AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **56** (2013), Pages 103–112

Criticality indices of Roman domination of paths and cycles*

Ahmed Bouchou

Mathematics and Computer Science Department
University of Médéa
Algeria
bouchou.ahmed@yahoo.fr

Mostafa Blidia

Lamda-RO, Mathematics Department
University of Blida
Algeria
m_blidia@yahoo.fr

Abstract

For a graph $G = (V, E)$, a Roman dominating function on G is a function $f : V(G) \to \{0, 1, 2\}$ such that every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v with $f(v) = 2$. The weight of a Roman dominating function is the value $f(V(G)) = \sum_{u \in V(G)} f(u)$. The minimum weight of a Roman dominating function on a graph G is called the Roman domination number of G, denoted by $\gamma_R(G)$. The removal criticality index of a graph G is defined as $ci^-_R(G) = (\sum_{e \in E(G)}(\gamma_R(G) - \gamma_R(G-e))/|E(G)|$ and the adding criticality index of G is defined as $ci^+_R(G) = (\sum_{e \in E(\overline{G})}(\gamma_R(G) - \gamma_R(G+e))/\left|E(\overline{G})\right|$ where $\overline{G}$ is the complement graph of G.

In this paper, we determine the criticality indices of paths and cycles.

1 Introduction

We consider simple graphs $G = (V(G), E(G))$ of order $|V(G)| = |V| = n(G)$ and size $|E(G)| = m(G)$. A graph is said to be *nontrivial* if $n(G) > 1$. The *complement* of G is the graph $\overline{G} = (V, E(\overline{G}))$, where $E(\overline{G}) = \{uv \mid uv \notin E\}$. The *neighborhood* of a vertex $v \in V$ is $N_G(v) = \{u \in V \mid uv \in E\}$. If S is a subset of vertices, then

* This work was supported by "Programmes Nationaux de Recherche: Code 8/u09/510".

its neighborhood is $N_G(S) = \cup_{v \in S} N_G(v)$. The *closed neighborhoods* of v and S are $N_G[v] = N_G(v) \cup \{v\}$ and $N_G[S] = N_G(S) \cup S$, respectively. The *degree* of a vertex v of G is $d_G(v) = |N(v)|$. The *maximum degree* of G is $\Delta(G) = \max\{d_G(v);\, v \in V\}$ and the *minimum degree* of G is $\delta(G) = \min\{d_G(v);\, v \in V\}$. The subgraph induced in G by a subset of vertices S is denoted by $G[S]$. The degree of vertex v in the subgraph induced in G by $S \subseteq V$ is denoted by $d_S(v) = |N(v) \cap S| = |N_S(v)|$ and the maximum degree of $G[S]$ is denoted by $\Delta(S)$. The path (respectively, the cycle) of order n is denoted by P_n (respectively, C_n).

A *dominating set* S is a set of vertices such that every vertex in $V - S$ has at least one neighbor in S, equivalently $N[S] = V$. A *total dominating set* S is a dominating set S with no isolated vertices in $G[S]$, equivalently $N(S) = V$. The minimum cardinality of a dominating set (respectively, a total dominating set) in G is called the *domination number* (respectively, the *total domination number*) of G, denoted by $\gamma(G)$ (respectively, $\gamma_t(G)$).

A *Roman dominating function* (RDF) on G is a function $f : V(G) \to \{0, 1, 2\}$ such that every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v with $f(v) = 2$. The weight of a Roman dominating function is the value $f(V(G)) = \sum_{u \in V(G)} f(u)$. The minimum weight of a Roman dominating function on a graph G is called the *Roman domination number* of G, denoted by $\gamma_R(G)$. A Roman dominating function of minimum weight is called a γ_R-function. If $f : V(G) \to \{0, 1, 2\}$ is a Roman dominating function, then let (V_0, V_1, V_2) be the ordered partition of $V(G)$ induced by f, where $V_i = \{v \in V(G) \mid f(v) = i\}$ for $i = 0, 1, 2$. Note that there is a 1-1 correspondence between the function $f : V(G) \to \{0, 1, 2\}$ and the ordered partition (V_0, V_1, V_2) of $V(G)$. Thus we will write $f = (V_0, V_1, V_2)$. Roman dominating functions have been introduced by Cockayne et al. [1].

Several studies have investigated the problem of the effect of some parameters of domination of a graph G when deleting or adding a vertex or edge. The study of critical, minimal or maximal graphs under vertex removal, edge removal or edge addition is classical (see [7, 8]). Sometimes, the concept of critical graph is presented in different aspects, depending on the effect of adding or deleting a vertex or edge; a survey on this concept is given in [3].

Sumner and Blitch [7] studied graphs for which $\gamma(G + e) = \gamma(G) - 1$ for each $e \notin E(G)$, and called these graphs domination edge critical. Haynes, Mynhardt and Van der Merwe [4] initiated the study of total domination edge critical graphs. A graph without isolated vertices is total domination critical, or just γ_t-critical if $\gamma_t(G + e) < \gamma_t(G)$ for any edge $e \in E(\overline{G})$ provided $E(\overline{G}) \neq \emptyset$. They remarked that while adding an edge can decrease the domination number by at most one, adding an edge can decrease the total domination number by as much as two.

Since any RDF of a spanning graph of G is also an RDF of G, we have $\gamma_R(G) \leq \gamma_R(G - e)$ for every $e \in E(G)$ and $\gamma_R(G + e) \leq \gamma_R(G)$ for every $e \notin E(G)$. On the other hand, it was shown in [5] that removing any edge from G can increase by at most one the Roman domination number of G. Also in [2], adding any edge to G can decrease by at most one the Roman domination number of G.

For a graph G, we define the *criticality index* of an edge $e \in E(G)$ as $ci_R^-(e) = \gamma_R(G) - \gamma_R(G-e)$, and the *removal criticality index* of a graph G as $ci_R^-(G) = \left(\sum_{i=1}^{m} ci_R^-(e_i)\right)/m(G)$. Similarly we define the *criticality index* of an edge $e \in E(\overline{G})$ as $ci_R^+(e) = \gamma_R(G) - \gamma_R(G+e)$, and the *adding criticality index* of a graph G as $ci_R^+(G) = \left(\sum_{i=1}^{\overline{m}} ci_R^+(e_i)\right)/m(\overline{G})$.

From the above, we mention that $ci_R^-(e) \in \{-1, 0\}$ for every $e \in E(G)$ and $ci_R^-(e) \in \{0, 1\}$ for every $e \notin E(G)$.

The criticality index of a graph is a value associated to a graph which measures the effect on a graph upon edge adding with respect to the parameter of a graph. Criticality index was introduced first in [4, 6] for the total domination number.

In this paper, we determine exact values of the criticality indices of G, $ci_R^-(G)$ and $ci_R^+(G)$ when G is a path or a cycle.

2 The removal criticality index of a cycle and a path

We begin by recalling the following result of Cockayne et al. that will be useful for what follows.

Theorem 1 (Cockayne et al. [1]) *If $G = C_n$ or P_n, then $\gamma_R(G) = \lceil 2n/3 \rceil$.*

As an immediate consequence of Theorem 1, we have the following.

Theorem 2 *For every cycle C_n, $ci_R^-(C_n) = 0$.*

Now we determine the value of the removal criticality index of a path. Let P_n be a path whose vertices are labeled $v_1, v_2, \ldots, v_n$. Note that when an edge v_iv_{i+1} is removed from the path P_n, we obtain two paths P_{p_i} and P_{n-p_i}. Hence the following observation is a direct consequence of Theorem 1.

Observation 3 *If P_n is a path, then for every i with $1 \le i \le n-1$,*

$$\gamma_R(P_n - v_iv_{i+1}) = \gamma_R(P_{p_i}) + \gamma_R(P_{n-p_i}) = \lceil 2p_i/3 \rceil + \lceil 2(n-p_i)/3 \rceil .$$

From the above observation, we deduce the following result.

Corollary 1 *For every nontrivial path P_n and $1 \le i \le n-1$,*

$$\gamma_R(P_n - v_iv_{i+1}) = \begin{cases} \lceil (2n+1)/3 \rceil & \text{if } i \equiv 1 \pmod 3 \\ \lceil (2n+2)/3 \rceil & \text{if } i \equiv 2 \pmod 3 \\ \lceil 2n/3 \rceil & \text{if } i \equiv 0 \pmod 3). \end{cases}$$

Theorem 4 *For every nontrivial path* P_n*,*

$$ci_R^-(P_n) = \begin{cases} -2n/3(n-1) & \text{for } n \equiv 0 \pmod 3 \\ -\lfloor n/3 \rfloor /(n-1) & \text{for } n \equiv 1 \pmod 3 \\ 0 & \text{for } n \equiv 2 \pmod 3. \end{cases}$$

Proof. By using Corollary 1, we can see that:

a) if $i \equiv 1 \pmod 3$, then $ci_R^-(v_iv_{i+1}) = \gamma_R(P_n) - \gamma_R(P_n - v_iv_{i+1})$

$$= \lceil 2n/3 \rceil - \lceil (2n+1)/3 \rceil = \begin{cases} -1 & \text{for } n \equiv 0 \pmod 3 \\ 0 & \text{for } n \equiv 1 \pmod 3 \\ 0 & \text{for } n \equiv 2 \pmod 3 \end{cases}$$

b) if $i \equiv 2 \pmod 3$, then $ci_R^-(v_iv_{i+1}) = \gamma_R(P_n) - \gamma_R(P_n - v_iv_{i+1})$

$$= \lceil 2n/3 \rceil - \lceil (2n+2)/3 \rceil = \begin{cases} -1 & \text{for } n \equiv 0 \pmod 3 \\ -1 & \text{for } n \equiv 1 \pmod 3 \\ 0 & \text{for } n \equiv 2 \pmod 3 \end{cases}$$

c) if $i \equiv 0 \pmod 3$, then $ci_R^-(v_iv_{i+1}) = \gamma_R(P_n) - \gamma_R(P_n - v_iv_{i+1}) = 0$.

Now we can establish the patterns for $ci_R^-(v_iv_{i+1})$; $1 \leq i \leq n-1$.

$$ci_R^-(v_iv_{i+1}) = \begin{cases} -1, \ -1, \ 0, \ \ldots, \ -1, \ -1, & \text{for } n \equiv 0 \pmod 3 \\ 0, \ -1, \ 0, \ \ldots, \ 0, \ -1, \ 0, & \text{for } n \equiv 1 \pmod 3 \\ 0, \ 0, \ 0, \ \ldots, \ 0, \ 0, \ 0, \ 0, & \text{for } n \equiv 2 \pmod 3. \end{cases}$$

which implies that if $n \equiv 0 \pmod 3$, then $ci_R^-(P_n) = -2n/3(n-1)$. If $n \equiv 1 \pmod 3$, then $ci_R^-(P_n) = -\lfloor n/3 \rfloor /(n-1)$, and if $n \equiv 2 \pmod 3$, then $ci_R^-(P_n) = 0$. ■

3 The adding criticality index of a cycle

In this section we give exact values of the adding criticality index of a cycle. Let G be a graph obtained from a cycle C_n by adding a chord such that G is formed from two cycles C_p and C_q (so $n = p + q - 2$).

The following proposition is fundamental in determining $ci_R^+(C_n)$.

Proposition 5 *Let* $G = C_n + uv$*, where the chord* uv *forms two cycles* C_p *and* C_q *sharing* uv*. Then*

$$\gamma_R(G) = \gamma_R(C_p) + \gamma_R(C_q) - 2.$$

Proof. Since C_n is vertex transitive, there exists a γ_R-function $f = (V_0, V_1, V_2)$ on C_p and a γ_R-function $f' = (V_0', V_1', V_2')$ on C_q such that $u \in V_2 \cap V_2'$ and $v \in V_0 \cap V_0'$. Then $(V_0 \cup V_0', V_1 \cup V_1', V_2 \cup V_2')$ is an RDF on G, so

$$\begin{aligned} \gamma_R(G) &\leq |V_1 \cup V_1'| + 2(|V_2 \cup V_2'|) \\ &= |V_1| + |V_1'| + 2(|V_2| + |V_2'| - 1) \\ &= \gamma_R(C_p) + \gamma_R(C_q) - 2. \end{aligned}$$

Now, we show that there exists a γ_R-function $f=(V_0,V_{1,}V_2)$ on G with $f(u)=2$ and $f(v)=0$ or $f(v)=2$ and $f(u)=0$. Let $C_p=(u,x_1,\ldots,x_{p-2},v)$ and $C_q=(u,y_1,\ldots,y_{q-2},v)$. Let $f=(V_0,V_{1,}V_2)$ be a γ_R-function on G. There are two cases:

Case 1. $f(v)\neq 2$ and $f(u)\neq 2$. Then f is a γ_R-function on C_n and so $\gamma_R(C_n)=\gamma_R(G)$. Since C_n is vertex transitive, there exists a γ_R-function $f_1=(V_0',V_1',V_2')$ on C_n, where $f_1(u)=2$. If $f_1(v)=0$, then we are done. If $f_1(v)=1$, then we form the new function $(V_0'\cup\{v\},V_1'-\{v\},V_2')$, which is an RDF for G with a smaller weight than f, a contradiction. If $f_1(v)=2$, then we form a new function $f_2=((V_0'\cup\{v\})-\{x_{p-2},y_{q-2}\},V_1\cup\{x_{p-2},y_{q-2}\},V_2-\{v\})$, which is a γ_R-function on G with $f_2(u)=f_1(u)=2$ and $f_2(v)=0$.

Case 2. $f(u)=2$ or $f(v)=2$. Suppose without loss of generality that $f(u)=2$. If $f(v)=0$, then we are done. If $f(v)=1$, then we form a new function $(V_0\cup\{v\},V_1-\{v\},V_2)$, which is an RDF with weight less than f, a contradiction. If $f(v)=2$, then we form a new function $f_3=((V_0\cup\{v\})-\{x_{p-2},y_{q-2}\},V_1\cup\{x_{p-2},y_{q-2}\},V_2-\{v\})$, which is a γ_R-function on G with $f_3(v)=f(v)=2$ and $f_3(v)=0$.

Therefore, in all cases there exists a γ_R-function $f=(V_0,V_{1,}V_2)$ on G with $f(u)=2$ and $f(v)=0$. Thus, $(V_0\cap C_p,V_1\cap C_{p,}V_2\cap C_p)$ is an RDF on C_p and $(V_0\cap C_q,V_1\cap C_q,V_2\cap C_q)$ is an RDF on C_q. Therefore,

$$\begin{aligned}\gamma_R(G)&=|V_1|+2|V_2|\\&=|(V_1\cap C_p)\cup(V_1\cap C_q)|+2(|(V_2\cap C_p)\cup(V_2\cap C_q)|-1)\\&=|(V_1\cap C_p)|+|(V_1\cap C_q)|+2|(V_2\cap C_p)|+2|(V_2\cap C_q)|-2\\&\geq\gamma_R(C_p)+\gamma_R(C_q)-2,\end{aligned}$$

and the desired equality is proved. ■

For the rest of this section, we label the vertices of C_n as $v_0,\ldots,v_{n-1}$. Without loss of generality, we consider v_0v_j as the edge added to C_n.

Proposition 6 *Let $G=C_n+v_0v_j$ and k a positive integer with $j+3k\leq n/2$. Then*

$$ci_R^+(v_0v_j)=ci_R^+(v_0v_{j+3k}).$$

Proof. Let C_p and C_q be the two induced cycles of G. By Theorem 1, $\gamma_R(C_{p+3k})=\lceil 2p/3\rceil+2k$ and $\gamma_R(C_{q-3k})=\lceil 2q/3\rceil-2k$, and by Proposition 5, $\gamma_R(C_n+v_0v_{j+3k})=\gamma_R(C_{p+3k})+\gamma_R(C_{q-3k})-2=\gamma_R(C_n+v_0v_j)$. Hence $ci_R^+(v_0v_{j+3k})=\gamma_R(C_n)-\gamma_R(C_n+v_0v_{j+3k})=\gamma_R(C_n)-\gamma_R(C_n+v_0v_j)=ci_R^+(v_0v_j)$. ■

We note that by symmetry, we have $ci_R^+(v_0v_j)=ci_R^+(v_0v_{n-j})$.

Theorem 7 *For every cycle C_n,*

$$ci_R^+(C_n)=\begin{cases}0 & \text{for } n\equiv 0(\bmod 3)\\ \lfloor n/3\rfloor/(n-3) & \text{for } n\equiv 1(\bmod 3)\\ 2\lfloor n/3\rfloor/(n-3) & \text{for } n\equiv 2(\bmod 3).\end{cases}$$

Proof. If $n \leq 7$, then it is easy to see that the theorem holds, so let $n \geq 8$. Proposition 6 implies that we need to consider the addition of only three edges, say v_0v_2, v_0v_3 and v_0v_4.

Case 1. $C_n + v_0v_2$. We have $\gamma_R(C_n + v_0v_2) = \gamma_R(C_3) + \gamma_R(C_{n-1}) - 2 = \gamma_R(C_{n-1})$. Now $\gamma_R(C_{n-1}) = \gamma_R(C_n)$ if $n \equiv 0 \pmod 3$ and $\gamma_R(C_{n-1}) = \gamma_R(C_n) - 1$ if $n \equiv 1, 2 \pmod 3$. Hence

$$ci_R^+(v_0v_2) = \begin{cases} 0 & \text{for } n \equiv 0 (\text{mod } 3) \\ 1 & \text{for } n \equiv 1, 2 (\text{mod } 3). \end{cases}$$

Case 2. $C_n + v_0v_3$. We have $\gamma_R(C_n + v_0v_3) = \gamma_R(C_4) + \gamma_R(C_{n-2}) - 2 = \gamma_R(C_{n-2}) + 1$.

Now $\gamma_R(C_{n-2}) = \gamma_R(C_n) - 1$ if $n \equiv 0, 1 \pmod 3$ and $\gamma_R(C_{n-2}) = \gamma_R(C_n) - 2$ if $n \equiv 2 \pmod 3$. Hence

$$ci_R^+(v_0v_3) = \begin{cases} 0 & \text{for } n \equiv 0, 1 (\text{mod } 3) \\ 1 & \text{for } n \equiv 2 (\text{mod } 3). \end{cases}$$

Case 3. $C_n + v_0v_4$. We have $\gamma_R(C_n + v_0v_4) = \gamma_R(C_5) + \gamma_R(C_{n-3}) - 2 = \gamma_R(C_{n-3}) + 2$. Now $\gamma_R(C_{n-3}) = \gamma_R(C_n) - 2$ for all n. Hence

$$ci_R^+(v_0v_4) = 0 \text{ for all } n.$$

Now we can establish the patterns for $ci^+(v_0v_j)$, $2 \leq j \leq n-2$.

$$ci_R^+(v_0v_j) = \begin{cases} 0,0,0,0,0,0,\ldots,0,0,0 & \text{for } n \equiv 0 (\text{mod } 3) \\ 1,0,0,1,0,0,\ldots,1,0,0,1 & \text{for } n \equiv 1 (\text{mod } 3) \\ 1,1,0,1,1,0,\ldots,1,1,0,1,1 & \text{for } n \equiv 2 (\text{mod } 3). \end{cases}$$

Note that $m(\overline{C_n}) = \left|E\left(\overline{C_n}\right)\right| = n(n-3)/2$. Hence

If $n \equiv 0 (\text{mod } 3)$, $ci_R^+(C_n) = 0$.

If $n \equiv 1 (\text{mod } 3)$, $ci_R^+(C_n) = (n/2)\lfloor n/3 \rfloor / (n/2)(n-3) = \lfloor n/3 \rfloor / (n-3)$.

If $n \equiv 2 (\text{mod } 3)$, $ci_R^+(C_n) = (n/2) 2 \lfloor n/3 \rfloor / (n/2)(n-3) = 2\lfloor n/3 \rfloor / (n-3)$, and the proof is complete. ■

4 The adding criticality index of a path

In this section we give exact values of the adding criticality index of a path. Let the path P_n have a chord forming two paths P_p, P_q and a cycle C_t (so $n = p + q + t$).

In order to determine the criticality index of a path, we first describe two procedures:

Procedure A :

Let P_n be the path with consecutive vertices $w_1, \ldots, w_n$ (where $n \geq 3$). Let H_1 be the unicyclic graph obtained from P_n by joining two non-adjacent vertices w_i and w_j with an edge, where $i < j$. Suppose H_1 has a cycle of length greater than 5. Then

H_1 has a subpath $P = v_1, u_1, u_2, u_3, v_2 = w_j$ of the cycle, and we form the graph H_2 from H_1 by deleting vertices u_1, u_2 and u_3 and joining vertices v_1 to v_2. We repeat this process until eventually we obtain a graph H_k having a cycle of order 3, 4 or 5.

Lemma 8 $\gamma_R(H_i) = \gamma_R(H_{i+1}) + 2$.

Proof. Suppose $f = (V_0, V_1, V_2)$ is a γ_R-function of H_{i+1}. Then $(V_0 \cup \{u_1, u_3\}, V_1, V_2 \cup \{u_2\})$ is an RDF of H_i, and so $\gamma_R(H_i) \leq |V_1| + 2|V_2 \cup \{u_2\}| = \gamma_R(H_{i+1}) + 2$.

Next suppose that $f' = (V_0', V_1', V_2')$ is a γ_R-function of H_i such that $f'(u_1) + f'(u_2) + f'(u_3)$ is minimum.

If $v_2 \in V_2'$ and $v_1 \in V_0'$ or $v_2 \in V_0'$ and $v_1 \in V_2'$ then, without loss of generality, $u_1 \in V_2'$ and $u_2, u_3 \in V_0'$ or $u_3 \in V_2'$ and $u_1, u_2 \in V_0'$. If $v_2 \in V_1'$ and $v_1 \in V_0'$ or $v_1 \in V_1'$ and $v_2 \in V_0'$ then $u_2 \in V_2'$ and $u_1, u_3 \in V_0'$, otherwise we have a contradiction with $f'(u_1) + f'(u_2) + f'(u_3)$ minimum. Finally, if $v_2 \in V_0'$, then $v_1 \in V_0'$ and so $u_2 \in V_2'$ and $u_1, u_3 \in V_0'$, for otherwise we obtain a contradiction with the choice of f'. So in all cases $(V_0' - \{u_1, u_3\}, V_1', V_2' - \{u_2\})$ is an RDF of H_{i+1} with weight $\gamma_R(H_{i+1}) \leq |V_1'| + 2|V_2'| - 2 = \gamma_R(H_i) - 2$. Hence $\gamma_R(H_{i+1}) = \gamma_R(H_i) - 2$. ■

<u>**Procedure B :**</u>

A 3-endpath in a graph is a path v, u_1, u_2, u_3 such that $d_G(v) \geq 2$, $d_G(u_1) = d_G(u_2) = 2$ and $d_G(u_3) = 1$. Consider the graph R_i with a 3-endpath v, u_1, u_2, u_3. Let R_{i+1} be the graph obtained from R_i by deleting vertices u_1, u_2 and u_3. We repeat this process until we obtain a graph R_l with no 3-endpath.

Lemma 9 $\gamma_R(R_i) = \gamma_R(R_{i+1}) + 2$.

Proof. Suppose that $f = (V_0, V_1, V_2)$ is a γ_R-function of R_{i+1}. Then $(V_0 \cup \{u_1, u_3\}, V_1, V_2 \cup \{u_2\})$ is an RDF of R_i, and so $\gamma_R(R_i) \leq |V_1| + 2|V_2 \cup \{u_2\}| = \gamma_R(R_{i+1}) + 2$.

Next suppose that $f' = (V_0', V', V_2')$ is a γ_R-function of R_i with $f'(u_1) + f'(u_2) + f'(u_3)$ is minimum. In this case it is not difficult to see that $u_1, u_3 \in V_0'$ and $u_2 \in V_2'$. So, $(V_0' - \{u_1, u_3\}, V_1', V_2' - \{u_2\})$ is an RDF of R_{i+1}, which implies that $\gamma_R(R_{i+1}) \leq |V_1'| + 2|V_2'| - 2 = \gamma_R(R_i) - 2$. Hence $\gamma_R(H_{i+1}) = \gamma_R(H_i) - 2$. ■

Theorem 10 *For a path P_n with $n \geq 3$,*

$$ci_R^+(P_n) = \begin{cases} 0 & \text{for } n \equiv 0 (\operatorname{mod} 3) \\ 2(n+2)/9(n-2) & \text{for } n \equiv 1 (\operatorname{mod} 3) \\ 5(n+1)/9(n-1) & \text{for } n \equiv 2 (\operatorname{mod} 3). \end{cases}$$

Proof. Let $M(n_1, c, n_2)$, where $n_1, n_2 \in \{0, 1, 2\}$ and $c \in \{3, 4, 5\}$, be the graph obtained from the cycle C of length c by joining one end-vertex of a path P_{n_1} to some vertex x of C and joining one end-vertex of the path P_{n_2} to a vertex y on C adjacent to x. The graph $M(n_1, c, n_2)$ will be called an elementary unicyclic graph.

By applying Procedures A and B on a $P_n + e$, where $e \in E\left(\overline{P_n}\right)$ on the resulting graphs as much as possible, at the end we obtain an elementary unicyclic graph $M(n_1, c, n_2)$ of order $n_1 + c + n_2$.

Let k_1, k_2 and k denote the number of groups of three vertices that were removed from $P_n + e$ to obtain the paths P_{n_1}, P_{n_2} and the cycle C, respectively, of the elementary unicyclic graph $M = M(n_1, c, n_2)$. Thus,

$$3(k_1 + k + k_2) = n - n(M),$$

and so

$$k_1 + k + k_2 = (n - n(M))/3. \tag{1}$$

The number of graphs $P_n + e$ corresponding to the elementary unicyclic graph M equals to the number of solutions of Equation (1),

$$\mathsf{C}^2_{(n-n(M))/3+2} = (n - n(M) + 3)(n - n(M) + 6)/18.$$

By Lemmas 8 and 9, we have that $ci^+_R(e) = \gamma_R(P_n) - \gamma_R(P_n + e) = \gamma_R(P_{n_1+c+n_2}) - \gamma_R(M)$ for some $e \in E\left(\overline{P_n}\right)$.

Let $\mathcal{M}_i$, for $i = 0, 1$, be the set of all elementary unicyclic graphs $M = M(n_1, c, n_2)$ for which $ci^+_R(e) = i$ and set $\mathcal{M} = \mathcal{M}_0 \cup \mathcal{M}_1$. So

$$\begin{aligned} ci^+_R(P_n) &= \left(\sum\nolimits_{e\in E(\overline{G})} ci^+_R(e)\right)/m\left(\overline{P_n}\right) \\ &= \left(\sum\nolimits_{M\in\mathcal{M}} (\#\text{ of graphs } P_n + e \text{ corresponding to } M)\right)/m\left(\overline{P_n}\right). \\ &= \left(\sum\nolimits_{M\in\mathcal{M}} (n - n(M) + 3)(n - n(M) + 6)/18\right)/m\left(\overline{P_n}\right). \end{aligned}$$

Note that $m\left(\overline{P_n}\right) = (n-1)(n-2)/2$, so

$$ci^+_R(P_n) = \left(\sum\nolimits_{M\in\mathcal{M}_1} (n - n(M) + 3)(n - n(M) + 6)/9\right)/(n-1)(n-2). \tag{2}$$

Then, by applying Procedures A and B, we distinguish three cases:

Case 1. $n \equiv 0 \pmod 3$. We have $n(M) \equiv 0 \pmod 3$. Note that $n(M) = n_1 + c + n_2 = 3$, 6 or 9 for each $M \in \mathcal{M}$. So,

$$\mathcal{M} = \left\{ \begin{array}{c} M(0,3,0), M(0,4,2), M(0,5,1), M(1,3,2), M(1,4,1), \\ M(1,5,0), M(2,3,1), M(2,4,0), M(2,5,2) \end{array} \right\}.$$

It is a routine matter to check that for all $M \in \mathcal{M}$, $M \in \mathcal{M}_0$. So, by Equation (2), we have $ci^+_R(P_n) = 0$.

Case 2. $n \equiv 1 \pmod 3$. We have $n(M) \equiv 1 \pmod 3$. Note that $n(M) = n_1 + c + n_2 = 4$ or 7 for each $M \in \mathcal{M}$. So,

$$\mathcal{M} = \left\{ \begin{array}{c} M(0,3,1), M(0,4,0), M(0,5,2), M(1,3,0), M(1,4,2), \\ M(1,5,1), M(2,3,2), M(2,4,1), M(2,5,0) \end{array} \right\}.$$

We can easily check that $\mathcal{M}_1 = \{M(0,3,1), M(1,3,0)\}$ and

$$\mathcal{M}_0 = \{M(0,4,0), M(0,5,2), M(1,4,2), M(1,5,1), M(2,3,2), M(2,4,1), M(2,5,0)\}.$$

So, by Equation (2), we have

$$ci_R^+(P_n) = 2(n-4+3)(n-4+6)/9(n-1)(n-2) = 2(n+2)/9(n-2).$$

Case 3. $n \equiv 2 \pmod 3$. We have $n(M) \equiv 2 \pmod 3$. Note that $n(M) = n_1+c+n_2 = 5$ or 8 for each $M \in \mathcal{M}$. So,

$$\mathcal{M} = \left\{ \begin{array}{c} M(0,3,2), M(0,4,1), M(0,5,0), M(1,3,1), M(1,4,0), \\ M(1,5,2), M(2,3,0), M(2,4,2), M(2,5,1) \end{array} \right\}.$$

Again, it is easy to see that

$$\mathcal{M}_1 = \{M(0,3,2), M(0,4,1), M(1,4,0), M(2,3,0)\}$$

and

$$\mathcal{M}_0 = \{M(0,5,0), M(1,3,1), M(1,5,2), M(2,4,2), M(2,5,1)\}.$$

So, by Equation (2), we have

$$ci_R^+(P_n) = 5(n-5+3)(n-5+6)/9(n-1)(n-2) = 5(n+1)/9(n-1),$$

and the proof is complete. ■

Acknowledgements

We would like to thank the anonymous referees for some comments and suggestions on the former version of this manuscript.

References

[1] E. J. Cockayne, P. M. Dreyer Jr., S. M. Hedetniemi and S. T. Hedetniemi, On Roman domination in graphs, *Discrete Math.* **278** (2004), 11–22.

[2] A. Hansberg, N. Jafari Rad and L. Volkmann, Vertex and edge critical Roman domination in graphs, *Utilitas Math.* (to appear).

[3] T. W. Haynes, S. T. Hedetniemi and P. J. Slater, *Fundamentals of Domination in Graphs, (Chapter 5)*, Marcel Dekker, New York, 1998.

[4] T. W. Haynes, C. M. Mynhardt and L. C. van der Merwe, Criticality index of total domination, *Congr. Numer.* **131** (1998), 67–73.

[5] N. Jafari Rad and L. Volkmann, Changing and unchanging the Roman domination number of a graph, *Util. Math.* **89** (2012), 79–95.

[6] J. H. Hattingh, E. J. Joubert and L. C. van der Merwe, The Criticality Index of Total Domination of path, *Utilitas Math.* (to appear).

[7] D. P. Sumner and P. Blitch, Domination Critical Graphs, *J. Combin. Theory Ser. B* **34** (1983), 65–76.

[8] H. B. Walikar and B. D. Acharya, Domination Critical Graphs, *Nat. Acad. Sci. Lett.* **2** (1979), 70–72.

(Received 28 May 2012; revised 7 Feb 2013)