{shack}(H,k)$ admits an (a,d)-H-super antimagic total labeling with $a = (mn+p)k + \frac{1}{2}m(m-3)k + \frac{1}{2}(n^2+n) + 2m - p$ and $d = n^2 - n + m - 2p$. □

Theorem 3 *Let H be a non-trivial connected graph, let k be an integer, $k \geq 2$, and $0 \leq p \leq m$. If* $\text{shack}(H, k)$ *contains exactly k subgraphs isomorphic to H, then* $\text{shack}(H, k)$ *is super (a, d)-H-antimagic, where $d = m^2 + n^2 - n - 2mp$.*

Proof. For $i \in [1, k]$, define

$$\begin{aligned} Z_i &= \{(n-1)k + (i-1)m + 1 + j_1, j_1 = 1, 2, 3, 4, \cdots, m + 1 - p\} \\ &\quad \cup \{(n-1)k + (k+2-i)m + j_2, j_2 = m + 2 - p, \cdots, m + 1\}. \end{aligned}$$

It is easy to verify that for each $i \in [1, k]$,

$$\biguplus_{i=1}^{k} Z_i = Z \ , \ Z_i \cap Z_j = \emptyset \text{ for } i \neq j,$$

and

$$\sum Z_i = (n + p - 1)mk + mp + \frac{1}{2}(3m - m^2) + (m^2 - 2mp)i.$$

Hence $\sum Z_1, \sum Z_2, \sum Z_3, \ldots, \sum Z_k$ form an arithmetic progression with common difference $m^2 - 2mp$.

Let $u_1, u_2, u_3, \ldots, u_{k-1}$ be the linkage vertices of $\text{shack}(H, k)$. Now, define a total labeling f_6 of $\text{shack}(H, k)$ as follows.

- For every $i \in [1, k-1]$, label the linkage vertex u_i with $ni - i + 1 \in X_i$.

- Label any vertex other than the linkage vertices in H_1 and H_k with 1 and $(n-1)k+1$, respectively.

- For every $i \in [1, k]$, label the remaining $n-2$ vertices of H_i with the elements of Y_i.

- For every $i \in [1, k]$, use the elements of Z_i to label edges of H_i.

Under the labeling f_6, we find

$$\begin{aligned} f_6(H_i) &= f_6(V(H_i)) + f_6(E(H_i)) \\ &= \sum X_i + \sum Y_i + \sum Z_i \\ &= (n + p - 1)mk + \frac{1}{2}(3m - m^2) + \frac{1}{2}(3n - n^2) + mp + \\ &\quad (m^2 + n^2 - n - 2mp)i. \end{aligned}$$

Hence $\text{shack}(H, k)$ admits an (a, d)-H-super antimagic total labeling with $a = (n + p - 1)mk + \frac{1}{2}(3m + m^2) + \frac{1}{2}(n + n^2) - mp$ and $d = m^2 + n^2 - n - 2mp$. □

4 Conclusion

In this paper we have presented results concerning super (a, d)-H-antimagic total labelings for shackles of a connected graph H. We have found labelings for all possible d up to $m + n$ (Theorem 1) and various values of d which include the maximum possible d (Theorems 2 and 3). The labelings for other possible values of d remain unknown and finding them might be seen as an interesting problem.

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