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# On 2-rainbow domination and Roman domination in graphs

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**Abstract**

A 2-rainbow dominating function of a graph $G$ is a function $g$ that assigns to each vertex a set of colors chosen from the set $\{1, 2\}$ so that for each vertex with $g(v) = \emptyset$ we have $\bigcup_{u\in N(v)} g(u) = \{1, 2\}$. The minimum of $g(V(G)) = \sum_{v\in V(G)} |g(v)|$ over all such functions is called the 2-rainbow domination number $\gamma_{2r}(G)$. A Roman dominating function on a graph $G$ is a function $f : V(G) \rightarrow \{0, 1, 2\}$ satisfying the condition that every vertex $u$ with $f(u) = 0$ is adjacent to at least one vertex $v$ of $G$ for which $f(v) = 2$. The minimum of $f(V(G)) = \sum_{u\in V(G)} f(u)$ over all such functions is called the Roman domination number $\gamma_R(G)$. We first prove that $\gamma_R(G)/\gamma_{r2}(G) \leq 3/2$ for every graph $G$ and we improve this ratio for all trees. Then we present some bounds for the 2-rainbow domination number in graphs. In particular, we give an upper bound on the 2-rainbow domination number for every tree of order at least three in terms of the number of vertices, stems and leaves of the tree.

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# 1 Introduction

We consider finite, undirected, and simple graphs $G$ with vertex set $V = V(G)$ and edge set $E = E(G)$. The number of vertices $|V(G)|$ of a graph $G$ is called the *order* of $G$ and is denoted by $n = n(G)$. The *open neighborhood* of a vertex $v \in V$ is $N(v) = N_G(v) = \{u \in V \mid uv \in E\}$ and the *degree* of $v$, denoted by $d_G(v)$, is the cardinality of its open neighborhood. A vertex of degree one is called a *leaf*, and its neighbor is called a *stem*. If $v$ is a stem of $G$, then $L_v$ will denote the set of the leaves attached at $v$.

A set $D \subseteq V(G)$ is a *dominating set* if every vertex of $V(G) - D$ has a neighbor in $D$. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of $G$. The concept of domination in graphs and its many variations are now well studied in graph theory (see for example [7]). Here we will focus on two variants called 2-rainbow domination and Roman domination introduced in [1] and [5], respectively.

A function $f : V(G) \to \{0, 1, 2\}$ is a *Roman dominating function* **(RDF)** on $G$ if every vertex $u$ of $G$ for which $f(u) = 0$ is adjacent to at least one vertex $v$ of $G$ for which $f(v) = 2$. The weight of an RDF is the value $f(V(G)) = \sum_{u\in V(G)} f(u)$. The *Roman domination number* $\gamma_R(G)$ is the minimum weight of an RDF on $G$.

Let $f$ be a function that assigns to each vertex a set of colors chosen from the set $\{1, 2\}$; that is $f : V(G) \to \mathcal{P}(\{1, 2\})$. If for each vertex $v \in V(G)$ such that $f(v) = \emptyset$, we have $\bigcup_{u\in N(v)} f(u) = \{1, 2\}$, then $f$ is called a 2-rainbow dominating function (2**RDF**) of $G$. The weight of a 2RDF $f$ is defined as $w(f) = \sum_{v\in V(G)} |f(v)|$. The minimum weight of a 2-rainbow dominating function is called the 2-*rainbow domination number* of $G$, denoted by $\gamma_{r2}(G)$. We say that a function $f$ is a $\gamma_{r2}(G)$-*function* if it is a 2RDF and $w(f) = \gamma_{r2}(G)$. Some papers on rainbow domination can be found, for example, in [2, 4, 8] and elsewhere.

In this paper, we determine a sharp upper bound on the ratio of the Roman domination and 2-rainbow domination numbers for every graph and we improve it for the class of trees. Then we present some bounds for the 2-rainbow domination number in graphs.

# 2 Main Results

We know from [10] that $\gamma_{r2}(G)/\gamma_R(G) \leq 1$ for every graph $G$. We can wonder whether there exists an upper bound for the ratio $\gamma_R(G)/\gamma_{r2}(G)$ for every graph $G$. The answer is positive as shown by the following result.

**Theorem 1** *For any graph $G$, $\frac{\gamma_R(G)}{\gamma_{r2}(G)} \leq \frac{3}{2}$.*

**Proof.** Let $f$ be a $\gamma_{r2}(G)$-function, and let $A_i$ be the set of all vertices $u$ for which $i \in f(u)$, for $i = 1, 2$. Clearly if a vertex of $G$ is assigned the set $\{1, 2\}$, then $A_1 \cap A_2 \neq$

$\emptyset$. Also $\gamma_{r2}(G) = |A_1| + |A_2|$. Assume, without loss of generality, that $|A_1| \leq |A_2|$. Then $|A_1| \leq \frac{|A_1|+|A_2|}{2} = \frac{\gamma_{r2}(G)}{2}$. Let $g : V(G) \longrightarrow \{0,1,2\}$ be defined by $g(x) = 0$ if $f(x) = \emptyset$, $g(x) = 1$ if $f(x) = \{2\}$, and $g(x) = 2$ if $1 \in f(x)$. Since $f$ is a 2RDF for $G$, we obtain that $g$ is an RDF for $G$, implying that $\gamma_R(G) \leq w(g) = 2|A_1| + |A_2|$. Consequently,

$$\gamma_R(G) \leq 2|A_1| + |A_2| = |A_1| + |A_1| + |A_2| \leq \frac{3}{2}\gamma_{r2}(G).$$

■

To see the sharpness of the ratio in Theorem 1, we form the graph $G_k$ from $(k-1)$ vertices $x_1, x_2, ..., x_{k-1}$ and $k$ disjoint copies of a cycle $C_8$ (where $y_i$ is a vertex of the $ith$ copy of $C_8$) by adding edges $x_iy_i$ and $x_iy_{i+1}$ for every $i$ with $1 \leq i \leq k-1$. Clearly, $\gamma_R(G_k) = 6k, \gamma_{r2}(G_k) = 4k$, and thus $\frac{\gamma_R(C_8)}{\gamma_{r2}(C_8)} = \frac{3}{2}$.

Before providing an improvement of the ratio $\gamma_R/\gamma_{r2}$ for the class of trees, we give a result that will be useful for the next. If a tree $T$ is a subdivision of a nontrivial tree $T'$, then we say that $T$ is a *subdivided tree*, and the $n(T') - 1$ new vertices resulting from the subdivision of the edges of $T'$ are called *subdivision vertices*. Note that a subdivided tree has an odd order at least three and at least one subdivision vertex. We also note that every stem in a subdivided tree is a subdivision vertex and has degree two. The *corona graph* of a graph $G$ is the graph constructed from a copy of $G$, where for each vertex $v \in V(G)$, a new vertex $v'$ and the edge $vv'$ are added.

**Lemma 2** *If $T$ is a subdivided tree, then $\gamma_R(T) \leq \frac{2(|V(T)|+1)}{3}$.*

**Proof.** We use an induction on the order $n$ of $T$. Clearly $n \geq 3$ and the result holds if $n = 3$. Let $n \geq 5$ and assume that every subdivided tree $T'$ of order $n'$ with $n' < n$ satisfies $\gamma_R(T') \leq \frac{2(n'+1)}{3}$. Let $T$ be a subdivided tree of order $n$. Note that $T$ has an even diameter at least four.

Now consider a diametrical path $u_0$-$u_1$-$u_2$-...-$u_{\mathrm{diam}(T)}$ chosen to maximize the degree of $u_2$. Note that $u_i$ is a subdivision vertex for every $i$ odd, and so for such a vertex $d_T(u_i) = 2$.

Let us first assume that $d_T(u_2) \geq 3$. If $\mathrm{diam}(T) = 4$, then $T$ is the subdivision tree of a star $K_{1,t}$ $(t \geq 3)$. In that case $T$ has order $2t+1$ and $\gamma_R(T) = 2+t \leq \frac{2(2t+2)}{3}$. Therefore the result is valid. Thus we can assume that $\mathrm{diam}(T) \geq 6$. Consider the subtrees $T_{u_3}$ and $T_{u_4}$ obtained from $T$ by deleting the edge $u_3u_4$, where $u_3 \in V(T_{u_3})$. Clearly $T_{u_3}$ is a corona of a star, where $n(T_{u_3}) = 2d_T(u_2)$ and $\gamma_R(T_{u_3}) = 1 + d_T(u_2)$. Also since $T_{u_4}$ is a subdivided tree of order $n(T_{u_4}) \geq 3$, by induction on $T_{u_4}$ we have $\gamma_R(T_{u_4}) \leq \frac{2(\left|V(T_{u_4})\right|+1)}{3}$. Now it is evident that $\gamma_R(T) \leq \gamma_R(T_{u_4}) + \gamma_R(T_{u_3})$, and by a simple calculation we obtain $\gamma_R(T) \leq \frac{2(|V(T)|+1)}{3}$.

Assume now that $d_T(u_2) = 2$, and let $T_{u_5}$ and $T_{u_6}$ be the subtrees obtained from $T$ by deleting the edge $u_5u_6$, where $u_5 \in V(T_{u_5})$. Since $d_T(u_2) = 2$, by our choice of the diametral path, every vertex of $T_{u_5}$ except possibly $u_4$ has degree one or two. Also every leaf in $T_{u_5}$ except $u_5$ is at distance two or four from $u_4$. So let $k$ and $r$ be the number of leaves in $T_{u_5}$ at distance four and two from $u_4$, respectively. Then $T_{u_5}$ has order $4k+2r+2$, where $k \geq 1$ and $r \geq 0$; and so $\gamma_R(T_{u_5}) = 2k+2+r$. Now if $\text{diam}(T) = 6$, then $T_{u_6}$ is a tree of order $|\{u_6\}|$ . Hence $\gamma_R(T) = 2k+2+r+1 \leq \frac{2(|V(T)|+1)}{3}$, and the result is valid. So we may assume that $\text{diam}(T) \geq 8$, that is $T_{u_6}$ is a subdivided tree of order $n(T_{u_3}) \geq 3$. By induction on $T_{u_6}$ we have $\gamma_R(T_{u_6}) \leq \frac{2(\left|V(T_{u_6})\right|+1)}{3}$. Clearly, $\gamma_R(T) \leq \gamma_R(T_{u_5})+\gamma_R(T_{u_6})$ and by a simple calculation we obtain $\gamma_R(T) \leq \frac{2(|V(T)|+1)}{3}$. ■

Notice that the bound of Lemma 2 is sharp for a path $P_5$.

**Theorem 3** *For every tree* $T$, $\frac{\gamma_R(T)}{\gamma_{r2}(T)} \leq \frac{4}{3}$.

**Proof.** We use an induction on the order $n$ of $T$. Clearly if $n \in \{1,2,3\}$, then $\gamma_R(T) = \gamma_{r2}(T)$. Hence $\frac{\gamma_R(T)}{\gamma_{r2}(T)} \leq \frac{4}{3}$, establishing the base cases.

Let $n \geq 4$ and assume that every tree $T'$ of order $n'$ with $n' < n$ satisfies $\frac{\gamma_R(T')}{\gamma_{r2}(T')} \leq \frac{4}{3}$. Let $T$ be a tree of order $n$. Among all $\gamma_{r2}(T)$-functions, let $f$ be one for which no leaf is assigned $\{1,2\}$. One can easily see that such a $\gamma_{r2}(T)$-function exists. Let $V_2$ be the set of vertices $u$ such that $f(u) = \{1,2\}$, $V_0$ the set of vertices $u$ such that $f(u) = \emptyset$, and $V_1 = V(T) - (V_2 \cup V_0)$.

Let $a$ and $b$ be any two adjacent vertices of $T$ such that either $f(a) = f(b) = \emptyset$ or $f(a) \neq \emptyset$ and $f(b) \neq \emptyset$. Let $T_a$ and $T_b$ be the subtrees obtained from $T$ by removing the edge $ab$. Then the restriction of $f$ on $V(T_a)$, denoted by $f|_{V(T_a)}$ is a 2RDF on $T_a$ and likewise $f|_{V(T_b)}$ for $T_b$. Hence $\gamma_{r2}(T_a) + \gamma_{r2}(T_b) \leq w(f|_{V(T_a)}) + w(f|_{V(T_b)}) = \gamma_{r2}(T)$. On the other hand, it is evident that $\gamma_R(T) \leq \gamma_R(T_a) + \gamma_R(T_b)$. Since each of $T_a$ and $T_b$ has order less than $n$, by induction we have $3\gamma_R(T_a) \leq 4\gamma_{r2}(T_a)$ and $3\gamma_R(T_b) \leq 4\gamma_{r2}(T_b)$. Combining all these inequalities we obtain:

$$\begin{aligned} 3\gamma_R(T) &\leq 3\gamma_R(T_a) + 3\gamma_R(T_b) \\ &\leq 4\gamma_{r2}(T_a) + 4\gamma_{r2}(T_b) \leq 4\gamma_{r2}(T). \end{aligned}$$

For the next, we can assume that the set of vertices assigned empty sets (respectively, non-empty sets) are independent. Now let $a$ be a vertex of $V_0$ such that either $d_T(a) \geq 3$ or $d_T(a) = 2$ but having a neighbor in $V_2$. In this case, let $b$ be a neighbor of $a$ such that $\bigcup_{u \in N(a)} f(u) = \{1,2\}$ in the tree $T - ab$. It is clear that such a vertex $b$ exists. Using the same argument to that used above for the tree $T - ab$ we obtain that $3\gamma_R(T) \leq 4\gamma_{r2}(T)$. Hence every vertex $x \in V_0$ has degree at most two. More precisely, either $x$ is a leaf adjacent to a vertex of $V_2$ or $x$ has degree two and has its two neighbors in $V_1$.

Suppose now that $V_2 \neq \emptyset$ and let $x \in V_2$. According to what it proceeds, all neighbors of $x$ are leaves and since each $V_i$, for $i = 0,1,2$ is an independent set, we conclude that $T$ is a star of center $x$. In that case the result holds. Hence we may assume that $V_2 = \emptyset$ and so all leaves of $T$ belong to $V_1$, each vertex of $V_0$ has degree two, $V_0$ and $V_1$ are independent sets. Note that since $V_2 = \emptyset$, we have $\gamma_{r2}(T) = |V_1|$. Thus $V_0$ can be seen as the set of the subdivision vertices resulting from the subdivision of the edges of some tree $T'$ of order $n(T') = |V_1|$. Therefore $T$ is a subdivided tree, where $|V_0| = \frac{n-1}{2}$ and $|V_1| = \frac{n+1}{2} = \gamma_{r2}(T)$. Now by Lemma 2, $\gamma_R(T) \leq \frac{2(n+1)}{3}$ and hence $\frac{\gamma_R(T)}{\gamma_{r2}(T)} \leq \frac{4}{3}$. ■

To see the sharpness of the ratio in Theorem 3, consider the path $P_5$.

We will now turn our attention to the 2-rainbow domination number. Our aim is to provide an upper bound on the 2-rainbow domination number for the class of trees improving the one given by Wu and Jafari Rad [9]. Let us first recall the following two upper bounds that can be found in [9] and [3], respectively.

**Theorem 4 (Wu and Jafari Rad [9])** *If $G$ is a connected graph of order $n \geq 3$, then $\gamma_{r2}(G) \leq 3n/4$.*

**Theorem 5 (Chambers et al. [3])** *If $G$ is a graph of order $n \geq 3$, then $\gamma_R(G) \leq 4n/5$.*

We also give the following useful observation.

**Observation 6** *Let $v$ be a stem of degree two in a graph $G$ and $u$ its leaf. Then there is a $\gamma_{r2}(G)$-function $f$ such that $|f(u)| = 1$ and $f(v) = \emptyset$.*

**Proof.** Let $w$ be the second neighbor of $v$ in $G$ and let $f$ be a $\gamma_{r2}(G)$-function. Clearly if $f(u) = \emptyset$, then $f(v) = \{1, 2\}$. Hence we can define a $\gamma_{r2}(G)$-function $h$ on $G$ such that $h(x) = f(x)$ if $x \notin \{u, v, w\}$, $h(v) = \emptyset$, and $h(u)$ and $h(w)$ are assigned sets so that $f(u) \cup f(w) = \{1, 2\}$ with $|f(u)| = |f(w)| = 1$. Now suppose that $f(u) \neq \emptyset$. Then depending on $f(v)$ and $f(w)$ we can define as previously a $\gamma_{r2}(G)$-function $h$ on $G$ such that $h(x) = f(x)$ if $x \notin \{u, v, w\}$, $h(v) = \emptyset$, and $h(u)$ and $h(w)$ are assigned sets so that $f(u) \cup f(w) = \{1, 2\}$ and $|f(u)| = |f(w)| = 1$. ■

Now we are ready to establish our next result.

**Theorem 7** *If $T$ is a tree of order $n \geq 3$ with $\ell$ leaves and $s$ stems, then $\gamma_{r2}(T) \leq (2n + \ell + s)/4$.*

**Proof.** We use an induction on the order $n$ of $T$. If $n = 3$, then $\gamma_{r2}(T) = 2 < (2n + \ell + s)/4 = 9/4$, establishing the base case.

Let $n \geq 4$, and assume that every tree $T'$ of order $n'$, where $3 \leq n' < n$ with $\ell'$ leaves and $s'$ stems satisfies $\gamma_{r2}(T') \leq (2n' + \ell' + s')/4$. Let $T$ be a tree of order $n$.

Since for stars $K_{1,p}$, we have $\gamma_{r2}(T) = 2 < (2n+\ell+s)/4$, we may assume that $T$ has diameter at least three. Suppose now that $T$ contains two adjacent vertices $u, v$, where each of $u$ and $v$ has degree at least three. Let $T(u)$ and $T(v)$ denote the subtrees of $T$ containing $u$ and $v$ respectively, obtained by removing the edge $uv$. Let $n_1, \ell_1, s_1$ be the order, the number of leaves and stems of $T(u)$, respectively, and likewise let $n_2, \ell_2, s_2$ for $T(v)$. Clearly $n_1 + n_2 = n$, and since $n_1$ and $n_2 \geq 3$, we have $\ell_1 + \ell_2 = \ell$, and $s_1 + s_2 = s$. Applying the inductive hypothesis to $T(u)$ and $T(v)$, we have $\gamma_{r2}(T(u)) \leq (2n_1+\ell_1+s_1)/4$ and $\gamma_{r2}(T(v)) \leq (2n_2+\ell_2+s_2)/4$. Let $f_1$ be a $\gamma_{r2}(T(u))$-function and likewise let $f_2$ be a $\gamma_{r2}(T(v))$-function. We define a 2RDF $f$ on $V(T)$ by letting $f(x) = f_1(x)$ if $x \in V(T(u))$ and $f(x) = f_2(x)$ if $x \in V(T(v))$. Clearly $f$ is a 2RDF of $T$ and so $\gamma_{r2}(T) \leq w(f_1) + w(f_2) \leq (2n_1 + \ell_1 + s_1)/4 + (2n_2 + \ell_2 + s_2)/4 = (2n + \ell + s)/4$. Thus from now on we may assume that all neighbors of every vertex of degree at least three have degree at most two.

Now consider a diametrical path $P : u_0\text{-}u_1\text{-}u_2\text{-}...\text{-}u_{\text{diam}(T)}$. Clearly $u_1$ is a stem. Also we note that if $\text{diam}(T) = 3$, then $3 \leq \gamma_{r2}(T) \leq 4$ and it can be checked easily that $\gamma_{r2}(T) \leq (2n + \ell + s)/4$. Hence we can assume that $\text{diam}(T) \geq 4$. Consider the following cases.

**Case 1.** $d_T(u_1) \geq 3$. Then as assumed previously, $d_T(u_2) = 2$. Let $T'$ be the tree resulting from $T$ by removing $u_1$, $u_2$ and all leaves of $u_1$. If $n' = 2$, then $n \geq 6$, $\ell' \geq 3$ and $s' = 2$, and so $\gamma_{r2}(T) = 4 < (2n + \ell + s)/4$. Thus let $n' \geq 3$. It follows that $n' = n - 2 - |L_{u_1}|, \ell' \leq \ell - 1$ and $s' \leq s$. If $f'$ is any $\gamma_{r2}(T')$-function, then define a 2RDF $f$ on $V(T)$ by letting $f(x) = f'(x)$ if $x \in V(T')$, $f(u_1) = \{1, 2\}$ and $f(x) = \emptyset$ if $x \in L_{u_1} \cup \{u_2\}$. It follows that $\gamma_{r2}(T) = w(f) \leq w(f') + 2$. Using the induction on $T'$, we obtain $\gamma_{r2}(T) \leq (2n' + \ell' + s')/4 + 2 < (2n + \ell + s)/4$.

**Case 2.** $d_T(u_1) = 2$. We first assume that $d_T(u_2) \geq 3$. Suppose there are two vertices $u'_1, u'_0$ so that $u'_0\text{-}u'_1\text{-}u_2\text{-}...\text{-}u_{\text{diam}(T)}$ is also a diametrical path. According to Case 1, we can assume that $d_T(u'_1) = 2$. Let $T'$ be the tree resulting from $T$ by removing $u_1$ and $u_0$. Then $n' = n-2 \geq 3, \ell' = \ell-1$ and $s' = s-1$. By Observation 6, there is a $\gamma_{r2}(T')$-function $f'$ such that $f'(u'_0) \neq \emptyset, f'(u_2) \neq \emptyset$ and $f'(u'_1) = \emptyset$, where $f'(u'_0) \cup f'(u_2) = \{1, 2\}$ and $|\, f'(u'_0)| = |f'(u_2)| = 1$. We define a 2RDF $f$ on $V(T)$ by letting $f(x) = f'(x)$ if $x \in V(T')$, $f(u_1) = \emptyset$, and $f(u_0) = \{1\}$ or $\{2\}$ depending on $f(u_2)$ so that $f(u_0) \cup f(u_2) = \{1, 2\}$. It follows that $\gamma_{r2}(T) \leq w(f) = w(f') + 1$. Using the induction on $T'$, we obtain $\gamma_{r2}(T) \leq (2n' + \ell' + s')/4 + 2 = (2n + \ell + s)/4$. Thus we can assume now that $P$ is the unique diametrical path containing $u_2$. Since $d_T(u_2) \geq 3$, $u_2$ is a stem and $d_T(u_3) = 2$. Thus the subtree induced by $u_1, u_2$ and their neighbors is a double star, say $S$, of order at least 5. Note that $\gamma_{r2}(S) = 3$. Let $T'$ be the tree obtained from $T$ by removing all vertices of $S$. Clearly, $n' = n-4-|L_{u_2}| \geq 1$ since $\text{diam}(T) \geq 4$. If $n' = 1$ or 2, then $\gamma_{r2}(T) = 4$ or 5, respectively, and the result is valid. So assume that $n' \geq 3$. Clearly, $\ell' \leq \ell - |L_{u_2}|$ and $s' \leq s - 1$. Also $\gamma_{r2}(T) \leq \gamma_{r2}(T') + \gamma_{r2}(S)$. Applying the inductive hypothesis to $T'$, we obtain $\gamma_{r2}(T) \leq (2n' + \ell' + s')/4 + 3 \leq (2n + \ell + s)/4$.

Finally assume that that $d_T(u_2) = 2$. If $d_T(u_3) \geq 3$, then let $T'$ be the subtree obtained from $T$ by removing $u_0$,$u_1$ and $u_2$. Then $n' = n - 3 \geq 3, \ell' = \ell - 1$ and

$s' = s - 1$. We also have $\gamma_{r2}(T) \le \gamma_{r2}(T') + 2$. Applying the inductive hypothesis to $T'$, we obtain the desired result. If $d_T(u_3) = 2$, then let $T'$ be the obtained tree from $T$ by removing $u_0$ and $u_1$. Then $n' = n - 2 \ge 3$. If $n' = 3$, then $T$ is a path $P_5$ and the result is valid. So assume that $n' \ge 4$. Then $\ell' = \ell$ and $s' = s$. Now let $f'$ be a $\gamma_{r2}(T')$-function satisfying Observation 6. We define a 2RDF $f$ on $V(T)$ by letting $f(x) = f'(x)$ if $x \in V(T')$, $f(u_1) = \emptyset$, and $f(u_0) = \{1\}$ or $\{2\}$ depending on $f'(u_2)$ so that $f(u_0) \cup f(u_2) = \{1, 2\}$. It follows that $\gamma_{r2}(T) \le \gamma_{r2}(T') + 1$. Now, applying the inductive hypothesis to $T'$, we obtain the desired result. ■

Note that since for trees of order $n \ge 3$, $\ell + s \le n$, the upper bound of Theorem 7 improves the upper bound of Theorem 4 for trees.

According to Theorems 3 and 7 we obtain the following upper bound on the Roman domination in trees that improves in some sense the bound in Theorem 5 for all trees $T$ with $\ell + s \le 2n/5$.

**Corollary 8** *If $T$ is a tree of order $n \ge 3$ with $\ell$ leaves and $s$ stems, then $\gamma_R(T) \le (2n + \ell + s)/3$.*

**Proof.** By Theorem 3, $\frac{3}{4}\gamma_R(T) \le \gamma_{r2}(T)$, and so by Theorem 7 we obtain $\frac{3}{4}\gamma_R(T) \le \gamma_{r2}(T) \le (2n + \ell + s)/4$. Hence $\gamma_R(T) \le (2n + \ell + s)/3$. ■

The following result established in [10] relates the 2-rainbow domination number of a graph $G$ to the domination number and the order of $G$.

**Proposition 9** *For any connected graph $G$ of order $n \ge 3$, then $\gamma_{r2}(G) + \frac{\gamma(G)}{2} \le n$.*

Recall that a set $R \subseteq V(G)$ is a *packing set* of $G$ if $N[x] \cap N[y] = \emptyset$ holds for any two distinct vertices $x, y \in R$. The *packing number* $\rho(G)$ is the maximum cardinality of a packing in $G$. Let $\delta$ denote the *minimum degree* of the graph $G$.

**Proposition 10** *If $G$ is a connected graph of order $n$, then $\gamma_{r2}(G) + (\delta - 1)\rho(G) \le n$.*

**Proof.** Obviously, the result holds if $n \in \{1, 2\}$. So assume that $n \ge 3$. Let $R$ be a maximum packing set of $G$, $A = N(R)$ and $B = V(G) - (A \cup R)$. Clearly $|A| \ge \delta\,|R|$ and $|B| = n - |A \cup R| \le n - (\delta + 1)\,|R|$. Now define a 2RDF $f$ on $V(G)$ by letting $f(x) = \{1, 2\}$ if $x \in R$; $f(x) = \emptyset$ if $x \in A$ and $f(x) = \{1\}$ or $\{2\}$ if $x \in B$. It follows that $\gamma_{r2}(G) \le w(f) = 2\,|R| + |B|$. Using the previous inequality we obtain the desired result. ■

A *hole* in a graph is an induced subgraph that is a cycle of length at least 4. A *chordal graph* is a graph with no hole. A graph is *strongly chordal* if it is chordal and every even cycle of length at least 6 has a *strong chord*, meaning a chord joining vertices whose distance along the cycle is odd. Farber [6] proved that the domination number and packing number are equal for any strongly chordal graph. Thus we have the following corollary to Proposition 10.

**Corollary 11** *For any connected strongly chordal graph $G$, we have $\gamma_{r2}(G) + (\delta - 1)\gamma(G) \leq n$.*

It is remarkable that since for any graph $G$, $\gamma(G) \leq \gamma_R(G)$, one may study a similar bound as Proposition 9 replacing $\gamma(G)$ by $\gamma_R(G)$. However, it is not the case that for any graph $G$, $\gamma_{r2}(G) + \frac{\gamma_R(G)}{2} \leq n$, as the path $P_4$ does not satisfy it. In the following we show that the difference $\gamma_{r2}(G) + \frac{\gamma_R(G)}{2} - |V(G)|$ in a graph $G$ can be arbitrarily large.

**Proposition 12** *The difference $\gamma_{r2}(G) + \frac{\gamma_R(G)}{2} - n$ in a graph $G$ of order $n$ can be arbitrarily large.*

**Proof.** Let $k \geq 1$ be a positive integer, and let $m = 2(k+1)$. Let $P_{14}, P_{24}, ..., P_{m4}$ be $m$ copies of a path $P_4$. For $1 \leq i \leq m$, let $x_i$ be a stem of $P_{i4}$. Let $T$ be a tree obtained from $P_{14}, P_{24}, ..., P_{m4}$ by adding a vertex $o$ and joining $o$ to every $x_i$ for $i = 1, 2, ..., m$. It is straightforward to see that $\gamma_R(T) = \gamma_{r2}(T) = 3m$. Now $\gamma_{r2}(T) + \frac{\gamma_R(T)}{2} - |V(T)| = k$. ■

However, Theorems 4 and 5 imply that for any connected graph $G$ of order $n \geq 3$, $\gamma_{r2}(G) + \frac{\gamma_R(G)}{2} \leq n + \frac{3n}{20}$. We close the paper with the following problem.

**Problem 13** *Find a sharp upper bound for $\gamma_{r2}(G) + \frac{\gamma_R(G)}{2}$ in a connected graph $G$ of order $n \geq 3$.*

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